# (1, 3) Optimal Linear Codes on Solid Bursts

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**Abstract -**The paper obtains a lower bound on the necessary number of parity-check digits in an $(n=n_1+n_2, k)$ linear code over GF(q) that corrects all solid burst errors of length $b_1$ or less in the first block of length $n_1$ and all solid burst errors of length $b_2$ or less in the second block of length $n_2$. Further, the author studies these codes over GF(2) that are optimal in a specific sense and gives a sufficient condition for the existence of such codes.

**Keywords-** *Parity check matrix, syndromes, solid burst, optimal codes.*

## 1.    Introduction

Perfect codes are considered to be very good codes for correcting errors occurred in the transmission of information. It was a good exercise for mathematicians for searching out perfect codes for several years. Then in 1971, Tietavanien and Perko[14] have established that there are no perfect codes other than the single error correcting Hamming codes[8], double and triple error correcting Golay codes ([6],[7]) and the Repetitive codes. Thereafter mathematician started to find codes that are not perfect in the usual sense but that correct certain types of error patterns and no others. They are called optimal codes. An attempt was given by Sharma and Dass[13] in the paper "Adjacent error correcting binary perfect codes". They studied codes that correct all solid burst errors of certain length and no others.

In certain memory systems (e.g. some spacecraft memories and supercomputer storage system), the most commonly occurred errors are solid burst errors.  By a solid burst error, we mean as follows:

**Definition:** *A solid burst error of length b is a vector with non zero entries in some b consecutive positions and zero elsewhere.*

Das[2] studied linear codes that detect and correct solid burst errors of certain length or less.  The author obtains bounds on the parity check for a linear code that detects and corrects such errors. A midway concept between error detection and error correction, known as Error Locating Codes, is also studied by Das[1]. This paper studies optimal codes that correct certain types of errors and no others.

Among the solid burst errors, the first most probable errors are solid burst error of length 1 and 2. And the next probable error is of solid burst of length 3. An effort is given to study these types of errors in this paper.

In many memory systems, the information is stored in different parts (sub-blocks) of the code and it is natural to expect that errors occur of different patterns in different sub-blocks. In this direction, Dass & Tyagi[4] explored a new type of binary (1, 2) optimal codes. Dass and Das[3] did a survey of such codes. Tuvi[15] also presented a paper which deals with the construction of such optimal codes. For more study on optimal codes, one may refer [5], [9], [10].

In this correspondence, this paper studies $(n=n_1+n_2, k)$ linear codes over GF(q) that correct all solid burst errors of length $b_1$ or less in the first block of length $n_1$ and all solid burst errors of length $b_2$ or less in the second block of length $n_2$. Section 2 obtains a lower bound on the number of parity check digits for such codes. Then, by fixing $b_1=1$ and $b_2=3$ in binary case, optimal codes for $n_1 \leq 21$ are investigated in section 3. In section 4, construction of such codes is presented in which sufficient condition for the existence of such codes

is also given. Section 5 gives the conclusion and open problem.

In what follows, the code length is taken to be $n$ over GF(q), consisting of two blocks of length $n_1$ and $n_2$, such that $n_1 + n_2 = n$. The distance between two vectors shall be considered in the Hamming sense.

## 2.    A Lower Bound

In the following, a lower bound on the number of parity check for an $(n = n_1 + n_2, k)$ linear code over GF(q) that corrects all *solid burst* errors of length $b_1$ or less in the first block of length $n_1$ and all *solid burst* errors of length $b_2$ or less in the second block of length $n_2$ are provided. The proof follows the technique used in the theorem 4.16, Peterson and Weldon [11].

**Theorem 1:** The number of parity check digits for an $(n=n_1+n_2, k)$ linear code over GF(q) that corrects all solid burst errors of length $b_1$ or less in the first block of length $n_1$ and all solid burst errors of length $b_2$ or less in the second block of length $n_2$ is at least

$$log_q \left\{ 1 + \sum_{i=1}^{b_1}(q-1)^i (n_1 - i + 1) + \sum_{j=1}^{b_2}(q-1)^j (n_2 - j + 1) \right\}$$

**Proof:** This proof is based on counting the number of errors of above specific type and comparing with the available cosets in the $(n=n_1+n_2, k)$ linear code over GF (q).

The number of *solid burst* errors of length $b_1$ or less in the first block of length $n_1$ is

$$\sum_{i=1}^{b_1}(q-1)^i (n_1 - i + 1).$$

The number of *solid burst* errors of length $b_2$ or less in the second block of length $n_2$

$$\sum_{j=1}^{b_2}(q-1)^j (n_2 - j + 1).$$

Thus the total number of errors, including the vector of all zero, is

$$1 + \sum_{i=1}^{b_1}(q-1)^i (n_1 - i + 1) + \sum_{j=1}^{b_2}(q-1)^j (n_2 - j + 1).$$

Hence

$$q^{n-k} \geq 1 + \sum_{i=1}^{b_1}(q-1)^i (n_1 - i + 1) + \sum_{j=1}^{b_2}(q-1)^j (n_2 - j + 1). \quad (1)$$

Now the equality of the inequality (1) gives us the optimal case i.e., we need to check for which parameters the bound is tight and the corresponding codes exist. In this communication, if we fixed $b_1=1$ and $b_2=3$, we obtain optimal codes. The codes are optimal in the sense that these codes correct all *solid burst* errors of length 1 or less in the first block of length $n_1$ and all *solid burst* errors of length 3 or less in the second block of length $n_2$ and no more. These codes are named as (1, 3) optimal codes.

## 3.    Optimal Codes

Considering the inequality (1) when the bound is optimal,

$$q^{n-k} = 1 + \sum_{i=1}^{b_1}(q-1)^i (n_1 - i + 1) + \sum_{j=1}^{b_2}(q-1)^j (n_2 - j + 1). \quad (2)$$

If we fix $b_1=1$ and $b_2=3$ for binary case, the equality (2) becomes

$$2^{n_1+n_2-k} = n_1 + 3n_2 - 2. \quad (3)$$

The codes obtained from different parameters of the equation (3) are called as binary (1, 3) optimal linear codes.

We now examine the possibility of existence of such codes for values of $n_1$, $n_2$, and $k$. This paper investigates for such codes for values of $n_1$ upto 21.

Now assigning the values of $n_1$ as 1, 2, 3, …,21 in equation (3), we shall find out possible values of the parameters $n_2$ and $k$.

**Let $n_1=1$.** Then the values of $n_2$ and k satisfying the equation (3) are

(3, 1), (11, 7), (43, 37), ……

This gives the possibility of the existence of (1+3, 1), (1+11, 7), (1+43, 37), …codes which may be binary (1, 3) linear optimal codes. Consider the matrix in the following

$$H = \begin{bmatrix} 1 & 100 \\ 0 & 010 \\ 1 & 001 \end{bmatrix}_{3\times(1+3)}$$

as the parity check matrix of a code which gives rise to (1+3, 1) code. This code can correct all *solid burst* errors of length 1 in the first block of length 1 and all *solid burst* errors of length 1 in the second block of length 3 or less and no other error pattern. It can be verified from the table 1 that the syndromes of all the errors are all distinct and exhaustive. This shows that the code under discussion is a binary (1, 3) optimal linear code.

Table 1

| Error patterns | Syndromes |
| --- | --- |
| 1000 | 101 |
| 0100 | 100 |
| 0010 | 010 |
| 0001 | 001 |
| 0110 | 110 |
| 0011 | 011 |
| 0111 | 111 |

**Let $n_1 = 2$.** Then the equation (3) reduces to
$$2^{2+n_2-k} = 3n_2.$$
It is clear that the above equation does not have integer solution for $n_2$. Therefore binary (1, 3) optimal linear code for $n_1 = 2$ cannot exist.

**Let $n_1 = 3$.** The equation (3) reduces to
$$2^{3+n_2-k} = 3n_2+1.$$
The various values of the parameter satisfying the above equation are
$$(n_2, k) = \{(5, 4), (21, 18), \dots\}.$$
This shows the possibility of the existence of (3+5, 4), (3+21, 18)…binary (1, 3) linear codes. Consider the matrix H,

$$H = \begin{bmatrix} 011 & 10001 \\ 111 & 01000 \\ 001 & 00101 \\ 111 & 00010 \end{bmatrix}_{4\times(3+5)}$$

The code obtained from the above matrix H as the parity check matrix is a (3+5, 4) linear code. The following error pattern-syndromes table 2 shows that all the syndromes of the errors are distinct and

exhaustive, thereby the code is a binary (1, 3) optimal linear code.

Table 2

| Error patterns | Syndromes |
| --- | --- |
| 10000000 | 0101 |
| 01000000 | 1101 |
| 00100000 | 1111 |
| 00010000 | 1000 |
| 00001000 | 0100 |
| 00000100 | 0010 |
| 00000010 | 0001 |
| 00000001 | 1010 |
| 00011000 | 1100 |
| 00001100 | 0110 |
| 00000110 | 0011 |
| 00000011 | 1011 |
| 00011100 | 1110 |
| 00001110 | 0111 |
| 00000111 | 1001 |

**Let $n_1 = 4$.** The equation (3) reduces to
$$2^{4+n_2-k} = 3n_2+2.$$
The various values of the parameter satisfying the above equation are
$$(n_2, k) = \{(2, 3), (10, 9), (42, 39) \dots\}, \text{ but}$$
(2, 3) does not fall under our case.
This shows the possibility of the existence of (4+10, 9), (4+42, 39)……...binary (1, 3) linear codes. Consider the matrix H as the parity check matrix of the code (4+10, 9):

$$H = \begin{bmatrix} 0110 & 1000010101 \\ 0011 & 0100001010 \\ 1101 & 0010000101 \\ 0100 & 0001010010 \\ 1011 & 0000101001 \end{bmatrix}_{5\times(4+10)}$$

It can be verified from the error pattern-syndromes table of the code that the (4+10, 9) code is a binary (1, 3) optimal linear code.

**Let $n_1 = 5$.** Then the equation (3) reduces to
$$2^{5+n_2-k} = 3(n_2+1).$$
The above equation does not have integer solution for $n_2$. Therefore binary (1, 3) optimal code for $n_1 = 5$ cannot exist.

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**Let $n_1= 6$.** The equation (3) reduces to
$$2^{6+n_2-k} = 3n_2+4.$$
The various values of the parameter satisfying the above equation are
$$(n_2, k) = \{(4, 6), (20, 20) \ldots\}.$$
This shows the possibility of the existence of (6+4, 6), (6+20, 20)…..binary (1, 3) linear codes. Consider the matrix H,

$$H=\begin{bmatrix} 101111 & 1000 \\ 010011 & 0100 \\ 100101 & 0010 \\ 011111 & 0001 \end{bmatrix}_{4\times(6+4)}$$

This matrix being the parity check matrix, gives rise to a binary (1, 3) optimal linear code. It can be verified from the error pattern-syndrome table.

**Let $n_1= 7$.** The equation (3) reduces to
$$2^{7+n_2-k} = 3n_2+5.$$
The various values of the parameter satisfying the above equation are
$$(n_2, k) = \{(9, 11), (41, 41),\ldots\}.$$
This shows the possibility of the existence of (7+9, 11), (7+41, 41),…… binary (1, 3) linear codes. The following matrix is an example of (7+9, 11) binary (1, 3) optimal linear code.

$$H=\begin{bmatrix} 0111001 & 100001010 \\ 0001111 & 010000101 \\ 1110011 & 001000010 \\ 0010101 & 000101001 \\ 1101111 & 000010100 \end{bmatrix}_{5\times(7+9)}$$

**Let $n_1= 8$.** The equation (3) reduces to
$$2^{8+n_2-k} = 3n_2+6.$$
The above equation does not have integer solution for $n_2$. Therefore binary (1, 3) optimal code for $n_1= 8$ can not exist.

**Let $n_1= 9$.** The equation (3) reduces to
$$2^{9+n_2-k} = 3n_2+7.$$
The various values of the parameter satisfying the above equation are
$$(n_2, k) = \{(3, 8), (19, 22), \ldots\}.$$
This shows the possibility of the existence of (9+3, 8), (9+19, 22),….. binary (1, 3) linear codes. The following parity check matrix gives rise to an (9+3, 8) binary linear code. It can be verified that

this code is a binary (1, 3) optimal linear code with the help of error pattern-syndrome table 2.

$$H=\begin{bmatrix} 101101001 & 100 \\ 011000011 & 010 \\ 100100111 & 001 \\ 011111111 & 000 \end{bmatrix}_{4\times(9+3)}$$

**Let $n_1= 10$.** The equation (3) reduces to
$$2^{10+n_2-k} = 3n_2+8.$$
The various values of the parameter satisfying the above equation are
$$(n_2, k) = \{(8, 13), (40, 43), \ldots..\}.$$
This shows the possibility of the existence of (10+8, 13), (10+40, 43), … binary (1, 3) linear codes. The following parity check matrix gives rise to (10+8, 13) binary linear code. It can be shown this code is a binary (1, 3) optimal linear code with the help of error pattern-syndrome table.

$$H=\begin{bmatrix} 0011101011 & 10000101 \\ 1100110011 & 01000010 \\ 0011011111 & 00100001 \\ 1101001011 & 00010100 \\ 0110111101 & 00001010 \end{bmatrix}_{5\times(10+8)}$$

**For $n_1= 11$.** The equation (3) reduces to
$$2^{11+n_2-k} = 3n_2+9.$$
This gives no solution, so no such (1, 3) optimal code can exist for $n_1= 11$.

Now, following is a table in which the first two pair suitable values of $n_2$ and $k$ for fixed value of **$n_1$ (upto 21)** are provided.

Table 3

| $n_1$ | $n_2$ | $k$ |
|---|---|---|
| 12 | 18 | 24 |
|  | 82 | 86 |
| 13 | 7 | 15 |
|  | 39 | 45 |
| 15 | 17 | 26 |
|  | 81 | 88 |

cont…

Table 3

| $n_1$ | $n_2$ | k |
|-------|-------|-----|
| 16 | 6 | 17 |
| | 38 | 47 |
| 18 | 16 | 28 |
| | 80 | 90 |
| 19 | 5 | 19 |
| | 37 | 49 |
| 21 | 15 | 30 |
| | 79 | 92 |

## 4.    Construction of (1, 3) optimal codes

For the construction of these binary (1, 3) optimal codes that correct all single errors in the first block of length $n_1$ and all *solid burst* errors of length 3 or less in the second block of length $n_2$ and no more, 2nd block construction will be sufficient because:

Suppose if 2nd block of length $n_2$ is constructed that means syndromes of all the *solid burst* errors of length 3 or less are all distinct. If the number of such syndromes is removed from the nonzero $(n-k)$ tuples, there will be left out with exactly $n_1$ nonzero $(n-k)$-tuples. If these $n_1$ nonzero $(n-k)$-tuples are considered as the columns of the first block irrespective of their order, they are going to correct all single errors in the first block.

The construction of the second block follows the technique used in the deriving Varshamov-Gilbert-Sacks bound (refer Sacks [12], also Theorem 4.7, Peterson and Weldon [11]). The construction of the second block is as follows.

Select any nonzero $(n-k)$ tuple as the first column of the requisite second block of length $n_2$. After having selected the first $j-1$ columns $h_1, h_2,.....,h_{j-1}$ appropriately,  we lay down the condition  to add $j$th column as follows ( refer [2] ):

$h_j$ should not be a linear sum of immediately preceding upto 2 consecutive columns $h_{j-1}, h_{j-2}$ , together with any 3 or fewer consecutive columns from amongst the first $j$-3 columns $h_1, h_2,...., h_{j-3}$. In other words,

$$h_j \neq (u_{j-1}\, h_{j-1} + u_{j-2}\, h_{j-2}) + (v_i\, h_i + v_{i+1}\, h_{i+1} + v_{i+2}\, h_{i+2}),    (4)$$

where $h_i$ 's in the second bracket are any 3 or less consecutive columns among the first $(j-1-l)$ columns, $l = 0,1,2$ and the coefficients $u_i, v_i\, \varepsilon\, GF(2)$ are non zero.

This condition ensures that there shall not be a code vector which can be expressed as sum (difference) of two solid bursts of length 3 or less each, thereby correcting all solid burst of length 3 or less. Thus, the coefficients $u_i$ form a solid burst of length $l$ and the coefficients $v_i$ form a solid burst of length 3 or less in a $(j-1-l)$-tuple, $l = 0,1,2$.

The number of ways in which the coefficients $u_i$'s and $v_i$'s on the R.H.S. of the expression (4), including the vector of all zero, is given by

$$1+\sum_{i=1}^{3}(j-1-i+1)+\sum_{i=1}^{3}(j-2-i+1)+\sum_{i=1}^{3}(j-3-i+1)$$
$$=1+9\,(j-3).$$

In fact this will give rise to the sufficient condition for the existence of such (1, 3) optimal linear codes. In view of this, we obtain the following result:

**Theorem 2:**    There exists an $(n=n_1+n_2,\ k)$ linear code over GF(2) that corrects all solid burst errors of length 1 in the first block of length $n_1$ and all solid burst errors of length 3 or less in the second block of length $n_2$ provided that

$$2^{n-k} > 1+9(n_2-3).$$

**Remark**

If we interchange the two sub blocks i.e., if we consider $(n=n_1+n_2,\ k)$ linear codes, these will form a class of (3, 1) optimal codes.

## 5.    Conclusion and Open question

It is well known that the use of optimal code improves the efficiency of the channel as optimal codes economize in the number of parity checks in

a code reducing thereby the redundancy and improving the rate of transmission. Therefore optimal codes are useful from application point of view in communication.

This paper investigates the integer solution of the equation (3) for $n_1 = 1, 2, \ldots, 21$. The equation (3) has integer solution for $n_1 = 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21$. For other values of $n_1$, the equation has no integer solution. The author gives a code corresponding to one of the solutions only. In view of the existence of other solutions of the equation (3), the existence of corresponding codes is an open problem.

Further, there may be a systematic way of constructing the parity check matrices of such binary (1, 3) optimal codes. Non binary (1, 3) optimal codes are also open problems.

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**References**

[1]. Das, P.K., Codes Detecting and Locating Solid Burst Errors, *Romanian Journal of Mathematics and Computer Science*, Vol. 2(2), pp. 1-10, 2012.

[2]. Das, P.K., Codes Detecting and Correcting Solid Burst Errors, *Bulletin of Electrical Engineering and Informatics,* Vol.1, No.3, pp. 225-232, 2012.

[3]. Dass, B.K. and Das, P.K., On perfect like binary and non binary perfect codes- A brief survey, *Bull. Malays. Maths. Sci. Soc.(2)*, 32(2), pp. 187-210, 2009.

[4]. Dass, B. K. and Tyagi, V., A New type of (1, 2)-optimal Codes Over GF(2), *Indian Journal of Pure & Applied Mathematics*, 13(7), pp.750-756, 1982.

[5]. Elspas, B., and Short, R.A., A note on optimum burst error-correcting codes, *IRE Trans. on Information Theory,* Vol. 8( 1), pp 39-42, 1962.

[6]. Golay, M.J.E., Notes on digital coding, *Proc. IRE*, 37, pp. 657, 1949.

[7]. Golay, M.J.E., Binary Coding, *IRE Trans., PGIT-4*, pp. 23-28, 1954.

[8]. Hamming, R.W., Error-detecting and error-correcting codes, *Bell System Technical Journal*, Vol. 29, pp. 147-160, 1950.

[9]. Kasami, T., Optimum shortened cyclic codes for burst-error correction, *IEEE Trans. Inform. Theory*, Vol. 9, pp. 105-109, 1963.

[10]. Khaled, A.S.A.G., Robert, J.M. and Andrew, M.O., On the Existence of Optimum Cyclic Burst-Correcting Codes, *IEEE Transaction on Information Theory*, Vol. IT-32, No. 6, pp. 768-775, 1986.

[11]. Peterson, W.W. and Weldon, E.J., *Error Correcting Codes*, 2nd edition, The M.I.T Press, 1972.

[12] Sacks, G. E., Multiple error correction by means of parity-checks, *IRE Trans. Inform. Theory IT,* Vol.4, pp. 145-147, 1958.

[13]. Sharma, B. D. and Dass, B. K., Adjacent error correcting binary perfect codes, *J. Cybernetics*, Vol. 7, pp. 9-13, 1977.

[14]. Tietavainen, A. and Perko, A., There are no unknown perfect binary codes, *Ann. Univ. Turion, Ser.* A, 148, 3-10, 1971.

[15]. Tuvi Etzion, Constructions for Perfect - Burst-Correcting Codes, *IEEE Transaction on Information Theory*, Vol. 47, No. 6, pp. 2553-2555, 2001.