Results in Nonlinear Analysis 4 (2021) No. 1, 21–32.
https://doi.org/10.53006/rna.852462
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

The rise and fall of MC-spaces

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Abstract

In 1994, Llinares introduced mc-spaces and began to study KKM theoretic results on them. Since 1998, he became an L-space theorist and repeated to claim that his mc-spaces generalize G-convex spaces without any justifications. Later he insisted that his mc-spaces are the same as L-spaces. Hence his study on mc-spaces is useless now as the L-space case shown by our previous works. The present article is a continuation of our previous works on L-spaces and concerns with the rise and fall of mc-spaces. This paper will be an important record for the history of the KKM theory.

Keywords: KKM theorem, Fan's 1961 KKM lemma, G-convex space, mc-space, L-space, abstract convex space, (partial) KKM space.
2010 MSC: 47H04, 47H10, 49J27, 49J35, 49J53, 52A01, 54H25, 55M20, 90C47, 91A13, 91B50.

1. Introduction

Since we introduced generalized convex spaces or G-convex spaces in the KKM theory initiated by ourselves, a large number of their imitations, modifications, and fake generalizations were appeared. In order to destroy or improve them, we introduced abstract convex spaces and (partial) KKM spaces. See [22], [23], [26]. One of them is the so-called L-spaces.

The present article is a continuation of our previous works [34], [35] on the rise and fall of the L-spaces due to Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [1] in 1998. Since then there have appeared some people concentrating mainly on L-spaces and they can be adequately called L-space theorists.

In [34], we showed that our KKM theory on abstract convex spaces improved typical results on L-spaces. Main topics there were related to extensions of the Himmelberg fixed point theorem. Since such studies were beyond of L-spaces, we cordially claimed that it was the proper time to quit the useless study on L-spaces and their variants FC-spaces.

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Received October 25, 2020, Accepted: December 20, 2020, Online: January 02, 2021.

In [35], various types of coercing families initiated by Ben-El-Mechaiekh, Chebbi, and Florenzano [2] are unified by a single coercivity condition. We showed that better forms of results using several coercing families can be deduced from a general KKM theorem on abstract convex spaces in our previous works. Consequently, all of known KKM theoretic results on L-spaces related coercing families were extended to corresponding better forms on abstract convex spaces.

On the other hand, in 1994, Llinares [8] introduced mc-spaces and began to study KKM theoretic results on them. In 1998, he became an L-space theorist and began to claim that his mc-spaces generalize G-convex spaces without any justifications. Later he insisted that his mc-spaces are same to L-spaces. Hence his study on mc-spaces are useless now as the L-spaces case as shown by our previous works.

The present article is a continuation of our previous [34], [35] and concerns with the rise and fall of mc-spaces. This will be an important record for the history of KKM theory.

This article is organized as follows: In Section 2, we recall origins of generalized convex (G-convex) spaces, mc-spaces, and L-spaces. Section 3 devotes why G-convex spaces were extended to the so called ϕ_A-spaces and the abstract convex spaces with their subclasses called (partial) KKM spaces. In Section 4, we give theorems on generalized KKM maps showing how to improve or destroy some typical results on mc-spaces. In Section 5, we extend a general result in [38] on the non-emptiness of choice functions. Section 6 devotes the history of mc-spaces in order to clarify the related authors' incorrect claims for reader's convenience. Finally, in Section 7, we clarify some other authors' misconceptions related to our subject.

2. G-convex spaces, mc-spaces and L-spaces

The KKM theory is first named by ourselves in 1992 as the study of applications of extensions or equivalents of the KKM theorem due to Knaster-Kuratowski-Mazurkiewicz in 1929. The KKM theory was first devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde, to H-spaces by Horvath and others, to G-convex spaces mainly by the present author. Since then a large number of works appeared by many authors on G-convex spaces and their imitations, modifications, or fake generalizations. In order to destroy them, in 2006-10, we proposed new concepts of abstract convex spaces and the partial KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory. Now the KKM theory becomes the study on abstract convex spaces including partial KKM spaces and we obtained a large number of new results in such frame; see the references at the end of this article.

The generalized convex spaces first appeared in the following:

[PK] S. Park and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. SNU **18** (1993) 1–21.

Here Coll. stands for College and SNU for Seoul National University, Seoul, Korea. This journal had very restricted distribution and short life. There we defined:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X and a nonempty map $\Gamma : \langle D\rangle \to 2^X \setminus \{\emptyset\}$ such that

(1) for each $A,\ B \in \langle D\rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and

(2) for each $A \in \langle D\rangle$ with the cardinality $|A| = n+1$, there exist a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J denotes the face of Δ_n corresponding to $J \in \langle A\rangle$. [For example, $\langle D\rangle$ denotes the class of nonempty finite subsets of D.] The monotonicity condition (1) was removed in 1998 [18]. Since then hundreds of articles appeared on G-convex spaces.

From the introduction of [PK] (without citations):

> Recently the first author introduced certain general classes of upper semicontinuous multimaps defined on convex spaces which were shown to be adequate to establish theories on fixed points, coincidence points, KKM maps, variational inequalities, best approximations, and many others.

Our admissible classes of multimaps (maps) include composites of important maps which appear in nonlinear analysis or algebraic topology. Examples of such maps are continuous functions, Kakutani maps, acyclic maps, Fan-Browder type maps, admissible maps in the sense of Górniewicz, permissible maps of Dzedzej, approachable maps, and many others.

Later we found that, in certain cases, the convex spaces can be replaced by new classes of more general spaces. Actually, our new concept of generalized convex spaces is a generalization of the usual convexity in a topological vector space, Michael's convex structure, Pasicki's S-contractible spaces, Komiya's convex spaces, Lassonde's convex spaces, Horvath's pseudoconvex spaces and c-structure or H-spaces, Bielawski's simplicial convexity, Joó's pseudoconvexity, and many others. Those general convexities were developed in connection mainly with the fixed point theory and the KKM theory.

In this paper we investigate fundamental properties of many examples of such classes of multimaps and generalized convex spaces.

In his Ph.D. Thesis [8] in 1994, the fourth L-space theorist Llinares introduced mc-spaces as follows:

Definition 1. *A topological space X is an mc-space (or has an mc-structure) if for any non-empty finite subset of X, $A \subset X$, there exists an ordering on it, namely $A = \{a_0, a_1, \ldots, a_n\}$, a family of elements $\{b_0, b_1, \ldots, b_n\} \subset X$, and a family of functions $P_i^A : X \times [0,1] \to X$, such that for $i = 0, 1, \ldots, n$,*

1. *$P_i^A(x, 0) = x, P_i^A(x, 1) = b_i$, for all $x \in X$.*
2. *The following function $G_A : [0,1]^n \to X$ given by*

$$G_A(t_0, t_1, \ldots, t_{n-1}) = P_0^A(\ldots(P_{n-1}^A(P_n^A(b_n, 1), t_{n-1}), \ldots), t_0),$$

is a continuous function.

After that he repeatedly stated that his mc-spaces generalize our G-convex spaces in [10], [13], [14], [38] without any justification and quoted our [PK].

Recall that L-spaces are originated from our generalized convex (G-convex) spaces. In 1998, apparently motivated by [PK], Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [1] stated:

Definition 2. *An L-structure on E is given by a nonempty set-valued map $\Gamma : \langle E \rangle \to E$ verifying:*

(*) For every $A \in \langle E \rangle$, say $A = \{x_0, x_1, \ldots, x_n\}$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, \ldots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j,\ j \in J\})$.

The pair (E, Γ) is then called an L-space and $X \subset E$ is said to be L-convex if $\forall A \in \langle X \rangle,\ \Gamma(A) \subset X$.

In particular, if Γ, as in Definition 2, verifies the additional condition

(**) For each $A, B \in \langle E \rangle,\ A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$,

then the pair (E, Γ) is what is called by Park and Kim [PK], a *G-convex space.*

This statement is incorrect. Our G-convex space is a triple $(X, D; \Gamma)$ and L-space is a pair (E, Γ). This statement leads many naive peoples to think L-spaces generalize G-convex spaces without checking [PK], [18] and hundreds of later works on G-convex spaces by ourselves and many followers.

Note that the L-spaces are motivated by [PK]: In fact, [1] states

"As noted by Park and Kim [PK], it follows from Theorem 1, Section 1 of Horvath [5] that if Γ defines an H-structure, then (X, Γ) is an L-space."

In [1], the original L-space theorists could give only B'- and B-simplicial convexities and H-spaces as examples of L-convexity comparing to the large number of examples of G-convex spaces in [PK]. Moreover they could fail to give a proper example of L-spaces not satisfying the so-called monotonicity. See also [25].

Recall that any (partial) KKM spaces including L-spaces have a large number of properties as shown in several works of ourselves, e.g. [26], [30]. But the L-space theorists could find only few of them for their L-spaces. Moreover, in his [14], [38], Llinares agreed that his mc-spaces are equivalent to L-spaces. Therefore, as we claimed in [34], mc-spaces as well as L-spaces should be destroyed.

3. Abstract convex spaces and partial KKM spaces

Since the appearance of G-convex spaces, many authors have tried to imitate, modify, or generalize them and published a large number of papers. In fact, there have appeared authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplices. Such examples are L-spaces, spaces having property (H), FC-spaces, convexity structures satisfying H-condition, pseudo-H-spaces, another L-spaces, M-spaces, GFC-spaces, simplicial spaces, FWC-spaces, and others; see [24, 25, 27, 28, 32]. Some authors also tried to generalize the KKM principle for their own settings. Some of them tried to rewrite certain results on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples. We found that most of such spaces can be subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$; see [24].

Definition 3. *A space having a family $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

In 2006 [19], we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces or ϕ_A-spaces and adequate to establish the KKM theory; see [20], [21].

Recall the following in [22, 23, 26] where we established the foundations of abstract convex space theory:

Definition 4. *An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.*

For any $D' \subset D$, the Γ-convex hull of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

Definition 5. . *Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap Z$ satisfies*

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a KKM map.

Definition 6. *The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The KKM principle is the statement that the same property also holds for any open-valued KKM map.*

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

In our works [22], [23], [26], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Recall the following well-known diagram for triples $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}$$

$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space}$$

$$\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}$$

Horvath spaces are newly defined one including c-spaces and H-spaces. Note that L-spaces and mc-spaces are pairs particular to G-convex spaces.

4. Generalized KKM maps

One of the topics in the KKM theory is related to generalized KKM maps initiated by Kassay-Kolumbán in 1990 and Chang-Zhang in 1991. Since then many authors studied generalized KKM maps on various types of spaces and applied them to extend or refine well-known previous results. In fact, it has been followed by Chang-Ma in 1993, Yuan in 1995, Cheng in 1997, Tan in 1997, Lin-Chang in 1998, Lee-Cho-Yuan in 1999, Kirk-Sims-Yuan in 2000 for various classes of abstract convex spaces; see [33]. All of those authors applied their results on KKM type theorems and others to extend or refine well-known previous results in the KKM theory; for example, variational or quasi-variational inequalities, fixed point theorems, the Ky Fan type minimax inequalities, the von Neumann type minimax or saddle point theorems, Nash equilibrium problems, and others.

More recently, we gave a unified account for generalized KKM maps in abstract convex spaces in [36]. There our results include the KKM type theorems and characterizations of generalized KKM maps, and can be applied to Hadamard manifolds, hyperbolic metric spaces, Riemanian manifolds, CAT(0) spaces, etc. due to other authors.

In this section, we give generalizations of some results on generalized KKM maps due to Llinares et al. [14], [38] based on our previous results including the KKM type theorems and characterizations of generalized KKM maps.

We begin with the following definition in [33]:

Definition 7. *Let $(X, D; \Gamma)$ be an abstract convex space and Y be a nonempty set such that, for each $A \in \langle Y \rangle$, there exists a function $\sigma_A : A \to D$. Then a new abstract convex space $(X, A; \Lambda_A)$ induced by Γ and A is defined by the following*

$$\Lambda_A(J) := \Gamma(\sigma_A(J)) \text{ for each } J \subset A.$$

Moreover, a multimap $T : Y \multimap X$ (called a generalized KKM map) reduces to a KKM map on $(X, A; \Lambda_A)$ for each $A \in \langle Y \rangle$ satisfying $\Lambda_A(J) \subset T(J)$ for each $J \subset A$.

Definition 7 was given first for G-convex spaces as a unification of previously given ones by several authors, and in [30] for the present form.

Independently to Definition 7, Sanchess et al. [38] in 2003 considered below an extension of the concept of the generalized KKM map:

Definition 8. ([38]) *Let X and Y be topological spaces such that X has an L-structure defined by $\Psi : \langle X \rangle \multimap X$ and by $f^B : \Delta_n \to \Psi(B)$ for each $B \in \langle X \rangle$. A correspondence $\Gamma : Y \multimap X$ is said to be a generalized KKM-correspondence, if for all $\{y_0, y_1, \ldots, y_n\} \in \langle Y \rangle$, there exists a subset $B = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$ such that for all $J \subseteq \{0, 1, \ldots, n\}$, it is satisfied that*

$$f^B(\Delta_J) \subseteq \bigcup_{j \in J} \Gamma(y_j).$$

Note that this definition is very particular to Definition 7.

We recall the following [33, Theorem C] with proof for the completeness:

Theorem 9. *Let $(X, D; \Gamma)$ be a partial KKM space [resp. KKM space], Y a nonempty set, and $T : Y \multimap X$ a map with closed [resp. open] values.*

(i) If T is a generalized KKM map, then the family of its values has the finite intersection property.

(ii) The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.

Proof. (i) Let $T : A \multimap X$ be a KKM map having closed [resp. open] values on $(X, A; \Lambda_A)$, that is,

$$\Lambda_A(J) \subset T(J) \quad \forall J \subset A.$$

Let $A = \{y_i\}_{i=1}^n$, $z_i = \sigma_A(y_i) \in D$, and $G(z_i) = T(y_i)$ for each $i = 1, \ldots, n$. Then

$$\Gamma(\sigma_A(J)) \subset G(\sigma_A(J)) \quad \forall J \subset A.$$

Hence $G : \sigma_A(A) \multimap X$ is a KKM map with closed [resp. open] values on $(X, A; \Gamma|_{\langle \sigma_A(A) \rangle})$ which is a (partial) KKM space. Hence $\{G(z_i)\}_{i=1}^n = \{T(y_i)\}_{i=1}^n$ has the finite intersection property.

(ii) Suppose that $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$. For any $A \in \langle Y \rangle$, by assumption, we have an $x^* \in \bigcap_{y \in A} T(y) \neq \emptyset$. Define a function $\sigma_A : A \to D = X$ by $\sigma_A(y) = x^*$ for all $y \in A$. Then for any nonempty subset J of A, we have

$$\Gamma_{\sigma_A(J)} = \Gamma_{\{x^*\}} = \{x^*\} \subset \bigcap_{y \in A} T(y) \subset T(J).$$

Therefore, T is a generalized KKM map. □ □

Consider the following related four conditions for a map $G : D \multimap X$ with a topological space X:

(a) $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z \in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$ (G is *intersectionally closed-valued* [16]).

(c) $\bigcap_{z \in D} \overline{G(z)} = \bigcap_{z \in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Luc et al. [16] noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d).

Theorem 10. *Let $(X, D; \Gamma)$ be an abstract convex space, Y a nonempty set, and $T : Y \multimap X$ a map with intersectionally closed values such that there exists $y^* \in Y$ with $\overline{T(y^*)}$ compact.*

(1) If $(X, D; \Gamma)$ is a partial KKM space and T is a generalized KKM map, then we have $\bigcap_{y \in Y} T(y) \neq \emptyset$.

(2) Conversely, if $\bigcap_{y \in Y} T(y) \neq \emptyset$, then X can be made into a G-convex space and T can be a generalized KKM map.

Proof. (1) Since $T : Y \multimap X$ a map with intersectionally closed values, we have

$$\bigcap_{y \in Y} \overline{T(y)} = \overline{\bigcap_{y \in Y} T(y)}.$$

So, it is sufficient to prove that $\bigcap_{y \in Y} \overline{T}(y) := \bigcap_{y \in Y} \overline{T(y)}$ is nonempty. To do so, we note that the family $\{\overline{T}(y)\}_{y \in Y}$ satisfies the finite intersection property by Theorem 4.3.(i). But this implies that family $\{\overline{T}(y) \cap \overline{T}(y^*)\}_{y \in Y}$, which is a family of closed subsets of the compact subset $\overline{T}(y^*)$, also satisfies this property. So, by compactness, we can ensure that $\bigcap_{y \in Y} \{\overline{T}(y) \cap \overline{T}(y^*)\} \neq \emptyset$ and since we can rewrite $\bigcap_{y \in Y} \{\overline{T}(y) \cap \overline{T}(y^*)\} = \bigcap_{y \in Y} \{\overline{T}(y)\}$, we obtain the required conclusion.

(2) Choose a point $x^* \in \bigcap_{y \in Y} T(y) \neq \emptyset$. Define a new $\Gamma : \langle X \rangle \multimap X$ by $\Gamma(N) = \{x^*\}$ for each $N \in \langle X \rangle$ with $|N| = n + 1$, and $\phi_N : \Delta_n \to \Gamma(N)$ such that $\phi_N(\Delta_n) = \{x^*\}$. Then $(X; \Gamma)$ is a G-convex space and hence a partial KKM space.

Now we show that $T : Y \multimap X$ is a generalized KKM map. Given $A \in \langle Y \rangle$, we have $\sigma_A : A \to X$ such that $\sigma_A(a) := x^*$ for all $a \in A$. Then

$$\Lambda_A(J) := \Gamma(\sigma_A(J)) = \Gamma(x^*) = \{x^*\} \textit{ for all } J \subset A.$$

Therefore T is a generalized KKM map. □ □

The following is [14, Theorem 6] in 2002:

Corollary 11. ([14]) *If X is a topological space and $\phi : X \multimap X$ is a nonempty valued correspondence with closed values, and there exists $x_0 \in X$ such that $\phi(x_0)$ is compact, then the following statements are equivalent:*

(i) $\bigcap_{x\in X} \phi(x) \neq \emptyset$;

(ii) X is an mc-space such that, for all finite subset $A = \{x_0, \dots, x_n\}$ of X, it is satisfied that for any family $\{i_0, \dots .i_k\} \subseteq \{0, 1, \dots, n\}$ of indices, then $G_{A|\{x_{i_0},\dots,x_{i_k}\}}([0, 1]^k) \subseteq \bigcup_{j=1}^{k} \phi(x_{i_j})$.

All subsequent results in [14] follow this corollary.

The following is [38, Theorem 1] in 2003:

Corollary 12. ([38]) *Let X and Y be topological spaces and $\Gamma : Y \multimap X$ a transfer closed-valued correspondence on Y such that there exists $y^* \in Y$ with $\mathrm{cl}[\Gamma(y^*)]$ compact. Then, the following conditions are equivalent:*

i) There exists an L-structure on X such that Γ is a generalized KKM correspondence.

ii) $\bigcap_{y\in Y} \Gamma(y) \neq \emptyset$.

All results in [38] are based on this Corollary. Hence they can be improved by adopting our Theorem 10.

5. Binary and non-binary choice functions

In this section we extend a general result in [38] on the non-emptiness of choice functions, and follow the notations of [38].

Throughout this section, X denotes a topological space and $\mathfrak{D}$ a family of non-empty subsets of X that represents the different feasible sets presented for choice. Given $A \in \mathfrak{D}$, $\mathfrak{D}_A$ denotes a family of non-empty subsets of A. A *choice function* is a correspondence $C : \mathfrak{D} \to X$ such that $C(A) \subseteq A$ for all $A \in \mathfrak{D}$.

From our Theorem 10, we have the following:

Theorem 13. *Let X be a topological space and $C : \mathfrak{D} \to X$ a choice function. If for $A \in \mathfrak{D}$ there exists a family $\mathfrak{D}_A$ and a multimap $\Omega_A : \mathfrak{D}_A \multimap A$ satisfying that:*

(i) $\bigcap_{D\in\mathfrak{D}_A} \Omega_A(D) \subseteq C(A)$;

(ii) Ω_A is intersectionally closed-valued;

(iii) A is a partial KKM space such that Ω_A is a generalized KKM map;

(iv) there exists $D^* \in \mathfrak{D}_A$ with $cl_A[\Omega_A(D^*)]$ compact on A.

Then $C(A) \neq \emptyset$.

Proof. It is sufficient to apply Theorem 10(1) to any $A \in \mathfrak{D}$ satisfying all of the hypothesis of Theorem 13, to ensure that

$$\bigcap_{D\in\mathfrak{D}_A} \Omega_A(D) \neq \emptyset.$$

Then, by applying (i), the conclusion is obtained. □

The following is the main result of [38, Theorem 2]:

Corollary 14. ([38]) *Let X be a topological space and $C : \mathfrak{D} \to X$ a choice function. If for $A \in \mathfrak{D}$ there exists a family $\mathfrak{D}_A$ and a correspondence $\Omega_A : \mathfrak{D}_A \to A$ satisfying that:*

(i) $\bigcap_{D\in\mathfrak{D}_A} \Omega_A(D) \subseteq C(A)$;

(ii) Ω_A is transfer closed-valued;

(iii) A is an L-space such that Ω_A is a generalized KKM-correspondence;

(iv) there exists $D^* \in \mathfrak{D}_A$ with $cl_A[\Omega_A(D^*)]$ compact on A.

Then $C(A) \neq \emptyset$.

In order to show in [38] that this result generalizes some results on the non-emptiness of usual choice functions (maximal elements, top cycle, uncovered set, non-binary choice functions,...), the authors consider the family $D_A = \langle A \rangle$ for every $A \in \mathfrak{D}$ and, given the choice function $C : \mathfrak{D} \to X$, they define correspondence $\Omega_A : \langle A \rangle \to A$ as follows:

$$\Omega_A(T) = \{a \in A \mid a \in C(T \cup \{a\})\} \text{ for all } T \in \langle A \rangle.$$

In order to present the results in a clear way, the authors analyze the case of binary and that of non-binary choice functions in two different subsections in [38].

6. The rise and fall of mc-spaces

From 1994 Llinarez began to study the so-called mc-spaces and, in 1998, became the fourth member of the original L-space theorists in [1]. In this section, we follow the history of mc-spaces from their birth to fall.

[I] In his 1994 Ph.D. Thesis [8], Llinarezs began to study the convexity in linear topological spaces and several extensions given by other authors. Then he introduced the K-convex structure with several examples, and stated that the contractibility condition and the condition of having a K-convex continuous structure are equivalent. He also defined mc-spaces as extensions of K-convex continuous structures. He noted that c-spaces due to Horvath are mc-spaces, and that mc-spaces have K-convex structure. And then some fixed point theorems, KKM type theorems, and applications of fixed point point theorems are added as routine in the KKM theory. Note that plenty of previously obtained results are introduced in this thesis.

[II] In 1995 [9], from his K-convex structure replacing the linear segments with a family of previously fixed paths joining up each two points, the author introduced a family of sets which generalizes the usual convex sets. In this context the author extends Sonnenschein's theorem on the existence of maximal elements and Browder's fixed point theorem.

[III] In 1998 [10], Llinares introduced mc-spaces that generalizes the notion of usual convexity. In Abstract:

> "It is presented as a powerful tool that allows many problems that have only been analyzed (previously) under convexity conditions to be solved."

[IV] Abstract of [11] in 1998:

> We analyze the existence of equilibrium in generalized games in a framework without any linear structure (where the usual convexity notion can not be defined) by using an abstract convexity structure called *mc*-spaces. In particular we replace the convexity condition on the strategy spaces and the images of preference and constraint correspondences by the notion of mc-set (which generalizes the notion of convex set). Among others, our results generalize those of Borglin and Keiding, Shafer and Sonnenschein, Border and Tulcea.

Comments: In Introduction, the author stated that "**mc-spaces** (see Llinares [10]) which generalizes usual convexity as well as other abstract convexity structures [as simplicial convexity (Bielawski), c-spaces (Horvath), or G-convex spaces (Park and Kim [PK])]."

His statement on G-convex spaces in [PK] is definitely false without any justification. Moreover, his false statement appears again p.3 with "see Llinares [8]." Note that [8] is his Ph.D. Thesis in 1994. However, nothing can be found there about G-convex spaces.

[V] In 1998 [12], LLinares stated that

> Although there are more abstract convexities (in connection to the field of fixed point theory) than the ones we are going to present (for instance, Michael's convex structure, Komiya convex spaces, etc.), most of them are particular cases of c-spaces or simplicial convexity (see Bielawski, or Park and Kim [PK]) and we will only focus on those that are more intuitive.

Furthermore it is immediate that the notion of G-convex spaces used by Park and Kim [PK], is a particular case of L-spaces since they require moreover (of definition ??) a monotone condition on the set-valued map Γ.

COMMENTS: This last part is exactly appeared with ?? in Page 7 of [12]. Here again the author repeats false statement and gives examples of L-spaces which are already given in [PK] as G-convex spaces.

[VI] ABSTRACT of [13] in 2000:

The aim of this paper is to prove the existence of equilibrium for generalized games or abstract economies in contexts where the convexity conditions on strategy spaces and preference correspondences are relaxed and an arbitrary number of agents is considered. The results are based on a fixed-point theorem in which the convexity condition on sets and images of correspondences is replaced by a general notion of abstract convexity, called mc-spaces, generalizing the notions of simplicial convexity, H-spaces, and G-convex spaces.

COMMENTS: As in the preceding paper, the author stated that

"mc-spaces (see Llinares, Ref. 6), generalizing the usual convexity structure as well as other abstract convexity structures that already exist in the literature: simplicial convexity (Ref. 7), H-spaces (Ref. 5), G-convex spaces (Ref. 8), and so on"

in Page 150, and

"Other abstract convexity structures that are generalized by the notion of mc-structure are simplicial convexity, H-spaces, and G-convex spaces (see Ref. 9)"

in Page 152. Moreover, the author stated in Page 159 that

"Furthermore, since mc-spaces generalize the notions of simplicial convexity, H-spaces, and G-convex spaces (among others), the previous results generalize also the corresponding ones in these structures."

All of these statements on G-convex are groundless false ones. Further, the author listed in References:

8. PARK, S., and KIM, H., Admissible Classes of Multifunctions on Generalized Convex Spaces, Proceedings of the Colloquium on Natural Sciences, Seoul National University, Vol. 18, pp.1–21, 1993.

9. LLINARES, J. V., Abstract Convexity, Some Relations, and Applications, CEPREMAP Report 98-03, 1998.

Note that "Colloquium on" should be "College of" as in [PK]. This is the evidence that the L-space theorist like others did not read or check [PK]. It is against ordinary scholastic behavior and academic justice.

[VII] Later in 2002 [14], Llinares recognized the following:

PROPOSITION 7. ([14]) *If X is an L-space, then X is an mc-space such that L-convex sets are mc-sets.*

The author repeated again the following false statement:

Furthermore, it is obvious that the notion of G-convex spaces, used in [PK], is a particular case of L-spaces since, to define the G-convex spaces, it is required that all of the conditions of Definition 9 [on L-spaces] be satisfied, together with a monotonicity condition on the set-valued map.

COMMENTS: This is definitely false. Actually, in a diagram in [14], Llinares expressed the equivalence of mc-spaces and L-spaces. Consequently, mc-spaces are pairs and particular to triples of G-convex spaces, and hence their mc-spaces should be discarded as well as L-spaces.

[VIII] From Summary of Sanchez et al. [38] in 2003:

By generalizing the classical KKM Theorem, we obtain a result that provides sufficient conditions to ensure the non-emptiness of several kinds of choice functions. This result generalizes well-known results on the existence of maximal elements for binary relations, on the non-emptiness of non-binary choice functions, and on the non-emptiness of some classical solutions for tournaments (top cycle and uncovered set) on non-finite sets.

In Section 2, the authors stated that

We are going to present our generalized KKM result by making use of a general abstract convexity structure called L-structure. This notion is equivalent to that of an mc-structure (see Llinares [12]), which generalizes the notion of usual convexity as well as other abstract convexity structures.

OUR CONCLUSION: Since the main result of this paper can be extended to abstract convex spaces as in Section 4, as like as L-spaces, all results on mc-spaces should be discarded.

7. Comments on some related articles

Here we give some comments on papers related to L-spaces not treated in [34], [35].

Horvath and Llinares Ciscar 1996 [6]

A semilattice is a partially ordered set X, with the partial ordering denoted by $\leq$, for which any pair (x, x') of elements has a least upper bound, denoted by $x \vee x'$. It is easy to see that any nonempty finite subset A of X has a least upper bound, denoted by sup A.

The following is [6, Theorem 2]:

Theorem 15. ([6]) *Let X be a topological semilattice with path-connected intervals, let $X_0 \subseteq X$ be a nonempty subset of X, and let $R \subseteq X_0 \times X$ be a binary relation such that*

(i) For each $x \in X_0$, the set $R(x) = \{y \in X : (x, y) \in R\}$ is not empty and closed in $R(X_0)$.

(ii) There exists $x_0 \in X_0$ such that the set $R(x_0)$ is compact.

(iii) For any nonempty finite subset $A \subseteq X_0$,

$$\bigcup_{x\in A} [x, \sup A] \subseteq \bigcup_{x\in A} R(x).$$

Then the set $\bigcap_{x\in X_0} R(x)$ is not empty.

COMMENTS: Let $\Gamma : \langle X_0\rangle \multimap X$ be defined by $\Gamma(A) := \bigcup_{x\in A}[x, \sup A]$ for each $A \in \langle X_0\rangle$. Then the above theorem shows that $R : X_0 \multimap X$ is a KKM map on the partial KKM space $(X, X_0; \Gamma)$. Therefore all KKM theoretic results in [26], [30] holds for a topological semilattice with path-connected intervals.

Luo 2001 [17]

ABSTRACT: In this paper, we obtain a generalized KKM theorem, a generalized Fan-Browder fixed theorem, and an existence theorem of Nash equilibria in topological ordered spaces.

COMMENTS: This paper is based on Theorem 2 in the preceding paper. Therefore all results of this paper follows from the KKM theoretic results in [26], [30].

González et al. 2007 [4]

Based on an incorrectly stated paper of Ding, the authors showed the following;

(1) In Definition 2, our original G-convex spaces are incorrectly called G-spaces.

(2) Theorem 4 is a particular KKM theorem for G-spaces originated from our work.

(3) Definitions 5 and 6 on a G-space are originally given by ourselves.

(4) A particular type of G-KKM maps is studied.

(5) L-spaces are defined like mc-spaces and a KKM theorem is given.

(6) For their L-spaces, they obtained a Fan type minimax inequality, a Fan-Browder fixed point theorem, and a Nash equilibrium theorem.

Since these steps are routine for abstract convex spaces, this paper seems to be not useful; see [26], [30].

Cain and González 2008 [3]

(1) From Abstract:

> We have introduced a new abstract convexity structure that generalizes the concept of a metric space with a convex structure, introduced by E. Michael in [E. Michael, Convex structures and continuous selections, Canad. J. Math. 11 (1959) 556–575] and called a topological space endowed with this structure an M-space. In an article by Shie Park and Hoonjoo Kim [S. Park, H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces, J. Math. Anal. Appl. 197 (1996) 173–187], the concepts of G-spaces and metric spaces with Michael's convex structure, were mentioned together but no kind of relationship was shown. In this article, we prove that G-spaces and M-spaces are close related. We also introduce here the concept of an L-space, which is inspired in the MC-spaces of J.V. Llinares [J.V. Llinares, Unified treatment of the problem of existence of maximal elements in binary relations: A characterization, J. Math. Econom. 29 (1998) 285–302], and establish relationships between the convexities of these spaces with the spaces previously mentioned.

(2) The authors are wrong: In our more earlier work [PK] in 1993, we clearly stated that a metric space X with convex structure (of Michael) becomes a G-convex space $(X;\Gamma)$. (Here we found more people who did not read [PK].)

(3) In [3], various subclasses of G-convex spaces are compared as in [PK]. All spaces in [3] are triples, not pairs as for L-spaces.

Kulpa and Szymanski 2008 [7]

In Section 6 of [7], its authors suggested a way of extending their results to a wider class of topological spaces that contains, in particular, the class of L-spaces due to Ben-El-Mechaiekh et al. [1] and defined an L*-structure on a topological space X by means of a map $L : \langle X\rangle \multimap X$ that satisfies the following condition:

(**) If $A \in \langle X\rangle$ and $\{U_x \mid x \in A\}$ is an open cover of X, then there exists $B \subset A$ such that $L(B)\cap\{U_x \mid x \in B\} \neq \emptyset$.

Accordingly, the authors call (X, L) an L*-space, and a non-empty subset Y of X to be L*-convex if for each non-empty finite subset A of Y, $L(A) \subset Y$. See also [31].

We note the following remarks on L*-spaces.

(1) An L*-space (X, L) is a particular case of a KKM-space $(E, D; \Gamma)$ for $X = E = D$ and $L = \Gamma$.

(2) Theorem 12 of [7] is a particular case of one of our results on KKM spaces. Therefore, according to Kulpa and Szymanski, our result enables transferring some results of [7] from simplicial spaces to KKM spaces.

(3) An example of a KKM space which is not a G-convex space is the extended long line $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$.

Lu 2009 [15]

From Abstract:

> the main purpose of this paper is to prove a section theorem, and next, as its applications, a weighted Nash equilibrium existence theorem and a Pareto equilibrium existence theorem for multi-objective games are obtained in topological ordered spaces. Our results improve and unify the corresponding results in the recently existing literatures.

Comments: This is based on the previous [6] and one of our old papers in 1996. This also can be extended to abstract convex spaces.

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