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Investigation of Stability Changes in a Neural Field Model

Berrak ÖZGÜR[1,*]

[1] *İzmir Democracy University, Department of Mathematics, Izmir, 35140, Turkey,* ***ORCID:*** *0000-0002-9709-7376*

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Abstract

In this paper, the stability analysis of the neural field model is studied. The special case for three neuron populations is considered. The work is conducted by finding the characteristic equation of the system first and then investigating the characteristic roots of the third-order equation by using the Routh-Hurwitz criterion and Sturm sequence. The main analysis is given in two parts considering the nonexistence and existence of the delay term. Some basic stability criteria in terms of coefficients of the system are given in the theorems.

1. Introduction

The term ‘dynamical system’ is used to determine a system varying with respect to time. In applied mathematics, in order to understand the general construction for a real-world phenomenon and analyse its future state, mathematical models are used. Differential equations, difference equations and functional equations are frequently used when writing mathematical models for dynamical systems representing the real phenomena. Hence, the important analyses related to them can be made by using convenient mathematical methods.

In sciences such as biology, engineering, economics, since time is very important, we generally use a time delay in writing more realistic models. The theory of delay differential equations has an important role in such fields.

The scientists aimed to model the activity of large neuron populations in the brain, use the neural field models. These models are constructed using integro-differential equations including a time delay. For the basic facts in neural field models, one can refer to the studies given in [1,2]. Besides these studies, the stability analysis of the neural field model and the existence and uniqueness of their solutions are studied in some papers [3-15].

In stability analysis, obtaining the characteristic equation of the system and determining the characteristic roots construct the important part. There are some studies on investigation of the stability analysis for this model including functional analysis and numerical methods [7,9,10]. In [11-15] the analysis is made by using the D-curves method and the Routh-Hurwitz criterion.

In this study, we are interested in the stability of a neural field model for three neuron populations. The general overview of the study is given in the following; the model is given in Section 2. The stability properties for the model are given in Section 3. The roles of the system parameters on the stability of the model are shown. This analysis is made by using the Sturm sequence since the corresponding characteristic equation of the model is third order. The conclusion of this study is in Section 4.

* Corresponding Author: berrak.ozgur@idu.edu.tr

2. Neural Field Model

The scientists use neural field equations to model dynamics of mean membrane potential for p neural populations on the space $\Omega \subset R^d$. This model given in [6,7,9,10] is given below

$$\left(\frac{d}{dt}+l_i\right)V_i(t,r) = \sum_{j=1}^{p}\int_{\Omega} J_{ij}(r,\bar{r})S\left[\sigma_j\left(V_j\left(t-\tau_{ij}(r,\bar{r}),\bar{r}\right)-h_j\right)\right]d\bar{r}+I_i^{ext}(r,t)\ ,\ t\geq 0\ ,\ 1\leq i\leq p$$
$$V_i(t,r)=\phi_i(t,r)\ ,\quad t\in[-T,0] \tag{1}$$

Here, we consider this model for three neuron populations ($p=3$). The synaptic inputs for large groups of neurons at position x and time t are represented by the functions $V_1(x,t)$, $V_2(x,t)$ and $V_3(x,t)$. We consider $\Omega=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. The relations of neurons on different populations are shown by the functions $J_{ij}(x,y)$. Some of the relations among them are restricted. The stability analysis is made in the special case that $J_{12}(x,y)\neq 0$, $J_{21}(x,y)\neq 0$, $J_{23}(x,y)\neq 0$, $J_{32}(x,y)\neq 0$. In this research, the delay term is considered constant as $\tau(x,y)=\tau$. Hence the linearized model near (0,0,0) is given below. For this model the functions $U_1(x,t)$, $U_2(x,t)$ and $U_3(x,t)$ are used.

$$\frac{d}{dt}U_1(x,t)+l_1U_1(x,t)=\sigma_2s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{12}(x,y)U_2\left(y,t-\tau(x,y)\right)dy$$

$$\frac{d}{dt}U_2(x,t)+l_2U_2(x,t)=\sigma_1s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{21}(x,y)U_1\left(y,t-\tau(x,y)\right)dy+\sigma_3s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{23}(x,y)U_3\left(y,t-\tau(x,y)\right)dy$$

$$\frac{d}{dt}U_3(x,t)+l_3U_3(x,t)=\sigma_2s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{32}(x,y)U_2\left(y,t-\tau(x,y)\right)dy \tag{2}$$

3. Stability Analysis

Considering the Fourier method, we are looking for the solutions as $U_1(x,t)=e^{ikx}u_1(t)$, $U_2(x,t)=e^{ikx}u_2(t)$, $U_3(x,t)=e^{ikx}u_3(t)$. Here $u_1(t)=c_1e^{\lambda t}$, $u_2(t)=c_2e^{\lambda t}$ and $u_3(t)=c_3e^{\lambda t}$. Writing them in the system (2), we get the following system

$$\lambda e^{ikx}u_1(t)+l_1e^{ikx}u_1(t)=\sigma_2s_1e^{-\lambda\tau}u_2(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{12}(x,y)e^{iky}dy$$

$$\lambda e^{ikx}u_2(t)+l_2e^{ikx}u_2(t)=\sigma_1s_1e^{-\lambda\tau}u_1(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{21}(x,y)e^{iky}dy+\sigma_3s_1e^{-\lambda\tau}u_3(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{23}(x,y)e^{iky}dy$$

$$\lambda e^{ikx}u_3(t)+l_3e^{ikx}u_3(t)=\sigma_2s_1e^{-\lambda\tau}u_2(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{32}(x,y)e^{iky}dy \tag{3}$$

The solutions of the system are the functions $cos(2nx)$ and $sin(2nx)$ [9]. Hence we have

$$\lambda u_1(t)+l_1u_1(t)=\sigma_2s_1e^{-\lambda\tau}u_2(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{12}(x,y)e^{iky}dy$$

$$\lambda u_2(t)+l_2u_2(t)=\sigma_1s_1e^{-\lambda\tau}u_1(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{21}(x,y)e^{iky}dy+\sigma_3s_1e^{-\lambda\tau}u_3(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{23}(x,y)e^{iky}dy$$

$$\lambda u_3(t)+l_3u_3(t)=\sigma_2s_1e^{-\lambda\tau}u_2(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{32}(x,y)e^{iky}dy \tag{4}$$

Considering the coefficient determinant of this system with respect to $u_1(t), u_2(t)$ and $u_3(t)$, we get the following characteristic equation arranged in terms of the powers of λ

$$\lambda^3+(l_1+l_2+l_3)\lambda^2+(l_2l_3+l_1l_3+l_1l_2)\lambda+l_1l_2l_3-K_1K_3F_3F_4e^{-2\lambda\tau}(\lambda+l_1)-K_1K_2F_1F_2e^{-2\lambda\tau}(\lambda+l_3)=0 \tag{5}$$

where $K_1=\sigma_2s_1, K_2=\sigma_1s_1,\ K_3=\sigma_3s_1$,

$$F_1=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{12}(x,y)e^{iky}dy\ ,$$

$$F_2=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{21}(x,y)e^{iky}dy,$$

$$F_3=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{23}(x,y)e^{iky}dy,$$

$$F_4=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}J_{32}(x,y)e^{iky}dy$$

If there is no delay term in the system, i.e., $\tau=0$, the characteristic equation turns into the following form

$$\lambda^3 + (l_1 + l_2 + l_3)\lambda^2 + (l_2 l_3 + l_1 l_3 + l_1 l_2 - K_1 K_3 F_3 F_4 - K_1 K_2 F_1 F_2)\lambda + l_1 l_2 l_3 - K_1 K_3 F_3 F_4 l_1 - K_1 K_2 F_1 F_2 l_3 = 0 \tag{6}$$

According to the Routh-Hurwitz criterion, we may give the following theorem.

Theorem: Consider the system (2). If the following conditions are satisfied

$$l_1 + l_2 + l_3 > 0\ ,$$

$$l_1 l_2 l_3 - K_1 K_3 F_3 F_4 l_1 - K_1 K_2 F_1 F_2 l_3 > 0$$

and

$$(l_1 + l_2 + l_3)(l_2 l_3 + l_1 l_3 + l_1 l_2 - K_1 K_3 F_3 F_4 - K_1 K_2 F_1 F_2) - (l_1 l_2 l_3 - K_1 K_3 F_3 F_4 l_1 - K_1 K_2 F_1 F_2 l_3) > 0$$

then the system is stable near $(0,0,0)$ in the absence of delay term.

Proof: If the conditions given above are satisfied then, according to the Routh-Hurwitz criterion, all roots of the characteristic equation have negative real parts and the system is stable near $(0,0,0)$ in the absence of delay term.

In case of existence of a delay term, we apply the procedure given in [16]. Because of the critical delays, some characteristic roots change from having negative real parts to having positive real parts. For this reason, we will examine the purely imaginary roots $\lambda = i\sigma$. To get the characteristic equation and see such a change, we substitute $\lambda = i\sigma$ in Eq. (5) , and separating the real and imaginary parts, we get

$$-\sigma^2(l_1 + l_2 + l_3) + l_1 l_2 l_3 - K_1 K_3 F_3 F_4 \sigma sin(2\sigma\tau) - K_1 K_3 F_3 F_4 l_1 cos(2\sigma\tau) - K_1 K_2 F_1 F_2 \sigma sin(2\sigma\tau) - K_1 K_2 F_1 F_2 l_3 cos(2\sigma\tau) = 0 \tag{7}$$

$$-\sigma^3 + \sigma(l_2 l_3 + l_1 l_3 + l_1 l_2) - K_1 K_3 F_3 F_4 \sigma cos(2\sigma\tau) + K_1 K_3 F_3 F_4 l_1 sin(2\sigma\tau) - K_1 K_2 F_1 F_2 \sigma cos(2\sigma\tau) + K_1 K_2 F_1 F_2 l_3 sin(2\sigma\tau) = 0 \tag{8}$$

Rearranging them we get the following two equations

$$-\sigma^2(l_1 + l_2 + l_3) + l_1 l_2 l_3 = K_1 K_3 F_3 F_4 \sigma sin(2\sigma\tau) + K_1 K_3 F_3 F_4 l_1 cos(2\sigma\tau) + K_1 K_2 F_1 F_2 \sigma sin(2\sigma\tau) + K_1 K_2 F_1 F_2 l_3 cos(2\sigma\tau) \tag{9}$$

$$-\sigma^3 + \sigma(l_2 l_3 + l_1 l_3 + l_1 l_2) = K_1 K_3 F_3 F_4 \sigma cos(2\sigma\tau) - K_1 K_3 F_3 F_4 l_1 sin(2\sigma\tau) + K_1 K_2 F_1 F_2 \sigma cos(2\sigma\tau) - K_1 K_2 F_1 F_2 l_3 sin(2\sigma\tau) \tag{10}$$

Taking squares of both sides in these two equations and adding them we get the polynomial equation given below

$$\sigma^6 + \left({l_1}^2 + {l_2}^2 + {l_3}^2\right)\sigma^4 + \left({l_2}^2{l_3}^2 + {l_1}^2{l_3}^2 + {l_1}^2{l_2}^2 - {K_1}^2{K_3}^2{F_3}^2{F_4}^2 - 2{K_1}^2 K_2 K_3 F_1 F_2 F_3 F_4 - {K_1}^2{K_2}^2{F_1}^2{F_2}^2\right)\sigma^2 + {l_1}^2{l_2}^2{l_3}^2 - {K_1}^2{K_3}^2{F_3}^2{F_4}^2{l_1}^2 - 2{K_1}^2 K_2 K_3 F_1 F_2 F_3 F_4 l_1 l_3 - {K_1}^2{K_2}^2{F_1}^2{F_2}^2{l_3}^2 = 0 \tag{11}$$

Now following the Routh-Hurwitz criterion, we replace μ by σ^2. Hence we have the following third-order polynomial equation to carry the stability analysis for the model.

$$\mu^3 + \left({l_1}^2 + {l_2}^2 + {l_3}^2\right)\mu^2 + \left({l_2}^2{l_3}^2 + {l_1}^2{l_3}^2 + {l_1}^2{l_2}^2 - {K_1}^2{K_3}^2{F_3}^2{F_4}^2 - 2{K_1}^2 K_2 K_3 F_1 F_2 F_3 F_4 - {K_1}^2{K_2}^2{F_1}^2{F_2}^2\right)\mu + {l_1}^2{l_2}^2{l_3}^2 - {K_1}^2{K_3}^2{F_3}^2{F_4}^2{l_1}^2 - 2{K_1}^2 K_2 K_3 F_1 F_2 F_3 F_4 l_1 l_3 - {K_1}^2{K_2}^2{F_1}^2{F_2}^2{l_3}^2 = 0 \tag{12}$$

For simplicity, we call the coefficients

$$A = {l_1}^2 + {l_2}^2 + {l_3}^2$$

$$B = {l_2}^2{l_3}^2 + {l_1}^2{l_3}^2 + {l_1}^2{l_2}^2 - {K_1}^2{K_3}^2{F_3}^2{F_4}^2 - 2{K_1}^2 K_2 K_3 F_1 F_2 F_3 F_4 - {K_1}^2{K_2}^2{F_1}^2{F_2}^2$$

$$C = {l_1}^2{l_2}^2{l_3}^2 - {K_1}^2{K_3}^2{F_3}^2{F_4}^2{l_1}^2 - 2{K_1}^2 K_2 K_3 F_1 F_2 F_3 F_4 l_1 l_3 - {K_1}^2{K_2}^2{F_1}^2{F_2}^2{l_3}^2 \tag{13}$$

Since the leading coefficient is positive, a positive real root may occur in two cases.

i) If $C < 0$ then the positive real root occurs.

ii) If $C > 0$ then a negative real root is guaranteed. To analyze the possibility to have two positive real roots, we use the Sturm sequence of the polynomial in Eq. (12).

The details of the method of Sturm sequence are constructed by determining whether a positive real root exists. After finding the functions in the Sturm sequence, the sign changes in endpoints of the considered interval must be determined. The number gives us the number of real roots. After this step, the conditions must be analyzed to see the positive real root.

We will follow the procedure given in [16]. Let us start with the polynomials

$$f_0 = \mu^3 + A\mu^2 + B\mu + C$$

and

$$f_1 = 3\mu^2 + 2A\mu + B$$

where $f_1 = f_0{}'$. Applying the division algorithm

$$f_0 = q_0 f_1 + f_2$$

$$f_1 = q_1 f_2 + f_3$$

we have

$$f_2 = \left(\frac{2}{9}A^2 - \frac{2}{3}B\right)\mu + C - \frac{1}{9}AB$$

$$f_3 = -\frac{9}{4}\frac{4B^3 - A^2B^2 - 18ABC + 4CA^3 + 27C^2}{(A^2-3B)^2}$$

By considering the sign changes at each endpoint of the interval $(-\infty, \infty)$, we may construct the following table given in [16] to have three sign changes, hence three real roots for the case (ii).

Table 1. The sign changes for the Sturm sequence.

	$-\infty$	∞
f_0	-	+
f_1	+	+
f_2	-	+
f_3	+	+

In order to get this table we need the following conditions,

$$A^2 - 3B > 0$$

and

$$4(B^2 - 3AC)(A^2 - 3B) - (9C - AB)^2 > 0$$

where the constants A, B, C are determined as in (13).

And for the case (ii), there exists one positive real root if $A < 0$ or $A > 0$ and $B < 0$ [16].

We will conclude this part by the following theorem based on the theorem in [16] in case of a delay term exists.

Theorem: Consider the characteristic equation (5) for the system (2) with a delay term. The system is unstable near (0,0,0) if and only if A, B and C are not all positive and either $C < 0$, or $C > 0$, $A^2 - 3B > 0$ and $4(B^2 - 3AC)(A^2 - 3B) - (9C - AB)^2 > 0$ is satisfied where A, B and C are given in (13).

Proof: In the existence of the conditions given above, we have three real characteristic roots and one of them is positive. Hence the system becomes unstable.

4. Conclusion

In this study, the stability properties of a neural field model are constructed in a special case. The linearized model for three neuron populations is considered and is investigated for the stability in an algebraic way. The main idea here is to determine the roots of the characteristic equation. Since the characteristic equation is third-order, the Routh-Hurwitz criterion and the Sturm sequence are used. These two methods give the chance to make the analysis in an efficient way. As shown in this study, the stability properties in terms of the coefficients on the system are determined in a quick way by two theorems.

Conflict of Interests

No conflict of interest was stated by the authors.

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Declaration of Ethical Standards

The authors of this article declares that the materials and methods used in this study do not require ethical committee permission and legal-special permission.

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