RESEARCH ARTICLE

BOUNDS FOR THE NORMS OF TOEPLITZ MATRICES WITH k-JACOBSTHAL AND k-JACOBSTHAL LUCAS NUMBERS

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ABSTRACT

This work is concerned with the spectral, Euclid norms of Toeplitz matrices with generalized k-Jacobsthal and k- Jacobsthal Lucas entries. k- Jacobsthal and k- Jacobsthal Lucas sequences are two generalizations of two very popular special integer sequences called Jacobsthal and Jacobsthal Lucas sequences. Upper and lower bounds for the spectral norms of these matrices, that is, the matrices of the forms $A = T(j_{k,0}, j_{k,1}, \ldots, j_{k,n-1})$ and $B = T(c_{k,0}, c_{k,1}, \ldots, c_{k,n-1})$ are obtained. The upper bounds for the Euclidean and spectral norms of Kronecker and Hadamard product matrices of Toeplitz matrices with k-Jacobsthal and the k- Jacobsthal Lucas numbers are computed.

Keywords: *Hadamard product, k -Jacobsthal numbers, k -Jacobsthal Lucas numbers, Kronecker product, Norm, Toeplitz matrix*

1. INTRODUCTION

Special matrices is a widely studied subject in matrix analysis. Especially special matrices whose entries are well-known number sequences have become a very interesting research subject in recent years and many authors have obtained some good results in this area. Sequences are the building blocks of special matrices such as circulant, Toeplitz, Hankel, geometric matrices. There have been many papers about the norms of special matrices. Recently, there has been much interest in investigation of some special matrices. Because of this, various number sequences are used as entries, and the properties of the resulting matrices are investigated. Research on these special matrices normally would revolve around the investigation of their determinants, eigenvalues, norms, inverses and bounds of norms.

In [6], the authors have studied bounds of the spectral norms of circulant matrices with Fibonacci numbers. In [7], Akbulak and Bozkurt studied the norms of Toeplitz matrices involving Fibonacci and Lucas numbers. Shen [8] investigated the upper and lower bounds for the spectral norms of Toeplitz matrices involving k-Fibonacci and k-Lucas numbers. In [9], Daşdemir demonsrated the norms of Toeplitz matrices with the Pell, Pell-Lucas and modified Pell numbers. Kocer [10] has given some properties of the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers, then she has defined the

circulant, negacyclic and semicirculant matrices with these numbers and she has investigated the norms, eigenvalues and determinants of these matrices. Raza and Ali [11] studied on the norms of some special matrices with generalized Fibonacci sequence. Uygun, constructed bounds for the norms of circulant matrices with the k-Jacobsthal and k-Jacobsthal Lucas numbers [13].

Considering the above articles, on the one hand, we obtain new lower and upper bounds estimates for the spectral norms of Toeplitz matrices with k-Jacobsthal and k-Jacobsthal Lucas numbers. Furthermore, Euclidean norms, and maximum row and maximum column norms of Toeplitz matrices with k-Jacobsthal and k-Jacobsthal Lucas numbers are computed. Then bounds for Euclidean and sprectral norms of Kronecker and Hadamard product of the matrices are calculated.

Now, we give some fundamental information related to our study. For $n \in Z$, the classic Jacobsthal and Jacobsthal Lucas sequences are defined respectively by the second order homogeneous linear recurrence relations

$$j_{n+2} = j_{n+1} + 2j_n, \quad j_0 = 0, \quad j_1 = 1,$$

$$c_{n+2} = c_{n+1} + 2c_n, \quad c_0 = 2, \quad c_1 = 1.$$

Many generalizations of the well-known Jacobsthal sequence have been introduced and studied. For example, the generalized k-Jacobsthal and k-Jacobsthal Lucas numbers have been studied [12]. For $n \geq 2, n \in \mathbb{N}$, the k-Jacobsthal $\{j_{k,n}\}_{n\in\mathbb{N}}$, and the k-Jacobsthal Lucas $\{c_{k,n}\}_{n\in\mathbb{N}}$ sequences are defined recurrently by

$$j_{k,n} = k\, j_{k,n-1} + 2\, j_{k,n-2}, \qquad j_{k,0} = 0, \ j_{k,1} = 1, \tag{1}$$

$$c_{k,n} = k\, c_{k,n-1} + 2\, c_{k,n-2}, \qquad c_{k,0} = 2, \ c_{k,1} = k. \tag{2}$$

respectively in [12]. The first k-Jacobsthal numbers are 0, 1, k, $k^2 + 2$, $k^3 + 4k$, $k^4 + 6k^2 + 4$ …The first k- Jacobsthal Lucas numbers are 2 , k , $k^2 + 4$, $k^3 + 6k$, $k^4 + 8k^2 + 8$ …

Recurrences (1) ve (2) involve the characteristic equation $x^2 - kx - 2 = 0$ with roots $\alpha = \frac{k + \sqrt{k^2 + 8}}{2}$, $\beta = \frac{k - \sqrt{k^2 + 8}}{2}$. Binet's formulas of the k-Jacobsthal and k- Jacobsthal Lucas are defined respectively by

$$j_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad c_{k,n} = \alpha^n + \beta^n. \tag{3}$$

Extension to negative values of n can be made, k-Jacobsthal and k-Jacobsthal Lucas sequence with negative indices are demonstrated by

$$j_{k,-i} = \frac{(-1)^{i+1} j_{k,i}}{2^i}, \qquad c_{k,-i} = \frac{(-1)^i c_{k,i}}{2^i}.$$

An $n \times n$ matrix $T = \{t_{ij}\} \in M_n(C)$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for each $i \ ; j = 1, \dots, n$.

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix} \tag{4}$$

Toeplitz matrix is determined by its first row (or column).

For any $A = [a_{ij}] \epsilon M_{\{m,n\}}(C)$, the largest absolute column sum (1-norm) and the largest absolute row sum(∞-norm) norms are

$$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|,$$

$$\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|.$$

For any $A = [a_{ij}] \epsilon M_{\{m,n\}}(C)$, the Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_E = \left(\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2 \right)^{1/2}, \tag{5}$$

and the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\frac{\max}{1 \leq i \leq n} \lambda_i(A^H A)} \ , \tag{6}$$

where $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$, and A^H is the conjugate transpose of matrix A.

The maximum column length norm $c_1 (A)$ and the maximum row length norm $r_1 (A)$ of a matrix of order $n \times n$ are defined as

$$c_1(A) = \max_j \sqrt{\sum_{i=1}^n |a_{ij}|^2},$$

$$r_1(A) = \max_i \sqrt{\sum_{j=1}^m |a_{ij}|^2}.$$

For any $A, B \epsilon M_{\{m,n\}}(C)$, the Hadamard product of A, B is entrywise product and defined by [3,4]

$$A \circ B = (a_{ij} b_{ij})$$

and have the following properties

$$\|A \circ B\|_2 \leq r_1(A) c_1(B), \quad \|A \circ B\| \leq \|A\| \|B\|. \tag{7}$$

Let $A \epsilon M_{\{m,n\}}(C)$, and $B \epsilon M_{\{p,q\}}(C)$ be given, then the Kronecker product of A, B is defined by [4,6,8]

$$\|A \otimes B\| = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

and have the following properties

$$\|A \otimes B\|_2 = \|A\|_2 \ \|B\|_2, \quad \|A \otimes B\|_E = \|A\|_E \ \|B\|_E. \tag{8}$$

Let $A \in M_{\{m,n\}}(C)$ be given, then the inequality is hold [1,2]

$$\frac{1}{\sqrt{n}} \ \|A\|_E \leq \|A\|_2 \leq \|A\|_E. \tag{9}$$

2. SOME SUM FORMULAS FOR k-JACOBSTHAL AND k-JACOBSTHAL LUCAS NUMBERS

Let $k \neq -1, 1$. The summation formulas for the *k*-Jacobsthal and *k*-Jacobsthal Lucas sequences are

$$\sum_{i=0}^{n-1} j_{k,i} = \frac{j_{k,n} + 2 j_{k,n-1} - 1}{k+1}, \tag{10}$$

$$\sum_{i=0}^{n-1} c_{k,i} = \frac{c_{k,n} + 2 \ c_{k,n-1} + k - 2}{k+1}. \tag{11}$$

The summation of the squares of *k*-Jacobsthal sequence and *k*-Jacobsthal Lucas sequence are

$$\sum_{i=1}^{n} j_{k,i}^2 = \frac{1}{k^2+8} \left[\frac{4c_{k,2n} - c_{k,2n+2} - c_{k,2} + 2}{5 - c_{k,2}} + 2 \ (-1)^{n+1} j_{n+1} \right], \tag{12}$$

$$\sum_{i=1}^{n} c_{k,i}^2 = \frac{4c_{k,2n} - c_{k,2n+2} - c_{k,2} + 2}{5 - c_{k,2}} - 2 \ (-1)^{n+1} j_{n+1} \ . \tag{13}$$

The summation of the squares of *k*-Jacobsthal and *k*-Jacobsthal Lucas sequence with negative indices are demonstrated

$$\sum_{i=1}^{n} \left(j_{k,-i} \right)^2 = \sum_{i=1}^{n} \left(\frac{j_{k,i}}{2^i} \right)^2 = \frac{1}{k^2+8} \left[\frac{1}{1-k^2} \left(\frac{c_{k,2n} - c_{k,2n+2}}{2^{2n}} - k^2 + 4 \right) - \frac{j_{n+1}}{2^{n-1}} \right], \tag{14}$$

$$\sum_{i=1}^{n} \left(c_{k,-i} \right)^2 = \sum_{i=1}^{n} \left(\frac{c_{k,i}}{2^i} \right)^2 = \frac{1}{1-k^2} \left(\frac{c_{k,2n} - c_{k,2n+2}}{2^{2n}} + 4 - k^2 \right) + \frac{j_{n+1}}{2^{n-1}} - 4. \tag{15}$$

Some summation formulas for *k*-Jacobsthal sequence and *k*-Jacobsthal Lucas sequence are as follows

$$\sum_{i=1}^{n-1} c_{k,2i} = \frac{4c_{k,2n-2} - c_{k,2n} + k^2 - 4}{1 - k^2}, \tag{16}$$

$$\sum_{i=1}^{n-1} \frac{c_{k,2i}}{4^i} = \frac{c_{k,2n-2} - c_{k,2n}}{4^{n-1}(1-k^2)} + \frac{k^2+2}{1-k^2}, \tag{17}$$

$$\sum_{i=1}^{n-1} \frac{c_{k,2i+2}}{4^i} = \frac{c_{k,2n} - c_{k,2n+2}}{4^{n-1}(1-k^2)} + 4 \frac{k^2+2}{1-k^2}, \tag{18}$$

$$\sum_{i=1}^{n-1} c_{k,2i+2} = \frac{4c_{k,2n} - c_{k,2n+2} + k^2 - 4}{1 - k^2} - (k^2 + 4), \tag{19}$$

$$\sum_{i=1}^{n-1} (-1)^{i+1} j_{i+1} = \frac{4(-1)^n J_{n-1} - n + 1}{3}, \tag{20}$$

$$\sum_{i=1}^{n-1} \frac{j_{i+1}}{2^{i-1}} = -\frac{j_{n-1}}{3.2^{n-2}} + \frac{4n-4}{3} . \tag{21}$$

3. LOWER AND UPPER BOUNDS OF TOEPLITZ MATRICES INVOLVING k-JACOBSTHAL NUMBERS

Theorem 1: Let $A = T\left(j_{k,0}, j_{k,1}, \dots, j_{k,n-1}\right)$ be a Toeplitz matrix with k-Jacobsthal numbers, then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are

$$\|A\|_1 = \|A\|_\infty = \frac{(k+2)\, j_{k,n} + 2j_{k,n-1} - 1}{k+1}.$$

Proof. Clearly, the explicit form of this matrix as follows:

$$A = \begin{bmatrix} j_{k,0} & j_{k,-1} & j_{k,-2} & \cdots & j_{k,1-n} \\ j_{k,1} & j_{k,0} & j_{k,-1} & \cdots & j_{k,2-n} \\ j_{k,2} & j_{k,1} & j_{k,0} & \cdots & j_{k,3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_{k,n-1} & j_{k,n-2} & j_{k,n-3} & \cdots & j_{k,0} \end{bmatrix} \tag{22}$$

By the definitions of 1-norm and ∞-norm, and (10), it is easily seen

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{i1}| = \sum_{i=0}^{n-1} j_{k,i} = \frac{j_{k,n} + 2j_{k,n-1} - 1}{k+1},$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{nj}| = \sum_{i=0}^{n-1} j_{k,i} = \frac{j_{k,n} + 2j_{k,n-1} - 1}{k+1}. \qquad \blacksquare$$

Theorem 2: Let $A = T\left(j_{k,0}, j_{k,1}, \dots, j_{k,n-1}\right)$ be a Toeplitz matrix, then the Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_E = \sqrt{\begin{array}{l} \frac{1}{(k^2+8)(1-k^2)^2}\left[c_{k,2n+2} + \frac{c_{k,2n+2}}{2^{2n-2}} - 8c_{k,2n} - \frac{2c_{k,2n}}{2^{2n-2}} + 16c_{k,2n-2} + \frac{c_{k,2n-2}}{2^{2n-2}} - 18\right] \\ + \frac{(n-1)(2k^2-2)-4-k^2}{(k^2+8)(k^2-1)} + \frac{1}{3(k^2+8)}\left[8(-1)^n j_{n-1} - \frac{4\, j_n}{2^n} - 6n + 8\right] . \end{array}}$$

(23)

Proof. Let A be an $n \times n$ matrix . Then by (5), (12), (14)

$$\|A\|_E{}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = n j_{k,0}{}^2 + \sum_{m=1}^{n-1}(n-m)j_{k,m}{}^2 + \sum_{m=1}^{n-1}(n-m)j_{k,-m}{}^2$$

$$= \sum_{m=1}^{n-1}\sum_{i=1}^{m} j_{k,i}{}^2 + \sum_{m=1}^{n-1}\sum_{i=1}^{m}\left(\frac{j_{k,i}}{2^i}\right)^2$$

$$= \frac{1}{(k^2+8)}\sum_{m=1}^{n-1}\left[\frac{4c_{k,2m}-c_{k,2m+2}-k^2-2}{1-k^2}+2(-1)^{m+1}j_{m+1}\right]$$

$$+\frac{1}{(k^2+8)}\sum_{m=1}^{n-1}\frac{1}{1-k^2}\left(\frac{c_{k,2m}-c_{k,2m+2}}{2^{2m}}-k^2+4\right)-\frac{j_{m+1}}{2^{m-1}}$$

And then by using the sum formulas (16), (19), (21), the following result is obtained:

$$= \frac{1}{(k^2+8)}\left[\frac{16c_{k,2n-2}-4c_{k,2n}+4k^2-16}{(1-k^2)^2}+\frac{c_{k,2n+2}-4c_{k,2n}-k^2+4}{(1-k^2)^2}-\frac{(k^2+4)}{k^2-1}\right.$$
$$\left.+\frac{(k^2+2)(n-1)}{k^2-1}+\frac{8(-1)^n j_{n-1}-2n+2}{3}\right]$$
$$+\frac{1}{(k^2+8)}\left\{\frac{1}{(1-k^2)^2}\left[\frac{-c_{k,2n}+c_{k,2n-2}}{2^{2n-2}}+2+k^2+\frac{-c_{k,2n}+c_{k,2n+2}}{2^{2n-2}}-8-4k^2\right]\right.$$
$$\left.+(n-1)\left[\frac{k^2-4}{k^2-1}\right]+\frac{j_{n-1}}{3.2^{n-2}}-\frac{4n-4}{3}\right\}$$
$$=\frac{1}{(k^2+8)(1-k^2)^2}\left[c_{k,2n+2}+\frac{c_{k,2n+2}}{2^{2n-2}}-8c_{k,2n}-\frac{2c_{k,2n}}{2^{2n-2}}+16c_{k,2n-2}+\frac{c_{k,2n-2}}{2^{2n-2}}-10\right]$$
$$+\frac{(n-1)(2k^2-2)-4-k^2}{(k^2+8)(k^2-1)}+\frac{1}{3(k^2+8)}\left[8(-1)^n j_{n-1}+\frac{j_{n-1}}{2^{n-2}}-6n+6\right]. \qquad \blacksquare$$

Theorem 3: Let $A = T\left(j_{k,0},j_{k,1},\dots,j_{k,n-1}\right)$ be Toeplitz matrix, then the lower and upper bounds for the spectral norm of A are obtained as

$$\sqrt{\begin{array}{l}\frac{1}{n(k^2+8)(1-k^2)^2}\left[c_{k,2n+2}+\frac{c_{k,2n+2}}{2^{2n-2}}-8c_{k,2n}-\frac{2c_{k,2n}}{2^{2n-2}}+16c_{k,2n-2}+\frac{c_{k,2n-2}}{2^{2n-2}}-10\right]\\+\frac{(n-1)(2k^2-2)-4-k^2}{n(k^2+8)(k^2-1)}+\frac{1}{3n(k^2+8)}\left[8(-1)^n j_{n-1}+\frac{j_{n-1}}{2^{n-2}}-6n+6\right]\end{array}} \le \|A\|_2 \qquad (24)$$

$$\|A\|_2 \le \sqrt{\begin{array}{l}\left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2}+2(-1)^{n-1}j_{n-1}\right)+1\right]\\\left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2}+2(-1)^n j_n\right)\right]\end{array}} \qquad (25)$$

Proof. By (23) and using the property (9), the left hand side of the inequality is completed.
On the other hand, let $A = B \circ C$ whereas

$$B = b_{ij} = \begin{cases} b_{ij} = 1 & j=1 \\ b_{ij} = j_{k,i-j} & j \ne 1 \end{cases} \text{ and } \qquad C = c_{ij} = \begin{cases} c_{ij} = j_{k,i-j} & j=1 \\ c_{ij} = 1 & j \ne 1 \end{cases}$$

$$B = \begin{bmatrix} 1 & j_{k,-1} & j_{k,-2} & \cdots & j_{k,1-n} \\ 1 & j_{k,0} & j_{k,-1} & \cdots & j_{k,2-n} \\ 1 & j_{k,1} & j_{k,0} & \cdots & j_{k,3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & j_{k,n-2} & j_{k,n-3} & \cdots & j_{k,0} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} j_{k,0} & 1 & 1 & \cdots & 1 \\ j_{k,1} & 1 & 1 & \cdots & 1 \\ j_{k,2} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_{k,n-1} & 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$r_1(B) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{\sum_{i=0}^{n-2} j_{k,i}^2 + 1}$$

$$= \sqrt{\frac{1}{k^2+8} \left(\frac{4c_{k,2n-4} - c_{k,2n-2} - k^2 - 2}{1-k^2} + 2(-1)^{n-1} j_{n-1} \right) + 1}$$

$$c_1(C) = \max_{1 \le i \le n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{j1}|^2} = \sqrt{\sum_{i=0}^{n-1} j_{k,i}^2}$$

$$= \sqrt{\frac{1}{k^2+8} \left(\frac{4c_{k,2n-2} - c_{k,2n} - k^2 - 2}{1-k^2} + 2(-1)^n j_n \right)} \ .$$

By using the property (7), the right hand side of the inequality is completed:

$$r_1(B)c_1(C) = \sqrt{\frac{\left[\frac{1}{k^2+8} \left(\frac{4c_{k,2n-4} - c_{k,2n-2} - k^2 - 2}{1-k^2} + 2(-1)^{n-1} j_{n-1} \right) + 1 \right]}{\left[\frac{1}{k^2+8} \left(\frac{4c_{k,2n-2} - c_{k,2n} - k^2 - 2}{1-k^2} + 2(-1)^n j_n \right) \right]}} \qquad \blacksquare$$

4. LOWER AND UPPER BOUNDS OF TOEPLITZ MATRICES INVOLVING k-JACOBSTHAL LUCAS NUMBERS

Theorem 4: Let the elements of the Toeplitz matrix be *k*-Jacobsthal Lucas numbers $A = T\left(c_{k,0}, c_{k,1}, \ldots, c_{k,n-1} \right)$, then 1-norm and ∞-norm of A are

$$\|A\|_1 = \|A\|_\infty = \frac{(k+2)\, j_{k,n} + 2 j_{k,n-1} - 1}{k+1}.$$

Proof. Clearly, the explicit form of this matrix as follows:

$$A = \begin{bmatrix} c_{k,0} & c_{k,-1} & c_{k,-2} & \cdots & c_{k,1-n} \\ c_{k,1} & c_{k,0} & c_{k,-1} & \cdots & c_{k,2-n} \\ c_{k,2} & c_{k,1} & c_{k,0} & \cdots & c_{k,3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k,n-1} & c_{k,n-2} & c_{k,n-3} & \cdots & c_{k,0} \end{bmatrix} . \tag{26}$$

By the definitions of 1-norm and ∞-norm, and (11), it is easily seen

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{i1}| = \sum_{i=0}^{n-1} c_{k,i} = \frac{c_{k,n} + 2\,c_{k,n-1} + k - 2}{k+1},$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{nj}| = \sum_{i=0}^{n-1} c_{k,i} = \frac{c_{k,n} + 2\,c_{k,n-1} + k - 2}{k+1}.$$

In the following theorem, we give the Euclidean (Frobenius) norm of the matrix involving *k*-Jacobsthal Lucas numbers.

Theorem 5: Let $A = T\,(c_{k,0}\,,c_{k,1}\,,\dots,c_{k,n-1})$ be Toeplitz matrix with *k*-Jacobsthal Lucas numbers, then the Frobenious (or Euclidean) norm of matrix *A* is

$$\|A\|_E = \sqrt{\begin{array}{l} \frac{16c_{k,2n-2} - 8c_{k,2n} + c_{k,2n+2} - k^2 - 20}{(1-k^2)^2} + \frac{c_{k,2n-2} - 2c_{k,2n} + c_{k,2n+2}}{2^{2n-2}(1-k^2)^2} \\ + \frac{(k^2+4)+(n-1)(6k^2-6)}{1-k^2} + \frac{-8(-1)^n j_{n-1} + 6n - 8}{3} + \frac{j_n}{3.2^{n-2}}. \end{array}} \tag{27}$$

Proof. Let *A* be an *n×n* matrix as in (26). Then by the definition of Frobenius norm and by (5), (13), (15), we can obtain the following equations for matrix A

$$\|A\|_E^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = n c_{k,0}^2 + \sum_{i=1}^{n-1}(n-i)c_{k,i}^2 + \sum_{i=1}^{n-1}(n-i)c_{k,-i}^2$$

$$= 4n + \sum_{m=1}^{n-1}\sum_{i=1}^{m} c_{k,i}^2 + \sum_{m=1}^{n-1}\sum_{i=1}^{m}\left(\frac{c_{k,i}}{2^i}\right)^2$$

$$= \sum_{m=1}^{n-1}\frac{4c_{k,2m} - c_{k,2m+2} - k^2 - 4}{1-k^2} - 2(-1)^{m+1}j_{m+1} + \sum_{m=1}^{n-1}\frac{c_{k,2m} - c_{k,2m+2}}{2^{2m}(1-k^2)} + \frac{3k^2}{1-k^2} + \frac{j_{m+1}}{2^{m-1}}$$

$$= \frac{16c_{k,2n-2} - 8c_{k,2n} + c_{k,2n+2} - (k^2-4)^2}{(1-k^2)^2} + \frac{c_{k,2n-2} - 2c_{k,2n} + c_{k,2n+2}}{2^{2n-2}(1-k^2)^2}$$

$$+ \frac{(-4k^2-8)+(n-1)(2k^2-4)}{1-k^2} + \frac{-8(-1)^n j_{n-1} + 6n - 6}{3} - \frac{j_{n-1}}{3.2^{n-2}}. \qquad \blacksquare$$

Theorem 6: Let $A = T\,(c_{k,0}\,,c_{k,1}\,,\dots,c_{k,n-1})$ be Toeplitz matrix, then the lower and upper bounds for the spectral norm of *A* are obtained as

$$\sqrt{\begin{array}{l} \frac{16c_{k,2n-2} - 8c_{k,2n} + c_{k,2n+2} - (k^2-4)^2}{n(1-k^2)^2} + \frac{c_{k,2n-2} - 2c_{k,2n} + c_{k,2n+2}}{2^{2n-2}n(1-k^2)^2} \\ + \frac{(-4k^2-8)+(n-1)(2k^2-4)}{(1-k^2)n} + \frac{-8(-1)^n j_{n-1} + 6n - 6}{3n} - \frac{j_{n-1}}{3n.2^{n-2}} \end{array}} \le \|A\|_2 \tag{28}$$

$$\|A\|_2 \le \sqrt{\begin{array}{l} \frac{4\,c_{k,2n-4} - c_{k,2n-2} - k^2 - 2}{1-k^2} - 2(-1)^{n-1}j_{n-1} + 5 \\ \frac{4\,c_{k,2n-2} - c_{k,2n} - k^2 - 2}{1-k^2} - 2(-1)^n j_n + 4 \end{array}} \tag{29}$$

Proof. By (27), and using the property (9), the left hand side of the inequality is completed. On the other hand, let $A = B \circ C$ whereas

$$B = b_{ij} = \begin{cases} b_{ij} = 1 & j = 1 \\ b_{ij} = c_{k,i-j} & j \neq 1 \end{cases} \quad \text{and} \quad C = c_{ij} = \begin{cases} c_{ij} = c_{k,i-j} & j = 1 \\ c_{ij} = 1 & j \neq 1 \end{cases}$$

$$B = \begin{bmatrix} 1 & c_{k,-1} & c_{k,-2} & \cdots & c_{k,1-n} \\ 1 & c_{k,0} & c_{k,-1} & \cdots & c_{k,2-n} \\ 1 & c_{k,1} & c_{k,0} & \cdots & c_{k,3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_{k,n-2} & c_{k,n-3} & \cdots & c_{k,0} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_{k,0} & 1 & 1 & \cdots & 1 \\ c_{k,1} & 1 & 1 & \cdots & 1 \\ c_{k,2} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k,n-1} & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then, by using the sum formula (13) and the definition of the maximum column length norm and the maximum row length norm, the following equalities are hold:

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{\sum_{i=0}^{n-2} c_{k,i}^2 + 1}$$

$$= \sqrt{\frac{4 c_{k,2n-4} - c_{k,2n-2} - k^2 - 2}{1-k^2} - 2(-1)^{n-1} j_{n-1} + 5},$$

$$c_1(C) = \max_{1 \leq i \leq n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{j1}|^2} = \sqrt{\sum_{i=0}^{n-1} c_{k,i}^2}$$

$$= \sqrt{\frac{4 c_{k,2n-2} - c_{k,2n} - k^2 - 2}{1-k^2} - 2(-1)^n j_n + 4}.$$

If we use the equations given in (7), the right hand side of the inequality is completed. ∎

Corollary 7: Let $A = T(j_{k,0}, j_{k,1}, \ldots, j_{k,n-1})$ and $B = T(c_{k,0}, c_{k,1}, \ldots, c_{k,n-1})$ be Toeplitz matrix with k-Jacobsthal and the k-Jacobsthal Lucas numbers, then the Euclidean norm of Kronecker product of these matrices is given as:

$$\|A \otimes B\|_E = \left(\sqrt{\begin{array}{c} \frac{1}{(k^2+8)(1-k^2)^2}\left[c_{k,2n+2} + \frac{c_{k,2n+2}}{2^{2n-2}} - 8c_{k,2n} - \frac{2c_{k,2n}}{2^{2n-2}} + 16c_{k,2n-2} + \frac{c_{k,2n-2}}{2^{2n-2}} - 18 \right] \\ + \frac{(n-1)(2k^2-2)-4-k^2}{(k^2+8)(k^2-1)} + \frac{1}{3(k^2+8)}\left[8(-1)^n j_{n-1} - \frac{4 j_n}{2^n} - 6n + 8 \right] \end{array}} \right.$$

$$\left. \cdot \sqrt{\begin{array}{c} \frac{16c_{k,2n-2} - 8c_{k,2n} + c_{k,2n+2} - k^2 - 20}{(1-k^2)^2} + \frac{c_{k,2n-2} - 2c_{k,2n} + c_{k,2n+2}}{2^{2n-2}(1-k^2)^2} \\ \frac{(k^2+4)+(n-1)(6k^2-6)}{1-k^2} + \frac{-8(-1)^n j_{n-1} + 6n - 8}{3} + \frac{j_n}{3 \cdot 2^{n-2}} \end{array}} \right)$$

Proof. The proof is seen easily by $\|A \otimes B\|_E = \|A\|_E \|B\|_E$ and (23), (27).

Corollary 8: Let $A = T\ (j_{k,0}\ , j_{k,1}\ , \dots, j_{k,n-1}\ )$ and $B = T\ (c_{k,0}\ , c_{k,1}\ , \dots, c_{k,n-1})$ be Toeplitz matrix with k-Jacobsthal and the k-Jacobsthal Lucas numbers, then the upper bound for the spectral norm of Kronecker product of these matrices is given as:

$$\|A \otimes B\|_2 \leq \sqrt{ \begin{array}{c} \left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2} + 2(-1)^{n-1}j_{n-1}\right) + 1\right] \\ \left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2} + 2(-1)^{n}j_{n}\right)\right] \end{array} } \cdot \sqrt{ \begin{array}{c} \frac{4\,c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2} - 2(-1)^{n-1}\,j_{n-1} + 5 \\ \frac{4\,c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2} - 2(-1)^{n}\,j_{n} + 4 \end{array} }$$

Proof. The proof is seen easily by $\|A \otimes B\|_2 = \|A\|_2\,\|B\|_2$ and (25), (29). ∎

Corollary 9 : Let $A = T\ (j_{k,0}\ , j_{k,1}\ , \dots, j_{k,n-1}\ )$ and $B = T\ (c_{k,0}\ , c_{k,1}\ , \dots, c_{k,n-1})$ be Toeplitz matrix with k-Jacobsthal and the k-Jacobsthal Lucas numbers, then the upper bound for the spectral norm of Hadamard product of the matrices is

$$\|A \circ B\|_2 \leq \sqrt{ \begin{array}{c} \left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2} + 2(-1)^{n-1}j_{n-1}\right) + 1\right] \\ \left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2} + 2(-1)^{n}j_{n}\right)\right] \end{array} } \cdot \sqrt{ \begin{array}{c} \left(\frac{4\,c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2} - 2(-1)^{n-1}\,j_{n-1} + 5\right) \\ \left(\frac{4\,c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2} - 2(-1)^{n}\,j_{n} + 4\right) \end{array} }$$

Proof. The proof is seen easily by $\|A \circ B\|_2 \leq \|A\|_2\,\|B\|_2$ and (25), (29). ∎

Corollary 10 : Let $A = T\ (j_{k,0}\ , j_{k,1}\ , \dots, j_{k,n-1}\ )$ and $B = T\ (c_{k,0}\ , c_{k,1}\ , \dots, c_{k,n-1})$ be Toeplitz matrix with k-Jacobsthal and the k-Jacobsthal Lucas numbers, then the upper bound for the Euclid norm of Hadamard product of the matrices is

$$\|A \circ B\|_E \leq \sqrt{ \begin{array}{c} \frac{1}{(k^2+8)(1-k^2)^2}\left[c_{k,2n+2} + \frac{c_{k,2n+2}}{2^{2n-2}} - 8c_{k,2n} - \frac{2c_{k,2n}}{2^{2n-2}} + 16c_{k,2n-2} + \frac{c_{k,2n-2}}{2^{2n-2}} - 18\right] \\ + \frac{(n-1)(2k^2-2)-4-k^2}{(k^2+8)(k^2-1)} + \frac{1}{3(k^2+8)}\left[8(-1)^n j_{n-1} - \frac{4\,j_n}{2^n} - 6n + 8\right] \end{array} } \cdot \sqrt{ \begin{array}{c} \frac{16c_{k,2n-2}-8c_{k,2n}+c_{k,2n+2}-k^2-20}{(1-k^2)^2} + \frac{c_{k,2n-2}-2c_{k,2n}+c_{k,2n+2}}{2^{2n-2}(1-k^2)^2} \\ + \frac{(k^2+4)+(n-1)(6k^2-6)}{1-k^2} + \frac{8(-1)^{n+1}j_{n-1}+6n-8}{3} + \frac{j_n}{3.2^{n-2}} \end{array} }$$

Proof. By $\|A \circ B\|_E \leq \|A\|_E\,\|B\|_E$ and (23), (27), the result is found. ∎

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