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# Binomial vanishing ideals

Research Article

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**Abstract:** In this note we characterize, in algebraic and geometric terms, when a graded vanishing ideal is generated by binomials over any field $K$.

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## 1. Introduction

Let $S = K[t_1, \ldots, t_s]$ be a polynomial ring over a field $K$ with the standard grading induced by setting $\deg(t_i) = 1$ for all $i$. By the *dimension* of an ideal $I \subset S$ we mean the Krull dimension of $S/I$. The affine and projective spaces over the field $K$ of dimensions $s$ and $s-1$ are denoted by $\mathbb{A}^s$ and $\mathbb{P}^{s-1}$, respectively. Points of $\mathbb{P}^{s-1}$ are denoted by $[\alpha]$, where $0 \neq \alpha \in \mathbb{A}^s$.

Given a set $\mathbb{Y} \subset \mathbb{P}^{s-1}$ define $I(\mathbb{Y})$, the *vanishing ideal* of $\mathbb{Y}$, as the graded ideal generated by the homogeneous polynomials in $S$ that vanish at all points of $\mathbb{Y}$. Conversely, given a homogeneous ideal $I \subset S$ define $V(I)$, the *zero set* of $I$, as the set of all $[\alpha] \in \mathbb{P}^{s-1}$ such that $f(\alpha) = 0$ for all homogeneous polynomial $f \in I$. The zero sets are the closed sets of the *Zariski topology* of $\mathbb{P}^{s-1}$. The Zariski closure of $\mathbb{Y}$ is denoted by $\overline{\mathbb{Y}}$.

We will use the following multi-index notation: for $a = (a_1, \ldots, a_s) \in \mathbb{Z}^s$, set $t^a = t_1^{a_1} \cdots t_s^{a_s}$. We call $t^a$ a *Laurent monomial*. If $a_i \geq 0$ for all $i$, $t^a$ is called a *monomial* of $S$. A *binomial* of $S$ is an element of the form $f = t^a - t^b$, for some $a, b$ in $\mathbb{N}^s$. An ideal $I \subset S$ generated by binomials is called a *binomial ideal*. A binomial ideal $I \subset S$ with the property that $t_i$ is not a zero-divisor of $S/I$ for all $i$ is called a *lattice ideal*.

In this note we classify binomial vanishing ideals in algebraic and geometric terms. There are some reasons to study vanishing ideals. They are used in algebraic geometry [5] and algebraic coding theory [4, 8]. They are also used in polynomial interpolation problems [3, 6, 11].

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The set $\mathcal{S} = \mathbb{P}^{s-1} \cup \{[0]\}$ is a monoid under componentwise multiplication, that is, given $[\alpha] = [(\alpha_1, \ldots, \alpha_s)]$ and $[\beta] = [(\beta_1, \ldots, \beta_s)]$ in $\mathcal{S}$, the product operation is given by

$$[\alpha] \cdot [\beta] = [\alpha \cdot \beta] = [(\alpha_1\beta_1, \ldots, \alpha_s\beta_s)],$$

where $[\mathbf{1}] = [(1, \ldots, 1)]$ is the identity element. Accordingly the affine space $\mathbb{A}^s$ is also a monoid under componentwise multiplication.

The contents of this note are as follows. In Section 2 we recall some preliminaries on projective varieties and vanishing ideals. Let $\mathbb{Y}$ be a subset of $\mathbb{P}^{s-1}$. If $\mathbb{Y} \cup \{[0]\}$ is a submonoid of $\mathbb{P}^{s-1} \cup \{[0]\}$, we show that $I(\mathbb{Y})$ is a binomial ideal (Theorem 3.2). The same type of result holds if $Y$ is a subset of $\mathbb{A}^s$ (Proposition 3.4). Then we show that $I(\mathbb{Y})$ is a binomial ideal if and only if $V(I(\mathbb{Y})) \cup \{[0]\}$ is a monoid under componentwise multiplication (Theorem 3.5). As a result if $\mathbb{Y}$ is finite, then $I(\mathbb{Y})$ is a binomial ideal if and only if $\mathbb{Y} \cup \{0\}$ is a monoid (Corollary 3.7). This essentially classifies all graded binomial vanishing ideals of dimension 1 (Corollary 3.8)

If $Y$ is a submonoid of an affine torus (see Definition 3.9), then $I(Y)$ is a non-graded lattice ideal [2, Proposition 2.3]. We give a graded version of this result, namely, if $\mathbb{Y}$ is a submonoid of a projective torus, then $I(\mathbb{Y})$ is a lattice ideal (Corollary 3.10).

Let $I(\mathbb{Y})$ be a vanishing ideal of dimension 1. According to [9, Proposition 6.7(a)] $I(\mathbb{Y})$ is a lattice ideal if and only if $\mathbb{Y}$ is a finite subgroup of a projective torus. We complement this result by showing that—over an algebraically closed field—$\mathbb{Y}$ is a finite subgroup of a projective torus if and only if there is a finite subgroup $H$ of $K^* = K \setminus \{0\}$ and $v_1, \ldots, v_s \in \mathbb{Z}^n$ that parameterize $\mathbb{Y}$ relative to $H$ (Proposition 3.12). For finite fields, this result was shown in [9, Proposition 6.7(b)].

Finally, we classify the graded lattice ideals of dimension 1 over an algebraically closed field of characteristic zero. It turns out that they are the vanishing ideals of finite subgroups of projective tori (Proposition 3.14).

For all unexplained terminology and additional information, we refer to [1, 5] (for algebraic geometry and vanishing ideals) and [2, 10, 12] (for binomial and lattice ideals).

## 2. Preliminaries

In this section, we present a few results that will be needed in this note. All results of this section are well-known.

**Definition 2.1.** *Let $K$ be a field. We define the projective space of dimension $s-1$ over $K$, denoted by $\mathbb{P}_K^{s-1}$ or $\mathbb{P}^{s-1}$ if $K$ is understood, to be the quotient space*

$$(K^s \setminus \{0\})/\sim$$

*where two points $\alpha$, $\beta$ in $K^s \setminus \{0\}$ are equivalent under $\sim$ if $\alpha = c\beta$ for some $c \in K$. It is usual to denote the equivalence class of $\alpha$ by $[\alpha]$. The affine space of dimension $s$ over the field $K$, denoted $\mathbb{A}_K^s$ or $\mathbb{A}^s$, is $K^s$.*

For any set $\mathbb{Y} \subset \mathbb{P}^{s-1}$ define $I(\mathbb{Y})$, the *vanishing ideal* of $\mathbb{Y}$, as the ideal generated by the homogeneous polynomials in $S$ that vanish at all points of $\mathbb{Y}$. Conversely, given a homogeneous ideal $I \subset S$ define its *zero set* as

$$V(I) = \left\{[\alpha] \in \mathbb{P}^{s-1} \,|\, f(\alpha) = 0, \forall f \in I \text{ homogeneous}\right\}.$$

A *projective variety* is the zero set of a homogeneous ideal. It is not difficult to see that the members of the family

$$\tau = \{\mathbb{P}^{s-1} \setminus V(I) \,|\, I \text{ is a graded ideal of } S\}$$

are the open sets of a topology on $\mathbb{P}^{s-1}$, called the *Zariski topology*. In a similar way we can define affine varieties, vanishing ideals of subsets of the affine space $\mathbb{A}^s$, and the corresponding Zariski topology of $\mathbb{A}^s$. The Zariski closure of $\mathbb{Y}$ is denoted by $\overline{\mathbb{Y}}$.

**Lemma 2.2.** *Let $K$ be a field.*

*(a) [1, pp. 191–192] If $Y \subset \mathbb{A}^s$ and $\mathbb{Y} \subset \mathbb{P}^{s-1}$, then $\overline{Y} = V(I(Y))$ and $\overline{\mathbb{Y}} = V(I(\mathbb{Y}))$.*

*(b) If $K$ is a finite field, then $Y = V(I(Y))$ and $\mathbb{Y} = V(I(\mathbb{Y}))$.*

***Proof.*** Part (b) follows from (a) because $\overline{Y} = Y$ and $\overline{\mathbb{Y}} = \mathbb{Y}$, if $K$ is finite. □

**Lemma 2.3.** *Let $K$ be a field. If $Y$ is a subset of $\mathbb{A}^s$ or a subset of $\mathbb{P}^{s-1}$ and $Z = V(I(Y))$, then $I(Z) = I(Y)$. In particular $I(Y) = I(\overline{Y})$.*

***Proof.*** Since $Y \subset Z$, we get $I(Z) \subset I(Y)$. As $I(Z) = I(V(I(Y))) \supset I(Y)$, one has equality. □

**Lemma 2.4.** *[1, Proposition 6, p. 441] If $\mathbb{Y} \subset \mathbb{P}^{s-1}$ and $\dim(S/I(\mathbb{Y})) = 1$, then $|\mathbb{Y}| < \infty$.*

The converse of Lemma 2.4 is true. This follows from the next result.

**Lemma 2.5.** *Let $\mathbb{Y}$ and $Y$ be finite subsets of $\mathbb{P}^{s-1}$ and $\mathbb{A}^s$ respectively, let $P$ and $[P]$ be points in $\mathbb{Y}$ and $Y$, respectively, with $P = (\alpha_1, \ldots, \alpha_s)$, and let $I_{[P]}$ and $I_P$ be the vanishing ideal of $[P]$ and $P$, respectively. Then*

$$I_{[P]} = (\{\alpha_k t_i - \alpha_i t_k \,|\, k \neq i \in \{1, \ldots, s\}\}), \quad I_P = (t_1 - \alpha_1, \ldots, t_s - \alpha_s), \tag{1}$$

*where $\alpha_k \neq 0$ for some $k$. Furthermore $I(\mathbb{Y}) = \bigcap_{[Q]\in\mathbb{Y}} I_{[Q]}$, $I(Y) = \bigcap_{Q\in Y} I_Q$, $I_{[P]}$ is a prime ideal of height $s - 1$ and $I_P$ is a prime ideal of height $s$.*

## 3. A classification of vanishing ideals generated by binomials

We continue to employ the notations and definitions used in Sections 1 and 2. In this part we classify vanishing ideals generated by binomials.

Let $(\mathcal{S}, \cdot, 1)$ be a monoid and let $K$ be a field. As usual we define a *character* $\chi$ of $\mathcal{S}$ in $K$ (or a *$K$-character* of $\mathcal{S}$) to be a homomorphism of $\mathcal{S}$ into the multiplicative monoid $(K, \cdot, 1)$. Thus $\chi$ is a map of $\mathcal{S}$ into $K$ such that $\chi(1) = 1$ and $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$ for all $\alpha, \beta$ in $\mathcal{S}$.

**Theorem 3.1.** *(Dedekind's Theorem [7, p. 291]) If $\chi_1, \ldots, \chi_m$ are distinct characters of a monoid $\mathcal{S}$ into a field $K$, then the only elements $\lambda_1, \ldots, \lambda_m$ in $K$ such that*

$$\lambda_1\chi_1(\alpha) + \cdots + \lambda_m\chi_m(\alpha) = 0$$

*for all $\alpha \in \mathcal{S}$ are $\lambda_1 = \cdots = \lambda_m = 0$.*

**Theorem 3.2.** *If $\mathbb{Y}$ is a subset of $\mathbb{P}^{s-1}$ and $\mathbb{Y}\cup\{[0]\}$ is a submonoid of $\mathbb{P}^{s-1}\cup\{[0]\}$ under componentwise multiplication, then $I(\mathbb{Y})$ is a binomial ideal.*

***Proof.*** The set $\mathcal{S} = \{x \in \mathbb{A}^s \,|\, [x] \in \mathbb{Y} \cup \{[0]\}\}$ is a submonoid of $\mathbb{A}^s$. Take a homogeneous polynomial $0 \neq f = \lambda_1 t^{a_1} + \cdots + \lambda_m t^{a_m}$ that vanishes at all points of $\mathbb{Y}$, where $\lambda_i \in K \setminus \{0\}$ for all $i$ and $a_1, \ldots, a_m$ are distinct non-zero vectors in $\mathbb{N}^s$. We set $a_i = (a_{i_1}, \ldots, a_{is})$ for all $i$. For each $i$ consider the $K$-character of $\mathcal{S}$ given by

$$\chi_i \colon \mathcal{S} \to K, \quad (\alpha_1, \ldots, \alpha_s) \mapsto \alpha_1^{a_{i1}} \cdots \alpha_s^{a_{is}}.$$

As $f \in I(\mathbb{Y})$, one has that $\lambda_1\chi_1 + \cdots + \lambda_m\chi_m = 0$. Hence, by Theorem 3.1, we get that $m \geq 2$ and $\chi_i = \chi_j$ for some $i \neq j$. Thus $t^{a_i} - t^{a_j}$ is in $I(\mathbb{Y})$. For simplicity of notation we assume that $i = 1$ and $j = 2$. Since $[\mathbf{1}] \in \mathbb{Y}$, we get that $\lambda_1 + \cdots + \lambda_m = 0$. Thus

$$f = \lambda_2(t^{a_2} - t^{a_1}) + \cdots + \lambda_m(t^{a_m} - t^{a_1}).$$

Since $f - \lambda_2(t^{a_2} - t^{a_1})$ is a homogeneous polynomial in $I(\mathbb{Y})$, by induction on $m$, we obtain that $f$ is a sum of homogeneous binomials in $I(\mathbb{Y})$. □

This result can be restated as:

**Theorem 3.3.** *Let $\mathbb{Y}$ be a subset of $\mathbb{P}^{s-1}$ such that $[\mathbf{1}] \in \mathbb{Y}$ and $[\alpha] \cdot [\beta] \in \mathbb{Y}$ for all $[\alpha]$, $[\beta]$ in $\mathbb{Y}$ with $\alpha \cdot \beta \neq 0$. Then $I(\mathbb{Y})$ is a binomial ideal.*

The next result was observed in the Remark after [2, Proposition 2.3].

**Proposition 3.4.** *[2] If $Y$ is a submonoid of $\mathbb{A}^s$ and $\tau \in K^*$, then $I(Y)$ is a binomial ideal and $I(\tau Y)$ is a non-pure binomial ideal.*

***Proof.*** That $I(Y)$ is a binomial ideal follows readily by adapting the proof of Theorem 3.2. Let $\{t^{b_i} - t^{c_i}\}_{i=1}^r$ be a set of generators of $I(Y)$ with $b_i, c_i$ in $\mathbb{N}^s$ for all $i$. If $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, we set $|a| = \sum_i a_i$. Then it is not hard to see that the set $\{t^{b_i}/\tau^{|b_i|} - t^{c_i}/\tau^{|c_i|}\}_{i=1}^r$ generates $I(\tau Y)$, that is, $I(\tau Y)$ is a non-pure binomial ideal. □

**Theorem 3.5.** *Let $K$ be a field and let $\mathbb{Y}$ be a subset of $\mathbb{P}^{s-1}$. Then $I(\mathbb{Y})$ is a binomial ideal if and only if $V(I(\mathbb{Y})) \cup \{[0]\}$ is a monoid under componentwise multiplication.*

***Proof.*** $\Rightarrow$) Consider an arbitrary non-zero binomial $f = t^a - t^b$ in $I(\mathbb{Y})$ with $a = (a_i)$ and $b = (b_i)$ in $\mathbb{N}^s$. As $I(\mathbb{Y})$ is graded, $f$ is homogeneous. First notice that $[\mathbf{1}] \in V(I(\mathbb{Y}))$ because $f$ vanishes at $[\mathbf{1}]$. Take $[\alpha]$, $[\beta]$ in $V(I(\mathbb{Y}))$ with $\alpha = (\alpha_i)$, $\beta = (\beta_i)$. Then

$$\alpha_1^{a_1} \cdots \alpha_s^{a_s} = \alpha_1^{b_1} \cdots \alpha_s^{b_s} \text{ and } \beta_1^{a_1} \cdots \beta_s^{a_s} = \beta_1^{b_1} \cdots \beta_s^{b_s},$$

and consequently $(\alpha_1\beta_1)^{a_1} \cdots (\alpha_s\beta_s)^{a_s} = (\alpha_1\beta_1)^{b_1} \cdots (\alpha_s\beta_s)^{b_s}$, i.e., $f$ vanishes at $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ if $\alpha \cdot \beta \neq 0$. Thus $[\alpha] \cdot [\beta] \in V(I(\mathbb{Y})) \cup \{[0]\}$.

$\Leftarrow$) Thanks to Theorem 3.2 one has that $I(V(I(\mathbb{Y})))$ is a binomial ideal. Recall that $V(I(\mathbb{Y}))$ is equal to $\overline{\mathbb{Y}}$ (see Lemma 2.2). On the other hand, by Lemma 2.3, $I(\mathbb{Y}) = I(\overline{\mathbb{Y}})$. Thus $I(\mathbb{Y})$ is a binomial ideal. □

**Remark 3.6.** *If $Y \subset \mathbb{A}^s$, then $I(Y)$ is a binomial ideal if and only if $V(I(Y))$ is a submonoid of $\mathbb{A}^s$ under componentwise multiplication. This follows by adapting the proof of Theorem 3.5.*

**Corollary 3.7.** *If $\mathbb{Y}$ is a subset of $\mathbb{P}^{s-1}$ which is closed in the Zariski topology, then $I(\mathbb{Y})$ is a binomial ideal if and only if $\mathbb{Y} \cup \{[0]\}$ is a submonoid of $\mathbb{P}^{s-1} \cup \{[0]\}$.*

***Proof.*** Thanks to Theorem 3.5 it suffices to recall that $V(I(\mathbb{Y}))$ is equal to $\overline{\mathbb{Y}}$ (see Lemma 2.2). □

**Corollary 3.8.** *If $\mathbb{Y}$ is a subset of $\mathbb{P}^{s-1}$ and $\dim(S/I(\mathbb{Y})) = 1$, then $I(\mathbb{Y})$ is a binomial ideal if and only if $\mathbb{Y} \cup \{[0]\}$ is a submonoid of $\mathbb{P}^{s-1} \cup \{[0]\}$.*

***Proof.*** This is a direct consequence of Lemma 2.4 and Corollary 3.7 because any finite set is closed in the Zariski topology. □

**Definition 3.9.** *The set $T = \{[(x_1, \ldots, x_s)] \in \mathbb{P}^{s-1} | \, x_i \in K^* \text{ for all } i\}$ is called a projective torus in $\mathbb{P}^{s-1}$, and the set $T^* = (K^*)^s$ is called an affine torus in $\mathbb{A}^s$, where $K^* = K \setminus \{0\}$.*

If $Y$ is a submonoid of an affine torus $T^*$, then $I(Y)$ is a non-graded lattice ideal (see [2, Proposition 2.3]). The following corollary is the graded version of this result.

**Corollary 3.10.** *If $\mathbb{Y}$ is a submonoid of a projective torus $T$, then $I(\mathbb{Y})$ is a lattice ideal.*

***Proof.*** By Theorem 3.2, $I(\mathbb{Y})$ is a binomial ideal. Thus it suffices to show that $t_i$ is not a zero-divisor of $S/I(\mathbb{Y})$ for all $i$. If $f \in S$ and $t_i f$ vanishes at all points of $\mathbb{Y}$, then so does $f$, as required. □

**Corollary 3.11.** *[9, Proposition 6.7(a)] If $\mathbb{Y} \subset \mathbb{P}^{s-1}$ and $\dim(S/I(\mathbb{Y})) = 1$, then the following are equivalent:*

*(a) $I(\mathbb{Y})$ is a lattice ideal.*

*(b) $\mathbb{Y}$ is a finite subgroup of a projective torus $T$.*

***Proof.*** (a) $\Rightarrow$ (b): By Lemma 2.4 the set $\mathbb{Y}$ is finite. Using Corollary 3.8 and Lemma 2.5 it follows that $\mathbb{Y}$ is a submonoid of $T$. As the cancellation laws hold in $T$ and $\mathbb{Y}$ is finite, we get that $\mathbb{Y}$ is a group.

(b) $\Rightarrow$ (a): This is a direct consequence of Corollary 3.10. □

**Proposition 3.12.** *Let $K$ be an algebraically closed field. If $\mathbb{Y} \subset \mathbb{P}^{s-1}$, then the following are equivalent:*

*(a) $\mathbb{Y}$ is a finite subgroup of a projective torus $T$.*

*(b) There is a finite subgroup $H$ of $K^*$ and $v_1, \ldots, v_s \in \mathbb{Z}^n$ such that*

$$\mathbb{Y} = \{[(x^{v_1}, \ldots, x^{v_s})] \,|\, x = (x_1, \ldots, x_n) \text{ and } x_i \in H \text{ for all } i\} \subset \mathbb{P}^{s-1}.$$

***Proof.*** (b) $\Rightarrow$ (a): It is not hard to verify that $\mathbb{Y}$ is a subgroup of $T$ using the parameterization of $\mathbb{Y}$ relative to $H$.

(a) $\Rightarrow$ (b): By the fundamental theorem of finitely generated abelian groups, $\mathbb{Y}$ is a direct product of cyclic groups. Hence, there are $[\alpha_1], \ldots, [\alpha_n]$ in $\mathbb{Y}$ such that

$$\mathbb{Y} = \left\{ [\alpha_1]^{i_1} \cdots [\alpha_n]^{i_n} \;\middle|\; i_1, \ldots, i_n \in \mathbb{Z} \right\}.$$

We set $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{is})$ for $i = 1, \ldots, n$. As $[\alpha_1], \ldots, [\alpha_n]$ have finite order, for each $1 \leq i \leq n$ there is $m_i = \mathrm{o}([\alpha_i])$ such that $[\alpha_i]^{m_i} = [\mathbf{1}]$. Thus

$$(\alpha_{i1}^{m_i}, \ldots, \alpha_{is}^{m_i}) = (\lambda_i, \ldots, \lambda_i)$$

for some $\lambda_i \in K^*$. Pick $\mu_i \in K^*$ such that $\mu_i^{m_i} = \lambda_i$. Setting, $\beta_{ij} = \alpha_{ij}/\mu_i$, one has $\beta_{ij}^{m_i} = 1$ for all $i, j$, that is all $\beta_{ij}$'s are in $K^*$ and have finite order. Consider the subgroup $H$ of $K^*$ generated by all $\beta_{ij}$'s. This group is cyclic because $K$ is a field. If $\beta$ is a generator of $(H, \cdot)$, we can write $\alpha_{ij}/\mu_i = \beta^{v_{ji}}$ for some $v_{ji}$ in $\mathbb{N}$. Hence

$$[\alpha_1] = [(\beta^{v_{11}}, \ldots, \beta^{v_{s1}})], \ldots, [\alpha_n] = [(\beta^{v_{1n}}, \ldots, \beta^{v_{sn}})].$$

We set $v_i = (v_{i1}, \ldots, v_{in})$ for $i = 1, \ldots, s$. Let $\mathbb{Y}_H$ be the set in $\mathbb{P}^{s-1}$ parameterized by the monomials $y^{v_1}, \ldots, y^{v_s}$ relative to $H$. If $[\gamma] \in \mathbb{Y}$, then we can write

$$[\gamma] = [\alpha_1]^{i_1} \cdots [\alpha_n]^{i_n} = [((\beta^{i_1})^{v_{11}} \cdots (\beta^{i_n})^{v_{1n}}, \ldots, (\beta^{i_1})^{v_{s1}} \cdots (\beta^{i_n})^{v_{sn}})]$$

for some $i_1, \ldots, i_n \in \mathbb{Z}$. Thus $[\gamma] \in \mathbb{Y}_H$. Conversely if $[\gamma] \in \mathbb{Y}_H$, then $[\gamma] = [(x^{v_1}, \ldots, x^{v_s})]$ for some $x_1, \ldots, x_n$ in $H$. Since any $x_k$ is of the form $\beta^{i_k}$ for some integer $i_k$, one can write $[\gamma] = [\alpha_1]^{i_1} \cdots [\alpha_n]^{i_n}$, that is, $[\gamma] \in \mathbb{Y}$. □

**Remark 3.13.** *The equivalence between (a) and (b) was shown in [9, Proposition 6.7(b)] under the assumption that $K$ is a finite field.*

**Proposition 3.14.** *Let $K$ be an algebraically closed field of characteristic zero and let $I$ be a graded ideal of $S$ of dimension 1. Then $I$ is a lattice ideal if and only if $I$ is the vanishing ideal of a finite subgroup $\mathbb{Y}$ of a projective torus $T$.*

***Proof.*** $\Rightarrow$) Assume that $I = I(\mathcal{L})$ is the lattice ideal of a lattice $\mathcal{L}$ in $\mathbb{Z}^s$. Since $I$ is graded and $\dim(S/I) = 1$, for each $i \geq 2$, there is $a_i \in \mathbb{N}_+$ such that $f_i := t_i^{a_i} - t_1^{a_i} \in I$. This polynomial has a factorization into linear factors of the form $t_i - \mu t_1$ with $\mu \in K^*$. In characteristic zero a lattice ideal is radical [12, Theorem 8.2.27]. Therefore $I$ is the intersection of its minimal primes and each minimal prime is generated by $s-1$ linear polynomials of the form $t_i - \mu t_1$. It follows that $I$ is the vanishing ideal of some finite subset $\mathbb{Y}$ of a projective torus $T$. By Corollary 3.7, $\mathbb{Y}$ is a submonoid of $T$. As the cancellation laws hold in $T$ and $\mathbb{Y}$ is finite, we get that $\mathbb{Y}$ is a group.

$\Leftarrow$) This implication follows at once from Corollary 3.10. $\square$

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## References

[1] D. Cox, J. Little and D. O'Shea, *Ideals, varieties, and algorithms*, Springer-Verlag, 1992.

[2] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J., 84(1), 1-45, 1996.

[3] M. Gasca and T. Sauer, *Polynomial interpolation in several variables*, Adv. Comput. Math., 12(4), 377-410, 2000.

[4] M. González-Sarabia, C. Rentería and H. Tapia-Recillas, *Reed-Muller-type codes over the Segre variety*, Finite Fields Appl., 8(4), 511-518, 2002.

[5] J. Harris, *Algebraic geometry: A first course*, Graduate Texts in Mathematics **133**, Springer-Verlag, New York, 1992.

[6] M. Hellus and R. Waldi, *Interpolation in affine and projective space over a finite field*, J. Commut. Algebra, to appear.

[7] N. Jacobson, *Basic algebra I*, Second Edition, W. H. Freeman and Company, New York, 1996.

[8] H. H. López, C. Rentería and R. H. Villarreal, *Affine cartesian codes*, Des. Codes Cryptogr., 71(1), 5-19, 2014.

[9] J. Neves, M. Vaz Pinto and R. H. Villarreal, *Regularity and algebraic properties of certain lattice ideals*, Bull. Braz. Math. Soc. (N.S.), 45(4), 777-806, 2014.

[10] L. O'Carroll, F. Planas-Vilanova and R. H. Villarreal, *Degree and algebraic properties of lattice and matrix ideals*, SIAM J. Discrete Math., 28(1), 394-427, 2014.

[11] A. Tochimani and R. H. Villarreal, *Vanishing ideals over rational parameterizations*, preprint, arXiv:1502.05451v1, 2015.

[12] R. H. Villarreal, *Monomial algebras, second edition*, Monographs and Research Notes in Mathematics, CRC Press, 2015.