J. Algebra Comb. Discrete Appl.
3(1) • 7–11

Received: 2 March 2015
Accepted: 10 November 2015

Journal of Algebra Combinatorics Discrete Structures and Applications

On the rank functions of $\mathcal{H}$-matroids

Research Article

Yoshio Sano

Abstract: The notion of $\mathcal{H}$-matroids was introduced by U. Faigle and S. Fujishige in 2009 as a general model for matroids and the greedy algorithm. They gave a characterization of $\mathcal{H}$-matroids by the greedy algorithm. In this note, we give a characterization of some $\mathcal{H}$-matroids by rank functions.

2010 MSC: 05B35, 90C27

Keywords: Matroid, $\mathcal{H}$-Matroid, Simplicial complex, Rank function

1. Introduction and main result

The notion of matroids was introduced by H. Whitney [10] in 1935 as an abstraction of the notion of linear independence in a vector space. Many researchers have studied and extended the theory of matroids (cf. [2, 4, 5, 8, 9]). In 2009, U. Faigle and S. Fujishige [1] introduced the notion of $\mathcal{H}$-matroids as a general model for matroids and the greedy algorithm. They gave a characterization of $\mathcal{H}$-matroids by the greedy algorithm. In this note, we give a characterization of the rank functions of $\mathcal{H}$-matroids that are *simplicial complexes*, for *any* family $\mathcal{H}$. Our main result is as follows.

Theorem 1.1. *Let E be a finite set and let $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ be a set function on E. Let $\mathcal{H}$ be a family of subsets of E with $\emptyset, E \in \mathcal{H}$. Then, ρ is the rank function of an $\mathcal{H}$-matroid $(E, \mathcal{I})$ if and only if ρ is a normalized unit-increasing function satisfying the $\mathcal{H}$-extension property.*

(E) ($\mathcal{H}$-extension property)
For $X \in 2^E$ and $H \in \mathcal{H}$ with $X \subseteq H$, if $\rho(X) = |X| < \rho(H)$,
then there exists $e \in H \setminus X$ such that $\rho(X \cup \{e\}) = \rho(X) + 1$.

Moreover, if ρ is a normalized unit-increasing set function on E satisfying the $\mathcal{H}$-extension property and $\mathcal{I} := \{X \in 2^E \mid \rho(X) = |X|\}$, then $(E, \mathcal{I})$ is an $\mathcal{H}$-matroid with rank function ρ and $\mathcal{I}$ is a simplicial complex.

This work was supported by JSPS KAKENHI Grant Number 15K20885.
Yoshio Sano; Division of Information Engineering, Faculty of Engineering, Information and Systems, University of Tsukuba, Japan (email: sano@cs.tsukuba.ac.jp).

http://dx.doi.org/10.13069/jacodesmath.26764 ISSN 2148-838X

This note is organized as follows. Section 2 gives some definitions and preliminaries on $\mathcal{H}$-matroids. In Section 3, we give a proof of Theorem 1.1 and an example which shows $\mathcal{H}$-matroids that are not simplicial complexes are not characterized only by their rank functions.

2. Preliminaries

Let E be a nonempty finite set and let 2^E denote the family of all subsets of E. For any family $\mathcal{I}$ of subsets of E, the *extreme-point operator* $\mathrm{ex}_{\mathcal{I}} : \mathcal{I} \to 2^E$ and the *co-extreme-point operator* $\mathrm{ex}^*_{\mathcal{I}} : \mathcal{I} \to 2^E$ associated with $\mathcal{I}$ are defined as follows:

$$\begin{aligned} \mathrm{ex}_{\mathcal{I}}(I) &:= \{e \in I \mid I \setminus \{e\} \in \mathcal{I}\} \quad (I \in \mathcal{I}), \\ \mathrm{ex}^*_{\mathcal{I}}(I) &:= \{e \in E \setminus I \mid I \cup \{e\} \in \mathcal{I}\} \quad (I \in \mathcal{I}). \end{aligned}$$

For any family $\mathcal{I} \subseteq 2^E$, we denote the set of maximal elements of $\mathcal{I}$ with respect to set inclusion by $\mathbf{Max}(\mathcal{I})$.

Let $\mathcal{I}$ be a nonempty family of subsets of a finite set E. The family $\mathcal{I}$ is called *constructible* if it satisfies

(C) $\mathrm{ex}_{\mathcal{I}}(I) \neq \emptyset$ for all $I \in \mathcal{I} \setminus \{\emptyset\}$.

Note that (C) implies $\emptyset \in \mathcal{I}$. We call $I \in \mathcal{I}$ a *base* of $\mathcal{I}$ if $\mathrm{ex}^*_{\mathcal{I}}(I) = \emptyset$. We denote by $\mathcal{B}(\mathcal{I})$ the family of bases of $\mathcal{I}$, i.e., $\mathcal{B}(\mathcal{I}) := \{I \in \mathcal{I} \mid \mathrm{ex}^*_{\mathcal{I}}(I) = \emptyset\}$. By definition, it holds that $\mathcal{B}(\mathcal{I}) \supseteq \mathbf{Max}(\mathcal{I})$.

A constructible family $\mathcal{I}$ induces a *(base) rank function* $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ via

$$\rho(X) = \max_{B \in \mathcal{B}(\mathcal{I})} |X \cap B| = \max_{I \in \mathcal{I}} |X \cap I| = \max_{I \in \mathbf{Max}(\mathcal{I})} |X \cap I|.$$

The following is easily verified by definitions.

Lemma 2.1. *The rank function ρ of a constructible family is normalized (i.e. $\rho(\emptyset) = 0$) and satisfies the unit-increase property*

(UI) $\rho(X) \leq \rho(Y) \leq \rho(X) + |Y \setminus X|$ *for all $X \subseteq Y \subseteq E$.*

Remark that, by putting $X = \emptyset$, we obtain

(UI)′ $0 \leq \rho(Y) \leq |Y|$ for all $Y \subseteq E$.

The *restriction* of $\mathcal{I}$ to a subset $A \in 2^E$ is the family $\mathcal{I}^{(A)} := \{I \in \mathcal{I} \mid I \subseteq A\}$. Note that every restriction of a constructible family is constructible.

A *simplicial complex* is a family $\mathcal{I} \subseteq 2^E$ such that $X \subseteq I \in \mathcal{I}$ implies $X \in \mathcal{I}$. We can easily check the following lemmas on simplicial complexes.

Lemma 2.2. *A family $\mathcal{I} \subseteq 2^E$ is a simplicial complex if and only if $\mathrm{ex}_{\mathcal{I}}(I) = I$ holds for any $I \in \mathcal{I}$.*

Proof. The lemma follows from the definitions of a simplicial complex and $\mathrm{ex}_{\mathcal{I}}(\cdot)$. □

Lemma 2.3. *Let $\mathcal{I} \subseteq 2^E$ be a simplicial complex and let $X \in 2^E$. Then,*

(a) $\mathcal{B}(\mathcal{I}) = \mathbf{Max}(\mathcal{I})$.

(b) *For $X \in 2^E$, $X \in \mathcal{I}$ if and only if $\rho(X) = |X|$.*

(c) *For $H \in 2^E$, the family $\mathcal{I}^{(H)} \subseteq 2^H$ is a simplicial complex.*

Proof. (a): Suppose that there exists an element $B \in \mathcal{B}(\mathcal{I}) \setminus \mathbf{Max}(\mathcal{I})$. Then, since B is not maximal in $\mathcal{I}$, there exists $I \in \mathcal{I}$ such that $B \subsetneq I$. For any $e \in I \setminus B$, we have $B \cup \{e\} \in \mathcal{I}$ since $B \cup \{e\} \subseteq I$ and $\mathcal{I}$ is a simplicial complex. Therefore $e \in \mathrm{ex}^*_{\mathcal{I}}(B)$. But this is a contradiction to $B \in \mathcal{B}(\mathcal{I})$.

(b): If $X \in \mathcal{I}$, then $\rho(X) = \max_{I \in \mathcal{I}} |X \cap I| = |X|$. Take $X \in 2^E$ with $\rho(X) = |X|$. Then there exists $I \in \mathcal{I}$ such that $|X \cap I| = \rho(X) = |X|$. Therefore, $X \subseteq I$. Since $\mathcal{I}$ is a simplicial complex, we have $X \in \mathcal{I}$.

(c): Take any $X \in 2^H$ and $I \in \mathcal{I}^{(H)} := \{I \in \mathcal{I} \mid I \subseteq H\}$ with $X \subseteq I$. Since $\mathcal{I}$ is a simplicial complex, $X \in \mathcal{I}$. Since $X \subseteq H$, we have $X \in \mathcal{I}^{(H)}$. □

We now recall the definitions of an $\mathcal{H}$*-independence system* and an $\mathcal{H}$*-matroid*, which were introduced by Faigle and Fujishige [1]. Let E be a finite set and let $\mathcal{H}$ be a family of subsets of E with $\emptyset, E \in \mathcal{H}$. A constructible family $\mathcal{I} \subseteq 2^E$ is called an $\mathcal{H}$*-independence system* if

(I) for all $H \in \mathcal{H}$, there exists $I \in \mathcal{I}^{(H)}$ such that $|I| = \rho(H)$.

An $\mathcal{H}$*-matroid* is a pair $(E, \mathcal{I})$ of the set E and an $\mathcal{H}$-independence system $\mathcal{I}$ satisfying the following property:

(M) for all $H \in \mathcal{H}$, all the bases B of $\mathcal{I}^{(H)}$ have the same cardinality $|B| = \rho(H)$.

3. Proof of Theorem 1.1

First, we see an example which shows that $\mathcal{H}$-matroids that are not simplicial complexes are not characterized by their rank functions.

Example 3.1. *Let $E = \{1, 2, 3\}$ and $\mathcal{H} = \{\emptyset, E\}$. Let*

$$\begin{aligned} \mathcal{I}_1 &= \{\emptyset, \{2\}, \{1,2\}, \{2,3\}\}, \\ \mathcal{I}_2 &= \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}\}, \\ \mathcal{I}_3 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}. \end{aligned}$$

Then $(E, \mathcal{I}_1)$, $(E, \mathcal{I}_2)$, and $(E, \mathcal{I}_3)$ are $\mathcal{H}$-matroids with the same rank function $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ such that $\rho(\emptyset) = 0$, $\rho(\{1\}) = \rho(\{2\}) = \rho(\{3\}) = \rho(\{1,3\}) = 1$, and $\rho(\{1,2\}) = \rho(\{2,3\}) = \rho(\{1,2,3\}) = 2$.

Therefore, we cannot distinguish $\mathcal{H}$-matroids in general by their rank functions. More generally, the following holds.

Proposition 3.2. *For any constructible families $\mathcal{I}$ and $\mathcal{I}'$ with $\mathbf{Max}(\mathcal{I}) = \mathbf{Max}(\mathcal{I}')$, the rank function ρ' associated with $\mathcal{I}'$ coincides with the rank function ρ associated with $\mathcal{I}$.*

Proof. For any $X \in 2^E$, it holds that

$$\rho(X) = \max_{I \in \mathbf{Max}(\mathcal{I})} |X \cap I| = \max_{I \in \mathbf{Max}(\mathcal{I}')} |X \cap I| = \rho'(X)$$

since $\mathbf{Max}(\mathcal{I}) = \mathbf{Max}(\mathcal{I}')$. □

In the following, we give a proof of Theorem 1.1.

Lemma 3.3. *For any constructible family, there exists a simplicial complex such that their rank functions are the same.*

Proof. Let $\mathcal{I} \subseteq 2^E$ be a constructible family. Define $\mathcal{I}' := \{X \in 2^E \mid X \subseteq I \text{ for some } I \in \mathcal{I}\}$. Then it is clear that $\mathcal{I}'$ is a simplicial complex. Obviously each $Y \in \mathbf{Max}(\mathcal{I})$ is maximal in $\mathcal{I}'$, and $\mathcal{I}'$ does not have new maximal members. Therefore $\mathbf{Max}(\mathcal{I}) = \mathbf{Max}(\mathcal{I}')$. Note that any simplicial complex is a constructible family. By Proposition 3.2, the rank functions of $\mathcal{I}$ and $\mathcal{I}'$ are the same. □

Lemma 3.4. *Let $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ be the rank function of an $\mathcal{H}$-matroid $(E, \mathcal{I})$, where $\mathcal{I}$ is a simplicial complex. Then ρ satisfies the $\mathcal{H}$-extension property.*

Proof. Take $X \in 2^E$ and $H \in \mathcal{H}$ with $X \subseteq H$, and suppose that $\rho(X) = |X| < \rho(H)$. By Lemma 2.3 (c), $\mathcal{I}^{(H)}$ is a simplicial complex since $\mathcal{I}$ is a simplicial complex. Note that $\mathcal{B}(\mathcal{I}^{(H)}) = \mathbf{Max}(\mathcal{I}^{(H)})$ by Lemma 2.3 (a). By Lemma 2.3 (b), $X \in \mathcal{I}$. Therefore $X \in \mathcal{I}^{(H)}$, and X is not a base of $\mathcal{I}^{(H)}$ by (I) and (M) since $\rho(X) < \rho(H)$. Thus there exists $B \in \mathcal{I}$ such that $X \subsetneq B \subseteq H$ and $|B| = \rho(H)$. Take any element $e \in B \setminus X \subseteq H \setminus X$. Then $X \cup \{e\} \in \mathcal{I}$ since $X \cup \{e\} \subseteq B \in \mathcal{I}$ and $\mathcal{I}$ is a simplicial complex. Hence it follows that $\rho(X \cup \{e\}) = |X \cup \{e\}| = |X| + 1 = \rho(X) + 1$. □

Lemma 3.5. *Let $\rho : 2^E \to \mathbb{Z}_{\geq 0}$ be a normalized unit-increasing function satisfying the $\mathcal{H}$-extension property for some family $\mathcal{H} \subseteq 2^E$ with $\emptyset, E \in \mathcal{H}$. Put*

$$\mathcal{I}_\rho := \{X \in 2^E \mid \rho(X) = |X|\}.$$

Then $(E, \mathcal{I}_\rho)$ is an $\mathcal{H}$-matroid and $\mathcal{I}_\rho$ is a simplicial complex.

Proof. First we show that $\mathcal{I}_\rho$ is a simplicial complex. Take any $I \in \mathcal{I}_\rho \setminus \{\emptyset\}$ and any $e \in I$. Then we have $\rho(I) = |I|$. Since ρ is unit-increasing, we have $\rho(I) \leq \rho(I \setminus \{e\}) + 1$ and thus $\rho(I \setminus \{e\}) \geq \rho(I) - 1 = |I| - 1 = |I \setminus \{e\}|$. By (UI) and $\rho(\emptyset) = 0$, we also have $\rho(I \setminus \{e\}) \leq 0 + |I \setminus \{e\}|$ and thus $\rho(I \setminus \{e\}) \leq |I \setminus \{e\}|$. Therefore we have $\rho(I \setminus \{e\}) = |I \setminus \{e\}|$ and thus $I \setminus \{e\} \in \mathcal{I}_\rho$. By Lemma 2.2, $\mathcal{I}_\rho$ is a simplicial complex. Hence it follows from definitions that $\mathcal{I}_\rho$ satisfies (C) and (I).

Now we show that $\mathcal{I}_\rho$ satisfies (M). Take any $H \in \mathcal{H}$. Suppose that there exist $B_1, B_2 \in \mathcal{B}(\mathcal{I}_\rho^{(H)})$ such that $|B_1| \neq |B_2|$. Without loss of generality, we may assume that $|B_1| < |B_2| \leq \rho(H)$. Note that $\rho(B_1) = |B_1|$ and $\rho(B_2) = |B_2|$. Then, by (E), there exists $e \in H \setminus B_1$ such that $\rho(B_1 \cup \{e\}) = \rho(B_1) + 1 = |B_1| + 1 = |B_1 \cup \{e\}|$. Thus we have $B_1 \cup \{e\} \in \mathcal{I}_\rho$ with $B_1 \cup \{e\} \subseteq H$. But this is a contradiction to the assumption that B_1 is a base of $\mathcal{I}_\rho^{(H)}$. Thus $\mathcal{I}_\rho$ satisfies (M). Hence $(E, \mathcal{I}_\rho)$ is an $\mathcal{H}$-matroid. □

Proof of Theorem 1.1. It follows from Lemmas 2.1, 3.3, 3.4, and 3.5. □

Remark 3.6. Strict cg-matroids *which were introduced by S. Fujishige, G. A. Koshevoy, and Y. Sano [3] in 2007 can be considered as $\mathcal{H}$-matroids $(E, \mathcal{I})$ where $\mathcal{H}$ is an abstract convex geometry and $\mathcal{I} \subseteq \mathcal{H}$. The rank functions $\rho : \mathcal{H} \to \mathbb{Z}_{\geq 0}$ of strict cg-matroids $(E, \mathcal{H}; \mathcal{I})$ are characterized in [6]. For more study on* cg-matroids*, see [7].*

Remark 3.7. *Faigle and Fujishige gave a characterization of the rank functions $\mathcal{H}$-matroids when $\mathcal{H}$ is a closure space (see [1, Theorem 5.1]).*

Acknowledgment: The author is grateful to the anonymous referees for careful reading and valuable comments.

References

[1] U. Faigle, S. Fujishige, A general model for matroids and the greedy algorithm, Math. Program. Ser. A 119(2) (2009) 353–369.

[2] S. Fujishige, Submodular functions and optimization, Annals of Discrete Mathematics, Elsevier, Amsterdam, 2005.
[3] S. Fujishige, G. A. Koshevoy, Y. Sano, Matroids on convex geometries (cg-matroids), Discrete Math. 307(15) (2007) 1936–1950.
[4] B. Korte, L. Lovász, R. Schrader, Greedoids, Algorithms and combinatorics, Vol. 4, Springer-Verlag, Berlin, 1991.
[5] J. Oxley, Matroid theory, Oxford University Press, Oxford, 1992.
[6] Y. Sano, Rank functions of strict cg-matroids, Discrete Math. 38(20) (2008) 4734–4744.
[7] Y. Sano, Matroids on convex geometries: subclasses, operations, and optimization, Publ. Res. Inst. Math. Sci. 47(3) (2011) 671–703.
[8] A. Schrijver, Combinatorial optimization. Polyhedra and Efficiency, Algorithms and Combinatorics, Vol. 24, Springer-Verlag, Berlin, 2003.
[9] D. J. A. Welsh, Matroid theory, Academic Press, London, 1976.
[10] H. Whitney, On the abstract properties of linear dependence, Amer. J. Math. 57(3) (1935) 509–533.