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THE GROUP OF UNITS OF GROUP ALGEBRAS OF GROUPS D_{30} AND $C_3 \times D_{10}$ OVER A FINITE FIELD OF CHARACTERISTIC 3

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Abstract. Let F be a finite field of characteristic p. There are three non-isomorphic non-abelian groups of order 30. The structure of $U(F(C_5 \times D_6))$ for $p = 3$ is given in [J. Gildea and R. Taylor, Int. Electron. J. Algebra, 24 (2018), 62-67]. In this article, we give the structure of $U(FD_{30})$ and $U(F(C_3 \times D_{10}))$ for $p = 3$.

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1. Introduction

Let $U(FG)$ be the group of units of the group algebra FG of a group G over a finite field F of characteristic p having $q = p^k$ elements. Let $J(FG)$ be the Jacobson radical of FG and let $V = 1+J(FG)$. We denote by D_n the dihedral group of order n. In this paper, we study the structure of $U(FG)$ where $G = D_{30}$ and $C_3 \times D_{10}$, for $p = 3$.

Let $V_1(FG) = \{\sum_{g\in G} r_g g \in U(FG) \mid \sum_{g\in G} r_g = 1\}$ be the group of normalized units of FG. It is well known that $U(FG) = V_1(FG) \times F^*$. If G is a finite abelian p-group, then $V_1(FG)$ is a finite p-group of order $|F|^{|G|-1}$. In [19], Sandling provides a basis for $V_1(FG)$. The map $* : FG \to FG$ defined by $(\sum_{g\in G} a_g g)^* = \sum_{g\in G} a_g g^{-1}$ is an antiautomorphism of FG and its order is 2. An element $v \in V_1(FG)$ is called a unitary unit if $v^* = v^{-1}$. Unitary units of some modular group algebras have been studied in [2,3]. In [16,17], the structure of the unitary subgroup of the group algebra FD_{2^n} and $F(QD_{16})$, where QD_{16} is the quasi-dihedral group of order 16 and $p = 2$, has been obtained.

Describing the group of units of group algebras is, in general, a hard task and it is more difficult when the group algebra is not semi-simple. Many authors have

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studied the structure of $U(FG)$ for the non-semisimple case, see([5]-[10]). In [4], Creedon provides a list of presentations of the unit groups $U(FG)$ of all group algebras FG with $|FG| < 1024$. The structure of $U(FD_6)$ for $p = 3$, is established in terms of split extensions of elementary abelian groups in [5]. The structure of $U(FS_5)$ for $p > 5$ is given in [12], where S_5 is symmetric group of degree 5. For $p = 3$, Monaghan[15], studied the structure of $U(FG)$ where G is a non-abelian group of 24 such that G has a normal subgroup of order 3. The structure of $U(FG)$ where G is a group of order 12 has been studied in [20] and [21]. Recently, Ansari and Sahai in [1], obtained the structure of $U(FG)$ for $G = C_{20}$, $C_{10} \times C_2$ and $GA(1,5)$ where $GA(1,5)$ is the general affine group of order 20. In the same paper, the structure of $U(FQ_{20})$ for the semisimple case is also given. In [18], they established the structure of the unit groups $U(FQ_{2^n})$ of the finite group algebras of the generalized quaternion groups Q_{2^n}, $p > 2$.

In [13], Makhijani et al. obtained the structure of the unit group of FD_{2n} for any odd $n \geq 3$ and $p = 2$. This is an extension of [10] in which they have studied the unit group of FD_{2p}, where p is a prime number. There are three non-abelian groups of order 30, namely, D_{30}, $C_3 \times D_{10}$ and $C_5 \times D_6$. In 2018, Gildea and Taylor [9] described the structure of $U(F(C_n \times D_6))$ for $p = 3$ which is an extension of [7]. In 2015, Makhijani et.al. [14] studied the structure of $U(FD_{30})$, but for $p = 3$ they provided only a preliminary description of the $U(FD_{30})$. Here in Section 1, we provide a complete characterization of $U(FD_{30})$ for $p = 3$. In Section 2, we give the structure of $U(F(C_3 \times D_{10}))$, again for $p = 3$ only.

2. Unit group of FD_{30}

Theorem 2.1. *Let F be a finite field of characteristic 3 with $|F| = q = 3^k$ and let $G = D_{30}$.*

(1) *If $q \equiv \pm 1$ mod 5, then $U(FG) \cong (C_3^{15k} \rtimes C_3^{5k}) \rtimes (C_{3^k-1}^2 \times GL(2,F)^2)$.*
(2) *If $q \equiv \pm 3$ mod 5, then $U(FG) \cong (C_3^{15k} \rtimes C_3^{5k}) \rtimes (C_{3^k-1}^2 \times GL(2,F_2))$.*

Proof. Let $G = \langle x, y \mid x^{15} = y^2 = 1, yxy = x^{-1}\rangle$. Let K be the normal subgroup of G generated by x^5. Then $G/K \cong H \cong \langle x^3, y\rangle$. Thus from the ring epimorphism $FG \to FH$ given by

$$\sum_{j=0}^{4}\sum_{i=0}^{2} x^{5i+3j}(a_{i+3j} + a_{i+3j+15}y) \to \sum_{j=0}^{4}\sum_{i=0}^{2} x^{3j}(a_{i+3j} + a_{i+3j+15}y),$$

we get a group epimorphism $\phi\colon U(FG) \to U(FH)$ and $ker\phi \cong 1 + J(FG) \cong V$. Further, from the ring monomorphism $FH \to FG$ given by

$$\sum_{i=0}^{4} x^{3i}(b_i + b_{i+5}y) \to \sum_{i=0}^{4} x^{3i}(b_i + b_{i+5}y),$$

we get a group monomorphism $\psi\colon U(FH) \to U(FG)$. Clearly, $\phi\psi = 1_{U(FH)}$ and $U(FG) \cong V \rtimes U(FD_{10})$.

If $u = \sum_{j=0}^{4}\sum_{i=0}^{2} x^{5i+3j}(a_{i+3j} + a_{i+3j+15}y) \in U(FG)$, then $u \in V$ if and only if $\sum_{i=0}^{2} a_i = 1$ and $\sum_{i=0}^{2} a_{i+3k} = 0$ for $k = 1, 2, \ldots, 9$. Hence

$$V = \{1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(x^{5i} - 1)x^{3j}(b_{i+2j} + b_{i+2j+10}y) \mid b_i \in F\},$$

$V^3 = 1$ and $|V| = 3^{20k}$.

Now we show that $V \cong C_3^{15k} \rtimes C_3^{5k}$. The centralizer of x^5 in V is

$$C_V(x^5) = \{v \in V \mid vx^5 = x^5v\}.$$

If $v = 1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(x^{5i} - 1)x^{3j}(b_{i+2j} + b_{i+2j+10}y) \in V$, then

$$vx^5 - x^5v = \widehat{x^5}\sum_{j=0}^{4}(b_{11+2j} - b_{12+2j})x^{3j}y.$$

Thus $v \in C_V(x^5)$ if and only if $b_i = b_{i+1}$ for $i = 11, 13, 15, 17$ and 19 and

$$C_V(x^5) = \{1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(x^{5i} - 1)c_{i+2j}x^{3j} + \widehat{x^5}\sum_{j=0}^{4} c_{j+11}x^{3j}y \mid c_i \in F\}.$$

Let W be a subset of V given by

$$W = \{1 + \sum_{j=0}^{4}\sum_{i=0}^{2} x^{5i+3j}(a_{j+1} + ia_{j+6}y) \mid a_i \in F\}.$$

It can easily be shown that W is an abelian group and $W \cong C_3^{10k}$. If

$$c = 1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(x^{5i} - 1)c_{i+2j}x^{3j} + \widehat{x^5}\sum_{j=0}^{4} c_{j+11}x^{3j}y \in C_V(x^5)$$

and

$$w = 1 + \sum_{j=0}^{4}\sum_{i=0}^{2} x^{5i+3j}(d_{j+1} + id_{j+6}y) \in W,$$

then

$$c^w = 1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(x^{5i} - 1)c_{i+2j}x^{3j} + \widehat{x^5}\sum_{j=0}^{4}(c_{j+11} - s_{j+1})x^{3j}y \in C_V(x^5)$$

where

$$s_1 = 2(c_1 - c_2)d_6 + ((c_3 - c_4) + (c_9 - c_{10}))(d_7 + d_{10}) + ((c_5 - c_6) + (c_7 - c_8))(d_8 + d_9)$$
$$s_2 = 2(c_1 - c_2)d_7 + ((c_3 - c_4) + (c_9 - c_{10}))(d_6 + d_8) + ((c_5 - c_6) + (c_7 - c_8))(d_9 + d_{10})$$
$$s_3 = 2(c_1 - c_2)d_8 + ((c_3 - c_4) + (c_9 - c_{10}))(d_7 + d_9) + ((c_5 - c_6) + (c_7 - c_8))(d_6 + d_{10})$$
$$s_4 = 2(c_1 - c_2)d_9 + ((c_3 - c_4) + (c_9 - c_{10}))(d_8 + d_{10}) + ((c_5 - c_6) + (c_7 - c_8))(d_6 + d_7)$$
$$s_5 = 2(c_1 - c_2)d_{10} + ((c_3 - c_4) + (c_9 - c_{10}))(d_6 + d_9) + ((c_5 - c_6) + (c_7 - c_8))(d_7 + d_8).$$

Now

$$R = C_V(x^5) \cap U = \{1 + \widehat{x^5} \sum_{j=0}^{4} d_{j+1} x^{3j} \mid d_i \in F\} \cong C_3^{5k}.$$

So, for some subgroup $T \cong C_3^{5k}$ of W, $W = R \times T \cong C_3^{5k} \times C_3^{5k}$. Obviously, $C_V(x^5) \cap T = 1$. Thus $V \cong C_V(x^5) \rtimes T \cong C_3^{15k} \rtimes C_3^{5k}$.

By [11, Theorem 2.1],

$$U(FD_{10}) \cong \begin{cases} C_{3^k-1}^2 \times GL(2,F)^2, & \text{if } q \equiv \pm 1 \text{ mod } 5; \\ C_{3^k-1}^2 \times GL(2,F_2), & \text{if } q \equiv \pm 3 \text{ mod } 5. \end{cases}$$

Hence

$$U(FG) \cong (C_3^{15k} \rtimes C_3^{5k}) \rtimes (C_{3^k-1}^2 \times GL(2,F)^2), \text{ if } q \equiv \pm 1 \text{ mod } 5$$

and

$$U(FG) \cong (C_3^{15k} \rtimes C_3^{5k}) \rtimes (C_{3^k-1}^2 \times GL(2,F_2)), \text{ if } q \equiv \pm 3 \text{ mod } 5. \qquad \square$$

3. Unit group of $F(C_3 \times D_{10})$

Theorem 3.1. *Let F be a finite field of characteristic 3 with $|F| = q = 3^k$ and let $G = C_3 \times D_{10}$.*

(1) *If $q \equiv \pm 1$ mod 5, then $U(FG) \cong V \rtimes (C_{3^k-1}^2 \times GL(2,F)^2)$,*
(2) *If $q \equiv \pm 3$ mod 5, then $U(FG) \cong V \rtimes (C_{3^k-1}^2 \times GL(2,F_2))$*

where $V \cong (((((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k)$.

Proof. Let $G = \langle x, y, z \mid x^2 = y^5 = z^3 = 1, xyx = y^{-1}, xz = zx, yz = zy\rangle$. Let K be the normal subgroup of G generated by z. Then $G/K \cong H \cong \langle x, y\rangle$. Thus from the ring epimorphism $FG \to FH$ given by

$$\sum_{j=0}^{4}\sum_{i=0}^{2} z^i y^j (a_{i+3j} + a_{i+3j+15}x) \to \sum_{j=0}^{4}\sum_{i=0}^{2} y^j (a_{i+3j} + a_{i+3j+15}x),$$

we get a group epimorphism $\phi\colon U(FG) \to U(FH)$ and $ker\phi \cong 1 + J(FG) \cong V$. Further, from the ring monomorphism $FH \to FG$ given by

$$\sum_{i=0}^{4} y^i(b_i + b_{i+5}x) \to \sum_{i=0}^{4} y^i(b_i + b_{i+5}x),$$

we get a group monomorphism $\psi\colon U(FH) \to U(FG)$. Clearly, $\phi\psi = 1_{U(FH)}$ and $U(FG) \cong V \rtimes U(FD_{10})$.

If $u = \sum_{j=0}^{4}\sum_{i=0}^{2} z^i y^j (a_{i+3j} + a_{i+3j+15}x) \in U(FG)$, then $u \in V$ if and only if $\sum_{i=0}^{2} a_i = 1$ and $\sum_{i=0}^{2} a_{i+3k} = 0$ for $k = 1, 2, \ldots, 9$. Hence

$$V = \{1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(z^i - 1)y^j(b_{i+2j} + b_{i+2j+10}x) \mid b_i \in F\},$$

$V^3 = 1$ and $|V| = 3^{20k}$. Now we complete the proof in following steps:

Step 1: Let H_1 be the subgroup of V given by

$$H_1 = \{1 + \sum_{i=1}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + a_{i+10}x) + \widehat{z}\sum_{i=1}^{4} a_{i+12}y^i x \mid a_i \in F\}.$$

Then $H_1 \cong C_3^{15k} \rtimes C_3^k$.

Let P_1 and Q_1 be the abelian subgroups of H_1 given by

$$P_1 = \{1 + b_1\widehat{z} + b_2 z(1 - z)x \mid b_i \in F\}$$

and

$$Q_1 = \{1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(z^i - 1)a_{i+2j}y^j + \widehat{z}\sum_{i=0}^{4} a_{i+11}y^i x \mid a_i \in F\}.$$

If

$$p_1 = 1 + b_1\widehat{z} + b_2 z(1 - z)x \in P_1$$

and

$$q_1 = 1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(z^i - 1)a_{i+2j}y^j + \widehat{z}\sum_{i=0}^{4} a_{i+11}y^i x \in Q_1,$$

then

$$\begin{aligned} q_1^{p_1} =& 1 + \sum_{j=0}^{4}\sum_{i=1}^{2}(z^i - 1)a_{i+2j}y^j + \widehat{z}\{a_{11} + (a_{12} + t_1)y + (a_{13} + t_2)y^2 \\ &+ (a_{14} - t_2)y^3 + (a_{15} - t_1)y^4\}x \in Q_1 \end{aligned}$$

where $t_1 = b_2\{(a_4 - a_3) - (a_{10} - a_9)\}$ and $t_2 = b_2\{(a_6 - a_5) - (a_8 - a_7)\}$.

Now

$$R_1 = P_1 \cap Q_1 = \{1 + b_1\widehat{z} \mid b_1 \in F\} \cong C_3^k.$$

So, for some subgroup $S_1 \cong C_3^k$ of P_1, $P_1 = R_1 \times S_1$. Clearly $Q_1 \cap S_1 = 1$. Hence

$$H_1 \cong Q_1 \rtimes S_1 \cong C_3^{15k} \rtimes C_3^k.$$

Step 2: Let H_2 be the subgroup of V given by

$$H_2 = \{1 + \sum_{i=1}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{1} a_{i+2j+10}y^j x) + \widehat{z}\sum_{i=2}^{4} a_{i+13}y^i x \mid a_i \in F\}.$$

Then $H_2 \cong (C_3^{15k} \rtimes C_3^k) \rtimes C_3^k$.

Let P_2 be the abelian subgroup of H_2 given by

$$P_2 = \{1 + b_1\widehat{z} + b_2 z(1 - z)yx \mid b_i \in F\}.$$

If

$$p_2 = 1 + b_1\widehat{z} + b_2 z(1 - z)yx \in P_2$$

and

$$h_1 = 1 + \sum_{i=0}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + a_{i+10}x) + \widehat{z}\sum_{i=1}^{4} a_{i+12}y^i x \in H_1,$$

then

$$\begin{aligned} h_1^{p_2} =& 1 + \sum_{i=1}^{2}(z^i - 1)\{a_i + (a_{i+2} - t_0)y + a_{i+4}y^2 + a_{i+6}y^3 + (a_{i+8} + t_0)y^4 \\ &+ (a_{i+10} - t_1)x\} + \widehat{z}\{a_{13}y + (a_{14} + t_1)y^2 + (a_{15} + t_2)y^3 + (a_{16} - t_2)y^4\}x \in H_1 \end{aligned}$$

where

$$\begin{aligned} t_0 &= b_2(a_{12} - a_{11}), \\ t_1 &= b_2\{(a_4 - a_3) - (a_{10} - a_9)\}, \\ t_2 &= b_2\{(a_6 - a_5) - (a_8 - a_7)\}. \end{aligned}$$

Now

$$R_2 = P_2 \cap H_1 = \{1 + b_1\widehat{z} \mid b_1 \in F\} \cong C_3^k.$$

So, for some subgroup $S_2 \cong C_3^k$ of P_2, $P_2 = R_2 \times S_2$. Clearly $H_1 \cap S_2 = 1$. Hence

$$H_2 \cong H_1 \rtimes S_2 \cong (C_3^{15k} \rtimes C_3^k) \rtimes C_3^k.$$

Step 3: Let H_3 be the subgroup of V given by

$$H_3 = \{1 + \sum_{i=1}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{2} a_{i+2j+10}y^j x) + \widehat{z}\sum_{i=3}^{4} a_{i+14}y^i x \mid a_i \in F\}.$$

Then $H_3 \cong ((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k$.

Let P_3 be the abelian subgroup of H_3 given by

$$P_3 = \{1 + b_1\widehat{z} + b_2 z(1 - z)y^2 x \mid b_i \in F\}.$$

If

$$p_3 = 1 + b_1\widehat{z} + b_2 z(1-z)y^2x \in P_3$$

and

$$h_2 = 1 + \sum_{i=0}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{1} a_{i+2j+10}y^jx) + \widehat{z}\sum_{i=2}^{4} a_{i+13}y^ix \in H_2,$$

then

$$\begin{aligned} h_2^{p_3} =& 1 + \sum_{i=1}^{2}(z^i - 1)\{a_i + (a_{i+2} - t_3)y + (a_{i+4} - t_0)y^2 + (a_{i+6} + t_0)y^3 + (a_{i+8} + t_3)y^4 \\ &+ (a_{i+10} - t_2)x + (a_{i+12} - t_1)yx\} + \widehat{z}\{a_{15}y^2 + (a_{16} + t_1)y^3 + (a_{17} + t_2)y^4\}x \in H_2 \end{aligned}$$

where

$$t_0 = b_2(a_{12} - a_{11}), \qquad t_1 = b_2\{(a_4 - a_3) - (a_{10} - a_9)\},$$
$$t_2 = b_2\{(a_6 - a_5) - (a_8 - a_7)\}, \qquad t_3 = b_2(a_{14} - a_{13}).$$

Now

$$R_3 = P_3 \cap H_2 = \{1 + b_1\widehat{z} \mid b_1 \in F\} \cong C_3^k.$$

So, for some subgroup $S_3 \cong C_3^k$ of P_3, $P_3 = R_3 \times S_3$. Clearly $H_2 \cap S_3 = 1$. Hence

$$H_3 \cong H_2 \rtimes S_3 \cong ((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k.$$

Step 4: Let H_4 be the subgroup of V given by

$$H_4 = \{1 + \sum_{i=1}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{3} a_{i+2j+10}y^jx) + \widehat{z}a_{19}y^4x \mid a_i \in F\}.$$

Then $H_4 \cong (((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k$.

Let P_4 be the abelian subgroup of H_4 given by

$$P_4 = \{1 + b_1\widehat{z} + b_2 z(1-z)y^3x \mid b_i \in F\}.$$

If

$$p_4 = 1 + b_1\widehat{z} + b_2 z(1-z)y^3x \in P_4$$

and

$$h_3 = 1 + \sum_{i=0}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{2} a_{i+2j+10}y^jx) + \widehat{z}\sum_{i=3}^{4} a_{i+14}y^ix \in H_3,$$

then

$$\begin{aligned}h_3^{p_4} =&1 + \sum_{i=1}^{2}(z^i - 1)\{a_i + (a_{i+2} - t_3)y + (a_{i+4} + t_0)y^2 + (a_{i+6} - t_0)y^3\\&+ (a_{i+8} + t_3)y^4 + (a_{i+10} + t_2)x + (a_{i+12} - t_2)yx + (a_{i+14} - t_1)y^2x\}\\&+ \widehat{z}\{a_{17}y^3 + (a_{18} + t_1)y^4\}x \in H_3\end{aligned}$$

where

$$t_0 = b_2\{(a_{12} - a_{11}) - (a_{14} - a_{13})\}, \qquad t_1 = b_2\{(a_4 - a_3) - (a_{10} - a_9)\},$$
$$t_2 = b_2\{(a_6 - a_5) - (a_8 - a_7)\}, \qquad t_3 = b_2(a_{16} - a_{15}).$$

Now

$$R_4 = P_4 \cap H_3 = \{1 + b_1\widehat{z} \mid b_1 \in F\} \cong C_3^k.$$

So, for some subgroup $S_4 \cong C_3^k$ of P_4, $P_4 = R_4 \times S_4$. Clearly $H_3 \cap S_4 = 1$. Hence

$$H_4 \cong H_3 \rtimes S_4 \cong (((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k.$$

Step 5: $V \cong ((((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k$.

Let P_5 be the abelian subgroup of V given by

$$P_5 = \{1 + b_1\widehat{z} + b_2z(1 - z)y^4x \mid b_i \in F\}.$$

If

$$p_5 = 1 + b_1\widehat{z} + b_2z(1 - z)y^4x \in P_5$$

and

$$h_4 = 1 + \sum_{i=0}^{2}(z^i - 1)(\sum_{j=0}^{4} a_{i+2j}y^j + \sum_{j=0}^{3} a_{i+2j+10}y^jx) + \widehat{z}a_{19}y^4x \in H_4,$$

then

$$\begin{aligned}h_4^{p_5} =&1 + \sum_{i=1}^{2}(z^i - 1)\{a_i + (a_{i+2} + t_3)y + (a_{i+4} + t_4)y^2 + (a_{i+6} - t_4)y^3\\&+ (a_{i+8} - t_3)y^4 + (a_{i+10} + t_1)x + (a_{i+12} + t_2)yx + (a_{i+14} - t_2)y^2x\\&+ (a_{i+16} - t_1)y^3x\} + \widehat{z}a_{19}y^4x \in H_4\end{aligned}$$

where

$$t_1 = b_2\{(a_4 - a_3) - (a_{10} - a_9)\}, \qquad t_2 = b_2\{(a_6 - a_5) - (a_8 - a_7)\},$$
$$t_3 = b_2\{(a_{12} - a_{11}) - (a_{18} - a_{17})\}, \qquad t_4 = b_2\{(a_{14} - a_{13}) - (a_{16} - a_{15})\}.$$

Now

$$R_5 = P_5 \cap H_4 = \{1 + b_1\widehat{z} \mid b_1 \in F\} \cong C_3^k.$$

So, for some subgroup $S_5 \cong C_3^k$ of P_5, $P_5 = R_5 \times S_5$. Clearly $H_4 \cap S_5 = 1$. Hence

$$V \cong H_4 \rtimes S_5 \cong ((((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k.$$

By [11, Theorem 2.1],

$$U(FG) \cong V \rtimes (C_{3^k-1}^2 \times GL(2,F)^2), \text{ if } q \equiv \pm 1 \bmod 5$$

and

$$U(FG) \cong V \rtimes (C_{3^k-1}^2 \times GL(2,F_2)), \text{ if } q \equiv \pm 3 \bmod 5$$

where $V \cong ((((C_3^{15k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k$. □

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References

[1] S. F. Ansari and M. Sahai, *Unit groups of group algebras of groups of order* 20, Quaest. Math., to appear, https://doi.org/10.2989/16073606.2020.1727583.

[2] A. Bovdi and L. Erdei, *Unitary units in modular group algebras of* 2*-groups*, Comm. Algebra, 28(2) (2000), 625-630.

[3] V. Bovdi and L. G. Kovács, *Unitary units in modular group algebras*, Manuscripta Math., 84(1) (1994), 57-72.

[4] L. Creedon, *The unit group of small group algebras and the minimum counterexample to the isomorphism problem*, Int. J. Pure Appl. Math., 49(4) (2008), 531-537.

[5] L. Creedon and J. Gildea, *The structure of the unit group of the group algebra* $F_{3^k}D_6$, Int. J. Pure Appl. Math., 45(2) (2008), 315-320.

[6] L. Creedon and J. Gildea, *The structure of the unit group of the group algebra* $F_{2^k}D_8$, Canad. Math. Bull., 54(2) (2011), 237-243.

[7] J. Gildea, *The structure of the unit group of the group algebra* $\mathbb{F}_{3^k}(C_3 \times D_6)$, Comm. Algebra, 38(9) (2010), 3311-3317.

[8] J. Gildea, *Units of the group algebra* $\mathbb{F}_{5^k}(C_5 \rtimes C_4)$, Int. Electron. J. Algebra, 9 (2011), 220-227.

[9] J. Gildea and R. Taylor, *Units of the group algebra of the group* $C_n \times D_6$ *over any finite field of characteristic* 3, Int. Electron. J. Algebra, 24 (2018), 62-67.

[10] K. Kaur and M. Khan, *Units in* F_2D_{2p}, J. Algebra Appl., 13(2) (2014), 1350090 (9 pp).

[11] M. Khan, *Structure of the unit group of* FD_{10}, Serdica Math. J., 35(1) (2009), 15-24.

[12] Y. Kumar, R. K. Sharma and J. B. Srivastava, *The structure of the unit group of the group algebra* $\mathbb{F}S_5$ *where* $\mathbb{F}$ *is a finite field with char*$(\mathbb{F}) = p > 5$, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 33(2) (2017), 187-191.

[13] N. Makhijani, R. K. Sharma and J. B. Srivastava, *Units in* $\mathbb{F}_{2^k}D_{2n}$, Int. J. Group Theory, 3(3) (2014), 25-34.

[14] N. Makhijani, R. K. Sharma and J. B. Srivastava, *The unit group of* $\mathbb{F}_q[D_{30}]$, Serdica Math. J., 41(2-3) (2015), 185-198.

[15] F. Monaghan, *Units of some group algebras of non-abelian groups of order* 24 *over any finite field of characteristic* 3, Int. Electron. J. Algebra, 12 (2012), 133-161.

[16] Z. Raza and M. Ahmad, *On the structure of the unitary subgroup of the group algebra* $\mathbb{F}_{2^q}D_{2^n}$, J. Algebra Appl., 13(4) (2014), 1350139 (8 pp).

[17] Z. Raza and M. Ahmad, *Structure of the unitary subgroup of the group algebra* $\mathbb{F}_{2^n}(QD)_{16}$, J. Algebra Appl., 17(4) (2018), 1850060 (7 pp).

[18] M. Sahai and S. F. Ansari, *Unit groups of the finite group algebras of generalized quaternion groups*, J. Algebra Appl., 19(6) (2020), 2050112 (5 pp).

[19] R. Sandling, *Units in the modular group algebra of a finite abelian p-group*, J. Pure Appl. Algebra, 33(3) (1984), 337-346.

[20] G. Tang and G. Yanyan, *The unit groups of FG of groups with order* 12, Int. J. Pure Appl. Math., 73(2) (2011), 143-158.

[21] G. Tang, Y. Wei and Y. Li, *Unit groups of group algebras of some small groups*, Czechoslovak Math. J., 64(1) (2014), 149-157.

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