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On tzitzeica surfaces in euclidean 3-space $\mathbb{E}^3$

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Abstract

In this study, we consider Tzitzeica surfaces (Tz-surface) in Euclidean 3-Space $\mathbb{E}^3$. We have been obtained Tzitzeica surfaces conditions of some surfaces. Finally, examples are given for these surfaces.

Keywords: *Tzitzeica condition, Tzitzeica surface, fundamental form, Gauss curvature.*

Öklid-3 uzayındaki tzitzeica yüzeyleri üzerine

Öz

Bu çalışmada Öklid-3 uzayındaki Tzitzeica yüzeylerini incelendi. Bazı yüzeylerin Tzitzeica yüzey şartları incelendi. Son olarak bu yüzeyler için örnekler verildi.

Anahtar kelimeler: *Tzitzeica şartı, Tzitzeica yüzeyi, temel form, Gauss eğriliği.*

1.Introduction

Gheorgha Tzitzeica, Romanian mathematician (1872-1939) introduced a class of curves, nowadays called Tzitzeica curves and a class of surfaces of the Euclidean 3-space called Tzitzeica surfaces. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x=x(s)$ with the Frenet frame $\{T, N_1, N_2\}$ and curvatures $\{k_1, k_2\}$ which the ratio of its torsion k_2 and the square of the distance d_{osc} from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e.,

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$$\frac{k_2}{{d_{osc}}^2} = a \tag{1}$$

where $d_{osc} = \langle N_2, x \rangle$ and $a \neq 0$ is a real constant, N_2 is the binormal vector field of x.

In [1], the authors gave the connections between Tzitzeica curve and Tzitzeica surface in Minkowski 3-space and the original ones from the Euclidean 3-space.

A Tzitzeica surface in $\mathbb{E}^3$ is a spatial surface M given with the parametrization *X(u,v)* for which the ratio of its gaussian curveture K and the distance d_{tan} from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e.,

$$\frac{K}{{d_{tan}}^4} = a_1 \tag{2}$$

for a constant $a_1 \neq 0$. The ortogonal distance from the origin to the tangent plane is defined by

$$d_{tan} = \langle X, N \rangle \tag{3}$$

where X is the position vector of surface and N is unit normal vector field of the surface.

The asimptotic lines of a tzitzeica surface with negative Gaussian curvature are Tzitzeica curves [2]. In [3], authors gave the necessary and sufficient condition for Cobb-Douglass production hypersurface to be a Tzitzeica hypersurface. In addition, a new Tzitzeica hypersurface was obtained in parametric, implicit and explicit forms in [4].

In this study, we consider Tzitzeica surface (Tz-surface) in Euclidean 3-space $\mathbb{E}^3$. We have been obtained Tzitzeica surface conditions of some surface.

Let M be a regular surface in $\mathbb{E}^3$ given with the parametrization $X(u, v): (u, v) \in D \subset \mathbb{E}^2$. The tangent space of M at an arbitrary point $p = X(u, v)$ is spanned by the vectors X_u and X_v. The first fundamental form coefficients of M are computed by

$$\begin{aligned} E &= \langle X_u, X_u \rangle \\ F &= \langle X_u, X_v \rangle \\ G &= \langle X_v, X_v \rangle \end{aligned} \tag{4}$$

where $\langle\, , \rangle$ is the scalar product of the Euclidean space. We consider the surface patch *X(u,v)* is regular, which implies that $W^2 = EG - F^2 \neq 0$.

The second fundamental form coefficient of M are computed by

$$\begin{aligned} e &= \langle X_{uu}, N \rangle \\ f &= \langle X_{uv}, N \rangle \\ g &= \langle X_{vv}, N \rangle \end{aligned} \tag{5}$$

where, N is unit normal vector field of the surface. The Gaussian curvature are given by

$$K = \frac{eg - f^2}{EG - F^2} \tag{6}$$

2.Tzitzeica surfaces in $\mathbb{E}^3$

Definition 2.1 Let $x: I \subset \mathbb{R} \to \mathbb{E}^2$ be a unit speed plane curve with curvatures $k_1(s) > 0$. If the curvature of x satisfies the condition

$$k_1(s) = a.{d_{osc}}^2, \tag{7}$$

for some real constant $a \neq 0$, then x is called planer Tz-curve, where $d_{osc} = \langle n, x\rangle$ and n is the unit normal vector field of x.

Proposition 2.2 Let M be a regular surface in $\mathbb{E}^3$ given with parametrization

$$X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big). \tag{8}$$

Then M is Tz-surface if and only if

$$(eg - f^2)(EG - F^2) = a_1.(det(X, X_u, X_v))^4 \tag{9}$$

Holds, where $a_1 \neq 0$ real constant and $x(u,v), y(u,v), z(u,v)$ is differentiable functions.
Proof. $N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$ is unit normal vector field of the surface. By the use of equations (2), (3), (5) we get (9).

Proposition 2.3 Let M be a regular surface in $\mathbb{E}^3$ with the parametrization (8). If M is Tz-surface then the equation

$$\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} - \begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}^2 = a_1 \begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}^4 \tag{10}$$

holds, where $a_1 \neq 0$ real constant.

Proof: Considering together (4), (5), (6) and the unit normal vector field of M

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{1}{W} \begin{vmatrix} e_1 & e_2 & e_3 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}, \tag{11}$$

we have,

$$K = \frac{eg - f^2}{EG - F^2}$$
$$= \frac{\langle X_{uu}, N\rangle\langle X_{vv}, N\rangle - \langle X_{uv}, N\rangle^2}{W^2}$$

$$= \frac{1}{W^2}\left\{\langle X_{uu}, \frac{X_u \times X_v}{\|X_u \times X_v\|}\rangle \langle X_{vv}, \frac{X_u \times X_v}{\|X_u \times X_v\|}\rangle - \langle X_{uv}, \frac{X_u \times X_v}{\|X_u \times X_v\|}\rangle^2\right\}$$
$$= \frac{1}{W^2}\left\{\frac{1}{(\|X_u \times X_v\|)^2}\left[det(X_{uu}, X_u, X_v)\det(X_{vv}, X_u, X_v) - (\det(X_{uv}, X_u, X_v))^2\right]\right\}$$
$$= \frac{1}{W^2}\frac{\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} - \begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}^2}{W^2}. \tag{12}$$

On the other hand

$$d_{tan} = \langle X, N\rangle$$
$$= \langle (x, y, z), \frac{X_u \times X_v}{\|X_u \times X_v\|}\rangle$$
$$= \frac{1}{W}\begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \tag{13}$$

is obtained. Substituting fourth exponent of (13) and (12) into (2), we get the result.

Proposition 2.4 Let M be a regular surface in $\mathbb{E}^3$ given with the parametrization (8). Then M is Tz-surface if and only if the equation

$$a^2(x_{uu}x_{vv} - x_{uv}^2) + b^2(y_{uu}y_{vv} - y_{uv}^2) + c^2(z_{uu}z_{vv} - z_{uv}^2)$$
$$+ab(x_{uu}y_{vv} + y_{uu}x_{vv} - 2x_{uv}y_{uv}) + ac(x_{uu}z_{vv} + z_{uu}x_{vv} - 2x_{uv}z_{uv})$$
$$+bc(y_{uu}z_{vv} + z_{uu}y_{vv} - 2y_{uv}z_{uv}) = a_1(ax + by + cz)^4 \tag{14}$$

holds, where

$$a(u, v) = y_u z_v - y_v z_u$$
$$b(u, v) = -x_u z_v + x_v z_u$$
$$c(u, v) = x_u y_v - x_v y_u$$

are differentiable functions and $a_1 \neq 0$ real constant.

Proof: The first and second derivatives of X are replaced by (4) and (5). By the use of (2), (3), (6) we obtained (14).

Definition 2.5 The equation given by (14) is called the *Tz-surface equations*.

3.Tz-Monge surface

Definition 3.1 A Monge patch is a patch $X: U \subset \mathbb{E}^2 \to \mathbb{E}^3$ of the form

$$X(u, v) = \big(u, v, f(u, v)\big) \tag{15}$$

where U is an open set in $\mathbb{E}^2$ and $f: U \to \mathbb{R}$ is a differentiable function [5].

Theorem 3.2 Let *M* be a regular surface in $\mathbb{E}^3$ given with the parametrization (15). Then *M* is a Tz-surface if and only if

$$a_1 = \frac{f_{uu}.f_{vv} - f_{uv}^2}{(-uf_u - vf_v + f)^4} \tag{16}$$

holds, where $a_1 \neq 0$ real constant.

Proof. Differentiating (15) with respect to *u* and *v* we obtain $X_u = (1,0,f_u)$ and $X_v = (0,1,f_v)$ respectively. We can find the coefficients of the first fundamentel form as follows:

$$E = 1 + f_u^2\,, \qquad F = f_u.f_v\,, \qquad G = 1 + f_v^2. \tag{17}$$

The unit normal vector field of *M* is given by the following vector field;

$$N = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}}(-f_u, -f_v, 1)\,. \tag{18}$$

The second partial derivatives of *X* are expressed as follows:

$$X_{uu} = (0,0,f_{uu})\,, \qquad X_{uv} = (0,0,f_{uv})\,, \qquad X_{vv} = (0,0,f_{vv}) \tag{19}$$

Using (18) and (19) we can get the coefficients of the second fundamental form

$$e = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}\,, \qquad f = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}}\,, \qquad g = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}\,. \tag{20}$$

Substituing (17) and (20) into (6) we obtain the Gaussian curvature as follows:

$$K = \frac{f_{uu}.f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2} \tag{21}$$

Substituing (18) into (3) we obtain

$$d_{tan} = \frac{-uf_u - vf_v + f}{\sqrt{1 + f_u^2 + f_v^2}}\,. \tag{22}$$

Consequently, by the use of (21) and (22) with (2) we get the result.

Example 3.3 Let *M* be a Monge patch in $\mathbb{E}^3$ with given by parametrization

$$X(u,v) = \left(u, v, \frac{-(3 + uv)}{(u + v)}\right),$$
$$f(u,v) = \frac{-(3 + uv)}{(u + v)} \tag{23}$$

is a differentiable function substituing by differentiating the equation (23) into (16) we obtain $a_1 = -\frac{1}{108}$ which means that *M* is a Tz-surface.

4. Tz-Translation surface

Definition 4.1 A surface *M* defined as the sum of two plane curves $\alpha(u) = (u, 0, f(u))$ and $\beta(v) = (0, v, g(v))$ is called a first type translation surface (is also known translation surface) in $\mathbb{E}^3$. So, a first type translation surface is defined by the parametrization

$$X(u, v) = \big(u, v, f(u) + g(v)\big). \tag{24}$$

A surface *M* defined as the sum of two plane curves (which are not lines) $\alpha(u) = (u, 0, f(u))$ and $\beta(v) = (v, g(v), 0)$ is called a second type translation surface in $\mathbb{E}^3$. So, a second type translation surface is defined by the parametrization

$$X(u, v) = \big(u + v, g(v), f(u)\big) \tag{25}$$

where *f* and *g* are smooth functions [6].

Theorem 4.2 Let *M* be a first type translation surface in $\mathbb{E}^3$ with given by parametrization (24). Then *M* is a Tz-surface if and only if

$$a_1 = \frac{f''g''}{(-uf' - vg' + f + g)^4} \tag{26}$$

holds, where $a_1 \neq 0$ real constant, *f* and *g* are smooth functions, α and β (which are not lines) are non-regular curves.

Proof. Differentiating (24) with respect to *u* and *v*, we obtain $X_u = (1,0,f')$ and $X_v = (0,1,g')$ respectively. We can find the coefficients of the first fundamental form as follow:

$$E = 1 + f'^2, \qquad F = f'.g', \qquad G = 1 + g'^2 \tag{27}$$

The unit normal vector field of *M* is given by the following vector field

$$N = \frac{(-f', -g', 1)}{\sqrt{1 + f'^2 + g'^2}}\ . \tag{28}$$

The second partial derivatives of *X* are expressed as follows:

$$X_{uu} = (0,0,f''), \qquad X_{uv} = (0,0,0), \qquad X_{vv} = (0,0,g''). \tag{29}$$

Using (28) and (29) we can get the coefficients of the second fundamental form

$$e = \frac{f''}{\sqrt{1 + f'^2 + g'^2}} \,, \qquad f = 0 \,, \qquad g = \frac{g''}{\sqrt{1 + f'^2 + g'^2}} \tag{30}$$

substituing (27) and (30) into (6) we obtain the Gaussian curvature as follows:

$$K = \frac{f''g''}{\left(1 + f'^2 + g'^2\right)^2} \tag{31}$$

substituing (28) into (3) we obtain

$$d_{tan} = \frac{(-uf' - vg' + f + g)}{\sqrt{1 + f'^2 + g'^2}} \;. \tag{32}$$

Consequently, by the use of (31) and (32) with (2) we get the result.

Theorem 4.3 Let *M* be a second type translation surface in $\mathbb{E}^3$ with given by the parametrization (25). Then *M* is a Tz-surface if and only if

$$a_1 = \frac{f'g'f''g''}{(-uf'g' - vf'g' + fg' + gf')^4} \tag{33}$$

holds, where $a_1 \neq 0$ real constant, *f* and g are smooth functions, α and β (which are not lines) are non-regular curves.

Proof. Differentiating (25) with respect to *u* and *v* we obtain $X_u = (1,0,f')$ and $X_v = (1,g',0)$ respectively. We can find coefficients of the first fundamental form as follow:

$$E = 1 + f'^2 \,, \qquad F = 1 \,, \qquad G = 1 + g'^2 \tag{34}$$

The unit normal vector field of *M* is given by the following vector field

$$N = \frac{(-f'g', f', g')}{\sqrt{f'^2 g'^2 + f'^2 + g'^2}} \;. \tag{35}$$

The second partial derivatives of *X* are expressed as follow:

$$X_{uu} = (0,0,f'') \,, \qquad X_{uv} = (0,0,0) \,, \qquad X_{vv} = (0,g'',0) \,. \tag{36}$$

Using (35) and (36) we can get the coefficients of the second fundamental form

$$e = \frac{g'f''}{\sqrt{f'^2 g'^2 + f'^2 + g'^2}} \,, \qquad f = 0 \,, \qquad g = \frac{f'g''}{\sqrt{f'^2 g'^2 + f'^2 + g'^2}} \tag{37}$$

substituing (34) and (37) into (6) we obtain the Gaussian curvature as follows:

$$K = \frac{f'f''g'g''}{\left(f'^2g'^2 + f'^2 + g'^2\right)^2} \quad . \tag{38}$$

Substituing (35) into (3) we obtain

$$d_{tan} = \frac{-(u+v)f'g' + f'g + g'f}{\sqrt{f'^2g'^2 + f'^2 + g'^2}} \quad . \tag{39}$$

Consequently, by the use of (38) and (39) with (2) we get the result.

Corollary 4.4 Let *M* be a first type Tz-translation surface in $\mathbb{E}^3$ with given by the parametrization (24). If $\alpha(u)$ and $\beta(v)$ are non-geodesic planar Tz-curves then

$$a_1 = \frac{f''g''}{\left(\frac{\sqrt{f''}}{\sqrt{a_\alpha}\left(1+f'^2\right)^{\frac{1}{4}}} + \frac{\sqrt{g''}}{\sqrt{a_\beta}\left(1+g'^2\right)^{\frac{1}{4}}}\right)^4} \tag{40}$$

holds, where $a_1 \neq 0$ real constant, a_α and a_β are planar Tz-curve constants of α and β curves respectively.

Proof. If $\alpha(u)$ and $\beta(v)$ are non-geodesic planar Tz-curves then by the use of (7) and $d_{osc} = \langle N_1, x\rangle$ equality, we get

$$a_\alpha = \frac{f''}{\sqrt{1+f'^2}(-uf'+f)^2} \tag{41}$$

and

$$a_\beta = \frac{g''}{\sqrt{1+g'^2}(-vg'+g)^2} \tag{42}$$

substituing (41) and (42) into (26) we get the result.

Corollary 4.5 Let *M* be a first type Tz-translation surface in $\mathbb{E}^3$ with given by parametrization (24). Let $\alpha(u)$ and $\beta(v)$ are non-geodesic planar Tz-curves. If

$$\sqrt{\left(1+f'^2\right)\left(1+g'^2\right)} = A.(4+A) + \frac{1}{A}\left(4+\frac{1}{A}\right) + 6 \tag{43}$$

then that is

$$a_\alpha . a_\beta = a_1 \tag{44}$$

where

$$A = \frac{-uf' + f}{-vg' + g} \tag{45}$$

a_α and a_β are planar Tz-curve constants of α and β curves respectively and a_1 is Tz-surface constant of the first type Tz-translation surface.

Proof: By the use of the equation (41) and (42) we get

$$a_\alpha . a_\beta = \frac{f''}{\sqrt{1 + f'^2}(-uf' + f)^2} . \frac{g''}{\sqrt{1 + g'^2}(-vg' + g)^2} \tag{46}$$

Substituing (43) and (45) into (46) we get the equation (26). Thus the proof is completed.

5. Tz-factorable surface

Definition 5.1 A surface *M* in $\mathbb{E}^3$ is called factorable surface if the parametrization of *M* can be written as

$$X(u, v) = \big(u, v, f(u). g(v)\big) \tag{47}$$

or

$$X(u, v) = (f(u). g(v), u, v) \tag{48}$$

or

$$X(u, v) = (u, f(u). g(v), v) \tag{49}$$

where *f* and *g* are smooth functions. The Factorable surfaces in the Euclidean Space, the pseudo Euclidean Space and Heisenberg group have been studied in [7-10].

Theorem 5.2 Let *M* be a regular surface in $\mathbb{E}^3$ given by the parametrization (47), (48) and (49). Then *M* is a Tz-surface if and only if

$$a_1 = \frac{ff''gg'' - (f'g')^2}{(-uf'g - vfg' + fg)^4} \tag{50}$$

holds, where $a_1 \neq 0$ real constant, *f* and *g* are smooth functions.

Proof. Differentiating (47), (48), (49) with respect to *u* and *v*, we can find the coefficients of the first and the second fundamental forms with (4) and (5). Substituing (3) and (6) into (2) we get the result.

Example 5.3 Let *M* be a Monge patch in $\mathbb{E}^3$ with given by the parametrization

$$X(u,v)=\left(u,v,\frac{1}{uv}\right).$$

$f(u)=\frac{1}{u}$ and $g(v)=\frac{1}{v}$ are differentiable functions. Substituing by differentiating equations *f* and *g* into (50) we obtain $a_1=\frac{1}{27}$ which means that M is a Tz-surface .

6. Tz-spherical product surface

Definition 6.1 Let $\alpha,\beta:\mathbb{R}\to\mathbb{E}^2$ be two Euclidean planar curves. Assume $\alpha(u)=(f_1(u),f_2(u))$ and $\beta(v)=(g_1(v),g_2(v))$. Then their spherical product immersions is given by,

$$X=\alpha\otimes\beta:\mathbb{E}^2\to\mathbb{E}^3$$
$$X(u,v)=\big(f_1(u),f_2(u)g_1(v),f_2(u)g_2(v)\big)\,, \tag{51}$$

$U_0<u<U_1, V_0<v<V_1$, which is a surface in $\mathbb{E}^3$ [11,12].

Theorem 6.2 The spherical product surface patch $X(u,v)=\alpha(u)\otimes\beta(v)$ of two planar curves α and β is a Tz-surface if and only if

$$a_1=\frac{-f_1'(f_1''f_2'-f_1'f_2'')(g_1''g_2'-g_1'g_2'')}{f_2(f_1f_2'-f_1'f_2)^4(g_1g_2'-g_1'g_2)^3} \tag{52}$$

holds, where $a_1\neq 0$ is real constant.

Proof. Differentiating (51) with respect to *u* and *v*, we obtain $X_u=(f_1',f_2'g_1,f_2'g_2)$ and $X_v=(0,f_2g_1',f_2g_2')$ respectively. We can find the coefficient of the first fundamental form as follow:

$$E=f_1'^2+f_2'^2(g_1^2+g_2^2), F=f_2f_2'(g_1g_1'+g_2g_2'), G=f_2^2\left(g_1'^2+g_2'^2\right) \tag{53}$$

The unit normal vector field of spherical product surface path is given by the following vector field

$$N=\frac{(f_2'(g_1g_2'-g_1'g_2),-f_1'g_2',f_1'g_1')}{\sqrt{f_1'^2\left(g_1'^2+g_2'^2\right)+f_2'^2(g_1g_2'-g_1'g_2)^2}}\ . \tag{54}$$

The second partial derivatives of *X* are expressed as follows:

$$X_{uu}=(f_1'',f_2''g_1,f_2''g_2)\,,X_{uv}=(0,f_2'g_1',f_2'g_2'),X_{vv}=(0,f_2g_1'',f_2g_2'') \tag{55}$$

Using (54) and (55) we can get the coefficient of the second fundamental form

$$e=\frac{(f_1''f_2'-f_1'f_2'')(g_1g_2'-g_1'g_2)}{\sqrt{f_1'^2\left(g_1'^2+g_2'^2\right)+f_2'^2(g_1g_2'-g_1'g_2)^2}}$$

$$f = 0 \tag{56}$$
$$g = \frac{f_1' f_2 (g_1' g_2'' - g_1'' g_2')}{\sqrt{f_1'^2 \left({g_1'}^2 + {g_2'}^2\right) + f_2'^2 (g_1 g_2' - g_1' g_2)^2}}.$$

Substituing (53) and (56) into (6) we obtain the Gaussian curvature as follows

$$K = \frac{(f_1'' f_2' - f_1' f_2'')(g_1 g_2' - g_1' g_2) f_1' (g_1' g_2'' - g_1'' g_2')}{f_2 \left(f_1'^2 \left({g_1'}^2 + {g_2'}^2\right) + f_2'^2 (g_1 g_2' - g_1' g_2)^2\right)^2} \tag{57}$$

Substituing (54) into (3) we obtain

$$d_{tan} = \frac{(f_1 f_2' - f_1' f_2)(g_1 g_2' - g_1' g_2)}{\sqrt{f_1'^2 \left({g_1'}^2 + {g_2'}^2\right) + f_2'^2 (g_1 g_2' - g_1' g_2)^2}}. \tag{58}$$

Consequently, by the use of (57) and (58) with (2) we get the result.

Corollary 6.3 Let $X(u,v) = \alpha(u) \otimes \beta(v)$ be the spherical product surface patch of two planar curves given with the parametrization (51). If α and β are unit speed curve then that is

$$a_1 = \frac{-f_1' k_{1\alpha} k_{1\beta}}{f_2 (f_1 f_2' - f_1' f_2)^4 (g_1 g_2' - g_1' g_2)^3} \tag{59}$$

where $k_{1\alpha} = (f_1'' f_2' - f_1' f_2'')$ and $k_{1\beta} = (g_1'' g_2' - g_1' g_2'')$ are curvatures of α and β curves, respectively.

Example 6.4 Let $\alpha, \beta : \mathbb{R} \to \mathbb{E}^2$ be two Euclidean planar curves. Assume $\alpha(u) = \left(f_1(u), f_2(u)\right) = (\cosh u, \sinh u)$ and $\beta(v) = \left(g_1(v), g_2(v)\right) = (\cosh v, \sinh v)$. Then the parametrization of spherical product surface M is given by

$$X(u,v) = (\cosh u, \sinh u \cosh v, \sinh u \sinh v).$$

Substituing the first and second derivatives of $f_1(u), f_2(u), g_1(v), g_2(v)$ into (52), we obtain $a_1 = -1$ which means that spherical product surface M is a Tz-surface .

Example 6.5 Let α and β be two Euclidean planar curves. Assume $\alpha(u) = (\cos(c+u), \sin(c+u))$ and $\beta(v) = (\sin(c_1+v), \cos(c_1+v))$. Then the parametrization of spherical product surface M is given by

$$X(u,v) = (\cos(c+u), \sin(c+u)\sin(c_1+v), \sin(c+u)\cos(c_1+v)) .$$

By using (59) we obtain $a_1 = 1$ which means that spherical product surface is a Tz-surface.

7. Tz-surface of revolution

Definition 7.1 Let $\alpha, \beta: \mathbb{R} \to \mathbb{E}^2$ be two Euclidean planar curves. Assume $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (\cos v, \sin v)$. Then their spherical product immersion is given by

$$X(u,v) = (f_1(u), f_2(u)\cos v, f_2(u)\sin v)\,. \tag{60}$$

The spherical product immersion given by the parametrization (60) is called surface of revolution.

Theorem 7.2 Surface of Revolution given by the parametrization (60) is a Tz-surface if and only if

$$a_1 = \frac{f_1'(f_1''f_2' - f_1'f_2'')}{f_2(f_1f_2' - f_1'f_2)^4\left({f_1'}^2 + {f_2'}^2\right)} \tag{61}$$

holds, where $a_1 \neq 0$ is real constant.

Proof. Differentiating (60) with respect to u and v, we obtain $X_u = (f_1', f_2'\cos v, f_2'\sin v)$ and $X_v = (0, -f_2\sin v, f_2\cos v)$ respectively. We can find the coefficient of the first fundamental form as follow:

$$E = {f_1'}^2 + {f_2'}^2\,, \qquad F = 0\,, \qquad G = f_2^2 \tag{62}$$

The unit normal vector field of surface of revolution is given by the following vector field

$$N = (f_2', -f_1'\cos v, -f_1'\sin v)\,. \tag{63}$$

The second partial derivatives of X are expressed as follows

$$\begin{aligned} X_{uu} &= (f_1'', f_2''\cos v, f_2''\sin v) \\ X_{uv} &= (0, -f_2'\sin v, f_2'\cos v) \\ X_{vv} &= (0, -f_2\cos v, -f_2\sin v) \end{aligned} \tag{64}$$

Using (63) and (64), we can get the coefficients of the second fundamental form

$$e = f_1''f_2' - f_1'f_2''\,, \qquad f = 0\,, \qquad g = f_1'f_2 \tag{65}$$

substituing (62) and (65) into (5) we obtain the Gaussian curvature as follows

$$K = \frac{f_1'(f_1''f_2' - f_1'f_2'')}{f_2\left({f_1'}^2 + {f_2'}^2\right)} \tag{66}$$

Substituing (63) into (3), we obtain

$$d_{tan} = f_1f_2' - f_1'f_2\,. \tag{67}$$

Consequently, by the use of (66) and (67) with (2) we get the result.

Example 7.3 Let $\alpha(u) = (\cosh u, \sinh u)$ and $\beta(v) = (\cos v, \sin v)$. Then the surface of revolution is given by the parametrization

$$X(u, v) = (\cosh u, \sinh u \cos v, \sinh u \sin v).$$

By the using (61), we obtain $a_1 = 1$ which means that *X* is aTz-surface.

Corollary 7.4 If $\alpha(u) = (f_1(u), f_2(u))$ is unit speed curve then that is $a_1 = \frac{-f_2''}{f_2(f_1 f_2' - f_1' f_2)^4}$ where $a_1 \neq 0$ is real constant.

8. Tz-ruled surface

Definition 8.1 A ruled surface is a surface that can be swept out by moving a line in space. It therefore has a parametrization of the form

$$X(u, v) = \alpha(u) + v\gamma(u) \tag{66}$$

where $\alpha(u)$ is called the ruled surfacee directrix (also called the base curve) and $\gamma(u)$ is the director curve and $\alpha'(u) \neq 0$.

Theorem 8.2 If ruled surface given with the parametrization (66) is a Tz-surface, then that is

$$a_1 = \frac{-\left(det(\alpha', \gamma', \gamma)\right)^2}{\left(det(\alpha, \gamma, X_u)\right)^4} \tag{67}$$

where $a_1 \neq 0$ is real constant.

Proof. Let $\alpha(u) = (x_1(u), y_1(u), z_1(u))$ and $\gamma(u) = (x_2(u), y_2(u), z_2(u))$. Then, we obtain

$$\begin{aligned} X(u, v) &= \alpha(u) + v\gamma(u) \\ &= \left(x_1(u) + v x_2(u), y_1(u) + v y_2(u), z_1(u) + v z_2(u)\right) \\ &= \left(x(u, v), y(u, v), z(u, v)\right). \end{aligned} \tag{68}$$

By using (10), we get the result.

Example 8.3 Let $\alpha(u) = (\cos u, \sin u, 0)$ and $\gamma(u) = \alpha'(u) + e_3$ where $e_3 = (0,0,1)$. Then the parametrization of the ruled surface *X* is given by

$$\begin{aligned} X(u, v) &= \alpha(u) + v\gamma(u) \\ &= (\cos u, \sin u, 0) + v\left((-\sin u, \cos u, 0) + (0,0,1)\right) \\ &= (\cos u - v \sin u, \sin u + v \cos u, v). \end{aligned}$$

By using (67), we obtain $a_1 = -1$ which means that *X* is a Tz-surface.

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