

TENSOR-CLOSED OBJECTS IN THE BGG CATEGORY OF A QUANTIZED SEMISIMPLE LIE ALGEBRA

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ABSTRACT. We consider the BGG category \mathcal{O} of a quantized universal enveloping algebra $U_q(\mathfrak{g})$. We call a module $M \in \mathcal{O}$ tensor-closed if $M \otimes N \in \mathcal{O}$ for any $N \in \mathcal{O}$. In this paper we prove that $M \in \mathcal{O}$ is tensor-closed if and only if M is finite dimensional. The method used in this paper applies to the unquantized case as well.

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1. Introduction

BGG category \mathcal{O} plays a central role in representation theory, see [3]. For a complex semisimple Lie algebra \mathfrak{g} we can consider its quantized universal enveloping algebra $U_q(\mathfrak{g})$ and the category \mathcal{O} of $U_q(\mathfrak{g})$ as in [1] and [7].

The large category $U_q(\mathfrak{g})\text{-Mod}$ has a tensor product but category \mathcal{O} is not closed under the tensor product. We call a module $M \in \mathcal{O}$ *tensor-closed* if $M \otimes N \in \mathcal{O}$ for any $N \in \mathcal{O}$. It is easy to show that finite dimensional modules are tensor-closed. Actually in [6] the authors used tensor products of finite dimensional $U_q(\mathfrak{g})$ -modules to construct the coordinate ring of the deformed flag variety of \mathfrak{g} . Therefore it is interesting to ask whether we can characterize finite dimensional $U_q(\mathfrak{g})$ -modules in a categorical way, to which we give an affirmative answer in this paper.

For (unquantized) complex semisimple Lie algebra \mathfrak{g} it is a folklore theorem that any tensor-closed module in \mathcal{O} must be finite dimensional, see [4] for an outline of the proof.

The main result of this paper is Theorem 4.4, which claims that $M \in \mathcal{O}$ of $U_q(\mathfrak{g})$ is tensor-closed if and only if M is finite dimensional. This result gives a categorical characterization of finite dimensional modules in category \mathcal{O} . The proof is based on the idea in [4] together with a careful study of rational expressions of formal

characters of modules in \mathcal{O} . We can apply the same proof to the unquantized case with little modification.

2. A review of the BGG category \mathcal{O} of a quantized universal enveloping algebra

2.1. A review of quantized universal enveloping algebras. We follow the notations in [7]. Please also see [1] for references. Let \mathfrak{g} a semisimple Lie algebra over \mathbb{C} of rank N . We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let Δ be the set of roots and we fix $\Sigma = \{\alpha_1, \dots, \alpha_N\} \subset \Delta$ the set of simple roots. We write $(\ , \)$ for the bilinear form on \mathfrak{h}^* obtained by rescaling the Killing form such that the shortest root α of \mathfrak{g} satisfies $(\alpha, \alpha) = 2$. For a root $\beta \in \Delta$ we set $d_\beta = (\beta, \beta)/2$ and let $\beta^\vee = \beta/d_\beta$ be the corresponding coroot. In particular let $d_i = (\alpha_i, \alpha_i)/2$ and hence $\alpha_i^\vee = d_i^{-1}\alpha_i$ for $i = 1, \dots, N$.

Denote by $\varpi_1, \dots, \varpi_N$ the fundamental weights of \mathfrak{g} , satisfying the relations $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$. We write

$$\mathbf{P} = \bigoplus_{j=1}^N \mathbb{Z}\varpi_j, \quad \mathbf{Q} = \bigoplus_{j=1}^N \mathbb{Z}\alpha_j, \quad \mathbf{Q}^\vee = \bigoplus_{j=1}^N \mathbb{Z}\alpha_j^\vee, \tag{1}$$

for the weight, root and coroot lattices of \mathfrak{g} , respectively. It is well-known that $\beta^\vee \in \mathbf{Q}^\vee$ for each $\beta \in \Delta$.

Let \mathbf{P}^+ denote the set of dominant integral weights and \mathbf{Q}^+ denote the set of non-negative integer combinations of the simple roots. Let $\Delta^+ = \mathbf{Q}^+ \cap \Delta$ be the set of positive roots.

As in the standard notation, let $(a_{ij})_{1 \leq i, j \leq N}$ be the Cartan matrix for \mathfrak{g} and let W be the Weyl group for \mathfrak{g} . See [2, Chapter III] for details.

Definition 2.1. [7, Definition 2.13] Let $q = e^h \in \mathbb{R}^\times$ be an invertible element for $h \in \mathbb{R}^\times$. It is clear q is not a root of 1. The algebra $U_q(\mathfrak{g})$ over \mathbb{C} has generators K_λ for $\lambda \in \mathbf{P}$, and E_i, F_i for $i = 1, \dots, N$, and the defining relations for $U_q(\mathfrak{g})$ are

$$K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_\lambda E_j K_\lambda^{-1} = q^{(\lambda, \alpha_j)} E_j, \quad K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda, \alpha_j)} F_j, \tag{2}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

for all $\lambda, \mu \in \mathbf{P}$ and all i, j , together with the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0. \tag{3}$$

In the above formulas we abbreviate $K_i = K_{\alpha_i}$ for all simple roots, and we use the notation $q_i = q^{d_i}$.

$U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication $\hat{\Delta} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by

$$\begin{aligned} \hat{\Delta}(K_\lambda) &= K_\lambda \otimes K_\lambda, \\ \hat{\Delta}(E_i) &= E_i \otimes K_i + 1 \otimes E_i \\ \hat{\Delta}(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \end{aligned} \tag{4}$$

counit $\hat{\epsilon} : U_q(\mathfrak{g}) \rightarrow \mathbb{C}$ given by $\hat{\epsilon}(K_\lambda) = 1, \hat{\epsilon}(E_j) = 0, \hat{\epsilon}(F_j) = 0$, and antipode $\hat{S} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by $\hat{S}(K_\lambda) = K_{-\lambda}, \hat{S}(E_j) = -E_j K_j^{-1}, \hat{S}(F_j) = -K_j F_j$.

Let $U_q(\mathfrak{n}_+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements E_1, \dots, E_N , and $U_q(\mathfrak{b}_+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by E_1, \dots, E_N and all K_λ for $\lambda \in \mathbf{P}$. We define $U_q(\mathfrak{n}_-)$ and $U_q(\mathfrak{b}_-)$ in the same way. Moreover we let $U_q(\mathfrak{h})$ be the subalgebra generated by the elements K_λ for $\lambda \in \mathbf{P}$. These algebras are Hopf subalgebras of $U_q(\mathfrak{g})$. By [7, Proposition 2.14] we know that there is a linear isomorphism

$$U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+) \cong U_q(\mathfrak{g}). \tag{5}$$

2.2. A review of the BGG category \mathcal{O} . Recall that $1 \neq q = e^h$ for an $h \in \mathbb{R}^\times$. We shall also use the notation $\hbar = \frac{h}{2\pi}$ hence $q = e^{2\pi\hbar}$.

As in [7, Section 2.3.1] we let $\mathfrak{h}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ and $\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ be the parameter space for weights. Here $i = \sqrt{-1}$. It is clear that there is an embedding $\text{Span}_{\mathbb{R}} \Delta \subset \mathfrak{h}_q^*$. In particular $\mathbf{Q} \subset \mathbf{P} \subset \mathfrak{h}_q^*$.

One says that a vector v in a left $U_q(\mathfrak{g})$ -module is a weight vector of weight $\lambda \in \mathfrak{h}_q^*$ if it is a common eigenvector for the action of $U_q(\mathfrak{h})$ with

$$K_\mu \cdot v = q^{(\lambda, \mu)} v, \quad \text{for all } \mu \in \mathbf{P}.$$

It is well defined: if $\lambda \in i\hbar^{-1}\mathbf{Q}^\vee$ then for any $\mu \in \mathbf{P}$ we have $q^{(\lambda, \mu)} = e^{2\pi\hbar(\lambda, \mu)} = 1$.

Definition 2.2. ([1, Definition 3.1], [7, Definition 4.1]) A left module M over $U_q(\mathfrak{g})$ is said to belong to the *BGG category \mathcal{O}* if

- a) M is finitely generated as a $U_q(\mathfrak{g})$ -module.
- b) M is a weight module, that is, a direct sum of its weight spaces M_λ for $\lambda \in \mathfrak{h}_q^*$.
- c) The action of $U_q(\mathfrak{n}_+)$ on M is locally nilpotent, that is, for each $v \in M$, the subspace $U_q(\mathfrak{n}_+) \cdot v$ of M is finite dimensional.

Morphisms in category \mathcal{O} are all $U_q(\mathfrak{g})$ -linear maps.

We list some basic properties of category \mathcal{O} .

Proposition 2.3. (1) \mathcal{O} is closed under submodules, quotient modules, and finite direct sums.

(2) All weight spaces of M in \mathcal{O} are finite dimensional.

(3) All finite dimensional weight modules of $U_q(\mathfrak{g})$ are in \mathcal{O} .

Definition 2.4. [7, Definition 2.31] The Verma module $M(\lambda)$ associated to $\lambda \in \mathfrak{h}_q^*$ is the induced $U_q(\mathfrak{g})$ -module

$$M(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{C}_\lambda \tag{6}$$

where \mathbb{C}_λ denotes the one-dimensional $U_q(\mathfrak{b}_+)$ -module \mathbb{C} with the action induced from the character χ_λ determined by

$$\chi_\lambda(K_\mu) = q^{(\lambda, \mu)} \text{ for all } \mu \in \mathbf{P}, \text{ and } \chi_\lambda(E_i) = 0, \ i = 1, \dots, N. \tag{7}$$

It is clear that $M(\lambda)$ belongs to category \mathcal{O} .

$M(\lambda)$ contains a unique maximal proper submodule $I(\lambda)$, namely the linear span of all submodules not containing the highest weight vector $v_\lambda = 1 \otimes 1 \in M(\lambda)$. The resulting simple quotient module $M(\lambda)/I(\lambda)$ will be denoted by $V(\lambda)$. It is again a module in \mathcal{O} .

Remark 2.5. In [1], $M(\lambda)$ and $V(\lambda)$ are denoted by $\Delta_q(\lambda)$ and $L_q(\lambda)$ respectively.

It is clear that every highest weight module of highest weight λ is isomorphic to a quotient of $M(\lambda)$ and every simple highest weight module of highest weight λ is isomorphic to $V(\lambda)$.

The following result characterizes finite dimensional weight modules of $U_q(\mathfrak{g})$

Proposition 2.6. [7, Corollary 2.100] We write \mathbf{X}_q for the set of weights $\omega \in \mathfrak{h}_q^*$ satisfying $q^{(\omega, \alpha)} = \pm 1$ for all $\alpha \in \mathbf{Q}$. We define

$$\mathbf{P}_q^+ = \mathbf{P}^+ + \mathbf{X}_q \subset \mathfrak{h}_q^*.$$

Then every finite dimensional weight module over $U_q(\mathfrak{g})$ decomposes into a direct sum of irreducible highest weight modules $V(\lambda)$ for weights $\lambda \in \mathbf{P}_q^+$.

Simple modules $V(\lambda)$ are the building blocks of modules in \mathcal{O} .

Proposition 2.7. [7, Theorem 4.3] Every module $M \in \mathcal{O}$ is both Artinian and Noetherian. Hence every module $M \in \mathcal{O}$ has a Jordan-Hölder decomposition series $0 = M_0 \subset M_1 \subset \dots \subset M_n = M$ such that all subquotients M_{j+1}/M_j are simple highest weight modules. Moreover, the number of subquotients isomorphic to $V(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$ is independent of the decomposition series and will be denoted by $[M : V(\lambda)]$.

To further study $[M(\mu) : V(\lambda)]$ for a Verma module $M(\mu)$ we need the following concept.

Definition 2.8. ([5, Section 8.3.2], [7, Definition 2.125]) We define

$$\mathbf{Y}_q = \{\zeta \in \mathfrak{h}_q^* \mid 2\zeta = 0\} \cong \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee / i\hbar^{-1}\mathbf{Q}^\vee. \quad (8)$$

It is clear that W acts on \mathbf{Y}_q . The *extended Weyl group* \hat{W} is defined as the semidirect product

$$\hat{W} = \mathbf{Y}_q \rtimes W \quad (9)$$

with respect to the action of W on \mathbf{Y}_q . \hat{W} is a finite group.

Explicitly, the product in \hat{W} is $(i\zeta, v)(i\eta, w) = (i\zeta + iv\eta, vw)$. We define two actions of \hat{W} on \mathfrak{h}_q^* by $(i\zeta, w)\lambda = w\lambda + i\zeta$ and

$$(i\zeta, w) \cdot \lambda = w \cdot \lambda + i\zeta = w(\lambda + \rho) - \rho + i\zeta, \quad (10)$$

for $\lambda \in \mathfrak{h}_q^*$, where ρ is the half sum of all positive roots. The latter is called the *shifted action* of \hat{W} on \mathfrak{h}_q^* .

Remark 2.9. See Theorem 2.128 of [7] for the relation between the $\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ -translation and the *Harish-Chandra map*, which plays an important role in the representation theory of $U_q(\mathfrak{g})$.

Definition 2.10. We say that $\mu, \lambda \in \mathfrak{h}_q^*$ are \hat{W} -linked if $\hat{w} \cdot \lambda = \mu$ for some $\hat{w} \in \hat{W}$.

Definition 2.11. We define a partial order \geq on \mathfrak{h}_q^* by saying that $\lambda \geq \mu$ if $\lambda - \mu \in \mathbf{Q}^+$. Here we are identifying \mathbf{Q}^+ with its image in \mathfrak{h}_q^* .

Lemma 2.12. [7, Section 4.1.1] *For any $\mu \in \mathfrak{h}_q^*$ we have $[M(\mu) : V(\mu)] = 1$, and moreover $[M(\mu) : V(\lambda)] = 0$ unless $\lambda \leq \mu$ and λ is \hat{W} -linked to μ . Since \hat{W} is a finite group, for each $\mu \in \mathfrak{h}_q^*$ there exists only finitely many $\lambda \in \mathfrak{h}_q^*$ such that $[M(\mu) : V(\lambda)] \neq 0$.*

3. Formal characters of modules in category \mathcal{O}

3.1. Basic properties of formal characters.

Definition 3.1. We define the formal character of M in \mathcal{O} as the formal sum

$$\text{ch}(M) = \sum_{\lambda \in \mathfrak{h}_q^*} \dim(M_\lambda) e^\lambda. \quad (11)$$

By Proposition 2.3 any module M in category \mathcal{O} satisfies $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}_q^*$. So $\text{ch}(M)$ is well-defined. We also have the following more general definition:

Definition 3.2. Let \mathcal{X} be the ring of formal sums of the form $\sum_{\lambda \in \mathfrak{h}_q^*} f(\lambda)e^\lambda$ where $f : \mathfrak{h}_q^* \rightarrow \mathbb{Z}$ is any integer valued function whose support lies in a finite union of sets of the form $\nu - \mathbf{Q}^+$ with $\nu \in \mathfrak{h}_q^*$. The product in \mathcal{X} is the convolution product given by

$$\left(\sum_{\lambda \in \mathfrak{h}_q^*} f(\lambda)e^\lambda \right) \left(\sum_{\mu \in \mathfrak{h}_q^*} g(\mu)e^\mu \right) = \sum_{\lambda, \mu \in \mathfrak{h}_q^*} f(\lambda)g(\mu)e^{\lambda+\mu}.$$

It is clear that the right hand side is still in \mathcal{X} .

Definition 3.3. We introduce an element $p \in \mathcal{X}$ as

$$p = \prod_{\beta \in \Delta^+} \left(\sum_{m=0}^{\infty} e^{-m\beta} \right). \quad (12)$$

Lemma 3.4. [7, Proposition 2.68] *For each $\mu \in \mathfrak{h}_q^*$, the formal character of the Verma module $M(\mu)$ is the convolution product of e^μ and p :*

$$\text{ch}(M(\mu)) = e^\mu p. \quad (13)$$

By Lemma 2.12 we have

$$\text{ch}(M(\mu)) = \sum_{\lambda \in \mathfrak{h}_q^*} [M(\mu) : V(\lambda)] \text{ch}(V(\lambda)). \quad (14)$$

where $[M(\mu) : V(\mu)] = 1$ and $[M(\mu) : V(\lambda)] = 0$ unless $\lambda \leq \mu$ and λ is \hat{W} -linked to μ .

We can obtain the following well-known result on $\text{ch}(V(\mu))$ by inverting the matrix $[M(\mu) : V(\lambda)]$:

Lemma 3.5. *For each $\mu \in \mathfrak{h}_q^*$, the formal character of the simple highest weight module $V(\mu)$ can be expressed as*

$$\text{ch}(V(\mu)) = \sum_{\lambda \in \mathfrak{h}_q^*} m_{\lambda, \mu} \text{ch}(M(\lambda)) = \sum_{\lambda \in \mathfrak{h}_q^*} m_{\mu, \lambda} e^\lambda p \quad (15)$$

where $m_{\mu, \lambda}$ are integers such that $m_{\mu, \mu} = 1$ and $m_{\mu, \lambda} = 0$ unless $\lambda \leq \mu$ and λ is \hat{W} -linked to μ .

Remark 3.6. If $\mu \in \mathbf{P}^+$ the set of dominant integral weights, then [7, Proposition 4.4] gives a more precise formula than (15).

Corollary 3.7. *For each $M \in \mathcal{O}$, there exists a finite set $\{\mu_1, \dots, \mu_m\} \subset \mathfrak{h}_q^*$ such that*

$$\text{ch}(M) = \sum_{i=1}^m \sum_{\lambda \in \mathfrak{h}_q^*} [M : V(\mu_i)] m_{\mu_i, \lambda} e^\lambda p. \quad (16)$$

where $m_{\mu_i, \lambda}$ are integers such that $m_{\mu_i, \mu_i} = 1$ and $m_{\mu_i, \lambda} = 0$ unless $\lambda \leq \mu_i$ and λ is \hat{W} -linked to μ_i .

Remark 3.8. Since \hat{W} is a finite group, the sums on the right hand side of (15) and (16) are both finite.

3.2. Reduced rational expressions of formal characters of modules in \mathcal{O} .

Notice that we can write the formal character $p = \prod_{\beta \in \Delta^+} (\sum_{m=0}^{\infty} e^{-m\beta})$ as

$$p = \frac{1}{\prod_{\beta \in \Delta^+} (1 - e^{-\beta})} \quad (17)$$

so by Corollary 3.7 for each $M \in \mathcal{O}$, we can write its formal character as

$$\text{ch}(M) = \frac{\sum_{i=1}^m \sum_{\lambda \in \mathfrak{h}_q^*} [M : V(\mu_i)] m_{\mu_i, \lambda} e^\lambda}{\prod_{\beta \in \Delta^+} (1 - e^{-\beta})} \quad (18)$$

We want to simplify $\text{ch}(M)$ to obtain a reduced fraction, which needs some work because the ring \mathcal{X} is not a UFD.

Let \mathcal{S} be the ring of \mathbb{Z} -coefficient polynomials generated by $e^{-\alpha_i}$, $i = 1, \dots, N$, where $\{\alpha_1, \dots, \alpha_N\}$ is the set of simple roots. It is clear that $\prod_{\beta \in \Delta^+} (1 - e^{-\beta}) \in \mathcal{S}$ but

$$\sum_{i=1}^m \sum_{\lambda \in \mathfrak{h}_q^*} [M : V(\mu_i)] m_{\mu_i, \lambda} e^\lambda$$

is not necessarily contained in \mathcal{S} .

We have the following definition.

Definition 3.9. Let \mathcal{X} be as in Definition 3.2. We say that $a \in \mathcal{X}$ can be written in reduced rational form if there exists a subset $T_a \subset \Delta^+$ and a finite collection $\{\mu_1, \dots, \mu_m\} \subset \mathfrak{h}_q^*$ such that

$$a = \frac{\sum_{i=1}^m e^{\mu_i} f_i}{\prod_{\beta \in T_a} (1 - e^{-\beta})^{n_\beta}} \quad (19)$$

where

- (1) $\mu_i - \mu_j$ is not in the root lattice \mathbf{Q} for each $i \neq j$;
- (2) f_i is a polynomial in \mathcal{S} with nonzero constant term for each i ;
- (3) n_β is a positive integer for each $\beta \in T_a$;
- (4) The numerator and denominator of (19) are coprime. More precisely, for each $\beta \in T_a$, there exists an f_i in the numerator such that $1 - e^{-\beta}$ is not a factor of f_i .

We call the set T_a the denominator roots of a .

Lemma 3.10. *For any $a \in \mathcal{X}$, the reduced rational form of a is unique if exists.*

Proof. It is clear from the definition and the fact that the polynomial ring \mathcal{S} is a UFD. \square

Not all elements in \mathcal{X} can be written in reduced rational form. Nevertheless for formal characters of modules in \mathcal{O} we have the following result.

Lemma 3.11. *Let $M \in \mathcal{O}$ be a nonzero module. Then the set $T_{ch(M)}$ of denominator roots exists and $ch(M)$ can be written uniquely in reduced rational form. Moreover we have*

$$ch(M) = \frac{\sum_{i=1}^m e^{\mu_i} f_i}{\prod_{\beta \in T_{ch(M)}} (1 - e^{-\beta})} \quad (20)$$

and Property 1, 2, 3, 4 in Definition 3.9 are satisfied with all $n_\beta = 1$. In the sequel we will denote $T_{ch(M)}$ by T_M and we will call T_M the set of denominator roots of M .

Proof. It is a direct consequence of Corollary 3.7, Lemma 3.10, and (18). \square

Example 3.12. By Lemma 3.11, for any $\alpha \in \Delta^+$ the formal power series

$$\frac{1}{(1 - e^{-\alpha})^2}$$

cannot be the formal character of any module in \mathcal{O} although $\frac{1}{(1 - e^{-\alpha})^2} \in \mathcal{X}$. Intuitively this is because the multiplicity of $e^{-n\alpha}$ grows too fast as n grows and here we have a precise criterion of this fact.

Corollary 3.13. *A nonzero module $M \in \mathcal{O}$ is finite dimensional if and only if its reduced rational form has denominator = 1, i.e. $T_M = \emptyset$.*

Proof. It is implied by the uniqueness of reduced rational form. \square

Example 3.14. For a Verma module $M(\mu)$, the reduced rational form of its formal character is

$$ch(M(\mu)) = \frac{e^\mu}{\prod_{\beta \in \Delta^+} (1 - e^{-\beta})}$$

hence $T_{M(\mu)} = \Delta^+$.

Example 3.15. For a simple highest weight module $V(\mu)$ we have the reduced rational form

$$ch(V(\mu)) = \frac{e^\mu f_{V(\mu)}}{\prod_{\beta \in T_{V(\mu)}} (1 - e^{-\beta})}$$

for some $f_{V(\mu)} \in \mathcal{S}$. In particular if $\mu \in \mathbf{P}_q^+$ then $T_{V(\mu)} = \emptyset$ and $ch(V(\mu))$ is given explicitly by the Weyl character formula. Actually [7, Proposition 4.4] only covers the $\mu \in \mathbf{P}^+$ case but the general case can be easily obtained by [7, Lemma 2.41].

Remark 3.16. For any $M \in \mathcal{O}$ it is clear that the denominator roots $T_M \subset \bigcup T_{V(\mu)}$ where the union is for all $V(\mu)$ such that $[M : V(\mu)] \neq 0$. The author does not know whether we have $T_M = \bigcup T_{V(\mu)}$ for all $V(\mu)$ such that $[M : V(\mu)] \neq 0$. Nevertheless we do not need this result in this paper.

4. Tensor closed objects in category \mathcal{O}

The category $U_q(\mathfrak{g})\text{-Mod}$ has a tensor product since $U_q(\mathfrak{g})$ is a Hopf algebra. Moreover we have the following lemma.

Lemma 4.1. *$U_q(\mathfrak{g})\text{-Mod}$ is a braided category. In particular for any left $U_q(\mathfrak{g})$ -modules V and W we have a $U_q(\mathfrak{g})$ -module isomorphism $V \otimes W \cong W \otimes V$.*

Proof. It is clear since $U_q(\mathfrak{g})$ is quasitriangular in the sense of [7, Theorem 2.108]. □

However category \mathcal{O} is not closed under tensor product.

Definition 4.2. We call a module $M \in \mathcal{O}$ *tensor-closed* if for any $N \in \mathcal{O}$, the tensor product $M \otimes N \cong N \otimes M$ is still in \mathcal{O} .

The following result is well-known.

Lemma 4.3. *Any finite dimensional module $V \in \mathcal{O}$ is tensor-closed.*

Proof. The proof is the same as that of [3, Theorem 1.1 (d)]. □

In this section we prove the following result.

Theorem 4.4. *A module $V \in \mathcal{O}$ is tensor-closed if and only if it is finite dimensional.*

To give the proof more rigorously we introduce the following auxiliary category.

Definition 4.5. A left module M over $U_q(\mathfrak{g})$ is said to belong to the category $\tilde{\mathcal{O}}$ if

- a) M is a weight module and all weight spaces of M are finite dimensional.
- b) There exist finitely many weights $\nu_1, \dots, \nu_l \in \mathfrak{h}_q^*$ such that

$$\text{supp}M \subset \bigcup_{i=1}^l (\nu_i - \mathbf{Q}^+),$$

where $\text{supp}M = \{\lambda \in \mathfrak{h}_q^* \mid M_\lambda \neq 0\}$.

Morphisms in category $\tilde{\mathcal{O}}$ are all $U_q(\mathfrak{g})$ -linear maps.

It is clear that \mathcal{O} is a full subcategory of $\tilde{\mathcal{O}}$. $\tilde{\mathcal{O}}$ is closed under tensor product and modules in $\tilde{\mathcal{O}}$ have formal characters in the ring \mathcal{X} in Definition 3.2. Moreover for $M, N \in \tilde{\mathcal{O}}$ we have

$$\text{ch}(M \otimes N) = \text{ch}(M)\text{ch}(N). \tag{21}$$

Lemma 4.6. *For two simple highest weight modules $V(\mu)$ and $V(\lambda)$, if $V(\mu) \otimes V(\lambda) \in \mathcal{O}$, then*

$$T_{V(\mu)} \cap T_{V(\lambda)} = \emptyset.$$

In particular for any infinite dimensional simple highest weight module $V(\mu)$ we have $V(\mu) \otimes V(\mu) \notin \mathcal{O}$.

Proof. For each simple highest weight module $V(\mu)$ we have the reduced rational form

$$\text{ch}(V(\mu)) = \frac{e^\mu f_{V(\mu)}}{\prod_{\beta \in T_{V(\mu)}} (1 - e^{-\beta})}$$

where $f_{V(\mu)}$ is in the polynomial ring \mathcal{S} such that $1 - e^{-\beta}$ is not a factor of $f_{V(\mu)}$ for each $\beta \in T_{V(\mu)}$. Therefore

$$\text{ch}(V(\mu) \otimes V(\lambda)) = \text{ch}(V(\mu))\text{ch}(V(\lambda)) = \frac{e^{\mu+\lambda} f_{V(\mu)} f_{V(\lambda)}}{\prod_{\beta \in T_{V(\mu)}} (1 - e^{-\beta}) \prod_{\gamma \in T_{V(\lambda)}} (1 - e^{-\gamma})}.$$

Assume $T_{V(\mu)} \cap T_{V(\lambda)} \neq \emptyset$ and let $\beta \in T_{V(\mu)} \cap T_{V(\lambda)}$, then $(1 - e^{-\beta})^2$ appears in the denominator and $1 - e^{-\beta}$ is not a factor of $f_{V(\mu)}$ nor $f_{V(\lambda)}$. Therefore $(1 - e^{-\beta})^2$ appears in the denominator of the reduced rational form of $\text{ch}(V(\mu) \otimes V(\lambda))$. On the other hand by Lemma 3.11, if $V(\mu) \otimes V(\lambda)$ is in \mathcal{O} then the reduced rational form of $\text{ch}(V(\mu) \otimes V(\lambda))$ cannot have squares in the denominator. Hence $V(\mu) \otimes V(\lambda) \in \mathcal{O}$ implies $T_{V(\mu)} \cap T_{V(\lambda)} = \emptyset$.

For infinite dimensional $V(\mu)$, we know $T_{V(\mu)} \neq \emptyset$ by Corollary 3.13, so $V(\mu) \otimes V(\mu) \notin \mathcal{O}$. \square

Lemma 4.7. *For any infinite dimensional simple highest weight module $V(\mu)$ and any Verma module $M(\lambda)$ we have $V(\mu) \otimes M(\lambda) \cong M(\lambda) \otimes V(\mu) \notin \mathcal{O}$.*

Proof. Similar to the proof of Lemma 4.6, we can show that $\text{ch}(V(\mu) \otimes M(\lambda))$ has squares in the denominator of its reduced rational form. \square

Remark 4.8. In general the product of two reduced rational forms needs not to be a reduced rational form since \mathcal{X} is not a UFD. For example for any $\alpha \in \Delta^+$ let

$$a = \frac{1 + e^{-\alpha/2}}{1 - e^{-\alpha}}, \quad b = \frac{1 - e^{-\alpha/2}}{1 - e^{-\alpha}}.$$

It is clear that both a and b are reduced rational forms but

$$ab = \frac{1 - e^{-\alpha}}{(1 - e^{-\alpha})^2}$$

is not reduced. The author does not know if we restrict to formal characters of modules in \mathcal{O} , whether or not the product of reduced rational forms must be a reduced rational form. Nevertheless we do not need this result in this paper.

Proof of Theorem 4.4. Let $M \in \mathcal{O}$ be infinite dimensional and we want to show that M is not tensor-closed. Actually by Proposition 2.7 there exists an infinite dimensional $V(\mu)$ in the Jordan-Hölder series of M . By Lemma 4.6, $V(\mu) \otimes V(\mu) \notin \mathcal{O}$. Since \mathcal{O} is closed under subquotients, $M \otimes V(\mu) \notin \mathcal{O}$ too. So M is not tensor-closed. \square

Remark 4.9. There exist two infinite dimensional modules with tensor product still in \mathcal{O} . Victor Ostrik gave the following example: Let $\mathfrak{g} = sl(2) \oplus sl(2)$. Let V be a Verma module for $U_q(sl(2))$ (with arbitrary highest weight). Using two projections $\mathfrak{g} \rightarrow sl(2)$ we can consider V as a $U_q(\mathfrak{g})$ -module in two different ways. Let us call the resulting $U_q(\mathfrak{g})$ -modules V_1 and V_2 . Then both V_1 and V_2 are in the category \mathcal{O} for $U_q(\mathfrak{g})$, and $V_1 \otimes V_2$ is a Verma module of $U_q(\mathfrak{g})$.

It is an interesting question if \mathfrak{g} is simple and $M, N \in \mathcal{O}$ are both infinite dimensional, is it always true that $M \otimes N \notin \mathcal{O}$. See [8] for a discussion in the cases that \mathfrak{g} is simple of type ADE.

Remark 4.10. All arguments and proofs in this paper work for the unquantized case as well. In particular we can prove Theorem 4.4 for the unquantized case using the method in this paper.

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