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# A New Generalization of Bernstein Polynomials

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Abstract. We will hereby introduce a new generalization of the Schurer, Stancu, Deo, and Izgi operators which are the modifications of the Bernstein polynomials and calculate the rate of approximation for the new operator with the help of the continuity module. Then, by using graphs and numerical values, we will demonstrate that the new general operator yields better results than the above classical operators which can be seen as the basis of the approximation theory.

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**Keywords:** Approximation properties, modulus of continuity, Bernstein operators.

## 1. Introduction

Polynomial approach and the classical approximation theory constitute a basic research area in applied mathematics. The development of the approximation theory played an important role in the numerical solution of partial differential equations, data processing sciences, and many other disciplines. For example, it is widely used in geometric modeling in the aerospace and automotive industries to calculate approximate values with basic functions. Work in this field goes back to the 18th century and still continues as a powerful tool in scientific calculations. Furthermore, it is used in civil engineering projects to analyze the energy efficiency and earthquake resistance data of different types of buildings in thermography calculations and earthquake engineering. The purpose of the approximation theory is to provide an approach between function spaces. In this context, the best approximation uses a linear positive operator. An operator that brings a function of positive value in one function space to another function of positive value in another function space is called a positive operator; whereas the operators that are both positive and linear are called linear positive operators. We will introduce a generalization of Bernstein operators that form the basis of linear positive operators. This new generalization to be defined will be a better version of Bernstein operators that contribute to all of the above mentioned fields of study. In this way, it is aimed to have a better approach. Before introducing the operator, if we need to talk about previous studies. Weierstrass, who laid the foundations of the approach with a linear positive operator, said in 1885 [17] that each continuous function as an element of C [a, b] was a sequence that could be approached with a polynomial in the same closed range, but he did not specify the properties of these sequences. In 1912 Bernstein [4] proved that the sequences in the Weierstrass theorem were the polynomials referred to by his name and exposed them as follows:

$$B_n(h;\varkappa) = \sum_{k=0}^{n} \binom{n}{k} \varkappa^k (1-\varkappa)^{n-k} h\left(\frac{k}{n}\right).$$

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There is a plethora of work on Bernstein polynomials and their generalizations [1,3,5,8–10,13,14]. The best known is the 1953 book "Bernstein Polynomials" by Lorentz [12]. In 1953, Korovkin [11] proved that these operators defined would satisfy the main conditions of approximation theory when they satisfy the Korovkin conditions formulated under his name. In the light of these studies, we will combine a generalization of Bernstein operators. We combined a modification of Bernstein polynomials, namely Schurer [15] 1962, Stancu [16] 1968, Deo et al. [6] 2008, and Izgi [7] 2012, to identify a new operator.

In this operator we defined: $\alpha, \beta, a, b, u, v, p \geq 0$ constant numbers and $a \leq b$, $\alpha + \beta = 1$, $p \in N_0$ is

$$A_n = \left(\frac{n+b}{n+a}\right)^{n+p} \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} h\left[\frac{n+a}{n+b}\left(\beta\frac{k}{n} + \alpha\frac{k+u}{n+p+v}\right)\right], \tag{1.1}$$

where $A_n := A_n(h; \alpha, \beta, a, b, u, v, p; \varkappa)$ and $\varkappa \in \left[0, \frac{n+a}{n+b}\right]$, $h \in C[0,1]$ .

In the operator we defined for $a \leq b$;

*i)* $\beta = 1$, $\alpha = p = 0$ and $a = b$ gives classical Bernstein [4] polynomials,

*ii)* $\alpha = 1, \beta = p = 0$ and $a = b$ gives the Bernstein-Stancu [16] polynomials,

*iii)* $\beta = 1$, $\alpha = 0$ and $a = b$ gives the Bernstein-Schurer [15] polynomials,

*iv)* $\beta = 1$ and $\alpha = p = 0$ gives the Bernstein-Izgi [7] polynomials,

*v)* $\beta = 1$, $\alpha = p = a = 0$ and $b = 1$ gives the Bernstein-Deo [6] polynomials.

It is seen that (1.1) is a generalization of some of the generalized and modified versions of the Bernstein polynomials. Hereinafter the following will be used for convenience:

$$\tilde{A}_n(h; \varkappa) := A_n(h; \alpha, \beta, a, b, u, v, p; \varkappa).$$

In this study, Korovkin conditions will be met and the approach speed of the operator $\tilde{A}_n(h; \varkappa)$ will be calculated with the help of continuity module for a generalization of the Bernstein operator defined in (1.1). Thereafter, we will prove the theorem which shows that the operator meets the Lipschitz condition. Finally, graphics and numerical values will be presented to ensure a better explanation of the approach.

## 2. Estimation Moments

In this section, the $\tilde{A}_n(t^m; \varkappa)$ values will be calculated for $m = 0, 1, 2$ as well as the central moments of $\tilde{A}_n((t-\varkappa)^m; \varkappa)$. Then, the theorem will be presented to demonstrate that the operator meets the Korovkin condition.

**Lemma 2.1.** *$\tilde{A}_n(h; \varkappa)$ the following equations are provided for these polynomials:*

i) $\tilde{A}_n(1; \varkappa) = 1$

ii) $\tilde{A}_n(t; \varkappa) = \varkappa + \left(\frac{\beta p}{n} - \frac{\alpha v}{n+p+v}\right)\varkappa + \alpha \frac{n+a}{n+b}\frac{u}{n+p+v}$

iii) $\tilde{A}_n(t^2; \varkappa) = \varkappa^2 + \left[\zeta_n + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n\right)(1+\zeta_n)\right]\varkappa^2 + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n + 2\alpha u n\vartheta_n + 1\right)\frac{\eta_n}{n}\varkappa + \kappa_n,$

*where*
$\zeta_n = \frac{(2p-1)n + (p-1)p}{n^2}$, $\eta_n = \frac{n+p}{n}\frac{n+a}{n+b}$, $\theta_n = \frac{p+v}{n+p+v}$, $\vartheta_n = \frac{2\beta\alpha u}{n+p+v} + 2\alpha^2 u \frac{n}{(n+p+u)^2}$
*and* $\kappa_n = \left(\frac{\alpha u}{n+p+u}\frac{n+a}{n+b}\right)^2$.

*Proof.*

$$\begin{aligned} i)\tilde{A}_n(1; \varkappa) &= \left(\frac{n+b}{n+a}\right)^{n+p} \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \\ &= \left(\frac{n+b}{n+a}\right)^{n+p} \left(\frac{n+a}{n+b}\right)^{n+p} = 1 \end{aligned}$$

$$\begin{aligned}
ii)\tilde{A}_n(t;\varkappa) &= \left(\frac{n+b}{n+a}\right)^{n+p}\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\left[\frac{n+a}{n+b}\left(\beta\frac{k}{n}+\alpha\frac{k+u}{n+p+v}\right)\right]\\
&= \left(\frac{n+b}{n+a}\right)^{n+p-1}\left[\beta\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{k}{n}\right.\\
&\quad\left.+\ \alpha\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{k+u}{n+p+v}\right]\\
&= \left(\frac{n+b}{n+a}\right)^{n+p-1}\left[\beta\sum_{k=1}^{n+p}\binom{n+p-1}{k-1}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{n+p}{n}\right.\\
&\quad+\alpha\sum_{k=1}^{n+p}\binom{n+p-1}{k-1}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{n+p}{n+p+v}\\
&\quad\left.+\ \alpha\sum_{k=1}^{n+p}\binom{n+p-1}{k-1}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{u}{n+p+v}\right]\\
&= \varkappa+\left(\frac{\beta p}{n}-\frac{\alpha v}{n+p+v}\right)\varkappa+\alpha\frac{n+a}{n+b}\frac{u}{n+p+v}
\end{aligned}$$

$$\begin{aligned}
iii)\tilde{A}_n(t^2;\varkappa) &= \left(\frac{n+b}{n+a}\right)^{n+p}\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\left[\frac{n+a}{n+b}\left(\beta\frac{k}{n}+\alpha\frac{k+u}{n+p+v}\right)\right]^2\\
&= \left(\frac{n+b}{n+a}\right)^{n+p-2}\left[\beta^2\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{k^2}{n^2}\right.\\
&\quad+2\alpha\beta\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\left(\frac{k}{n}\frac{k+u}{n+p+v}\right)\\
&\quad\left.+\ \alpha^2\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\left(\frac{k+u}{n+p+v}\right)^2\right].
\end{aligned}$$

In order to avoid confusion, we will solve each sum in the last statement separately $k^2 = k(k-1)+k$ using the equation,

$$\begin{aligned}
I_1 &= \sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{k(k-1)}{n^2}+\sum_{k=0}^{n+p}\binom{n+p}{k}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{k}{n^2}\\
&= \sum_{k=2}^{n+p}\binom{n+p-2}{k-2}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{(n+p)(n+p-1)}{n^2}\\
&\quad+\sum_{k=1}^{n+p}\binom{n+p-1}{k-1}\varkappa^k\left(\frac{n+a}{n+b}-\varkappa\right)^{n+p-k}\frac{(n+p)}{n^2}.
\end{aligned}$$

Using the equation $k(k+u) = k^2 + uk = k(k-1) + k + uk = k(k-1) + k(u+1)$,

$$\begin{aligned} I_2 &= \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{k(k-1)}{n^2+np+nv} \\ &+ \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{k(u+1)}{n^2+np+nv} \\ &= \sum_{k=2}^{n+p} \binom{n+p-2}{k-2} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{(n+p)(n+p-1)}{n^2+np+nv} \\ &+ \sum_{k=1}^{n+p} \binom{n+p-1}{k-1} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{(n+p)(u+1)}{n^2+np+nv} \end{aligned}$$

is obtained.

Using the equation $(k+u)^2 = k^2 + 2uk + u^2 = k(k-1) + k(2u+1) + u^2$,

$$\begin{aligned} I_3 &= \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{k(k-1)}{(n+p+v)^2} \\ &+ \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{k(2u+1)}{(n+p+v)^2} \\ &+ \sum_{k=0}^{n+p} \binom{n+p}{k} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{u^2}{(n+p+v)^2} \\ &= \sum_{k=2}^{n+p} \binom{n+p-2}{k-2} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{(n+p)(n+p-1)}{(n+p+v)^2} \\ &+ \sum_{k=1}^{n+p} \binom{n+p-1}{k-1} \varkappa^k \left(\frac{n+a}{n+b} - \varkappa\right)^{n+p-k} \frac{(n+p)(2u+1)}{(n+p+v)^2} + \frac{n+a}{n+b} \frac{u^2}{(n+p+v)^2} \end{aligned}$$

is obtained. If the required operation is completed and the $I_1$,$I_2$ and $I_3$ is written in its place in the main equation and $\zeta_n = \frac{(2p-1)n+(p-1)p}{n^2}$, $\eta_n = \frac{n+p}{n}\frac{n+a}{n+b}$, $\theta_n = \frac{p+v}{n+p+v}$, $\vartheta_n = \frac{2\beta\alpha u}{n+p+v} + 2\alpha^2 u \frac{n}{(n+p+u)^2}$ and $\kappa_n = \left(\frac{\alpha u}{n+p+u}\frac{n+a}{n+b}\right)^2$ reductions are used, then;

$$\tilde{A}_n(t^2;\varkappa) = \varkappa^2 + \left(\zeta_n + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n\right)(1+\zeta_n)\right)\varkappa^2 + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n + 2\alpha u n\vartheta_n + 1\right)\frac{\eta_n}{n}\varkappa + \kappa_n$$

is obtained and the proof is completed. □

If $\beta = 1$, $\alpha = p = 0$, $a = b$ taken, then $\zeta_n = -\frac{1}{n}$, $\eta_n = 1$, $\theta_n = \frac{v}{n+v}$, $\vartheta_n = \kappa_n = 0$ and

$$\tilde{A}_n(t^2;\varkappa) = B_n(t^2;\varkappa) = \varkappa^2 + \frac{\varkappa(1-\varkappa)}{n} \tag{2.1}$$

is obtained. As seen, where variables are attributed the required values at each step, the same values are obtained with the classical Bernstein polynomial.

**Lemma 2.2.** *The following moments are provided for the polynomials* $\tilde{A}_n(h;\varkappa)$

*i)* $\tilde{A}_n((t-\varkappa)^0;\varkappa) = 1$

*ii)* $\tilde{A}_n((t-\varkappa)^1;\varkappa) = \left(\frac{\beta p}{n} - \frac{\alpha v}{n+p+v}\right)\varkappa + \alpha\frac{n+a}{n+b}\frac{u}{n+p+v}$

*iii)* $\tilde{A}_n((t-\varkappa)^2;\varkappa) = U_n\varkappa^2 - V_n\varkappa,$

*where*

$$U_n = \left(\zeta_n + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n\right)(1+\zeta_n) - 2\left(\frac{\beta p}{n} - \frac{\alpha v}{n+p+v}\right)\right) \tag{2.2}$$

*and*

$$V_n = 2\alpha\frac{n+a}{n+b}\frac{u}{n+p+v} - (\alpha^2\theta_n^2 - 2\alpha\theta_n + +2\alpha u n\vartheta_n + 1)\frac{\eta_n}{n}. \tag{2.3}$$

Lemma 2.1 and Lemma 2.2 yield the following results:

$$\delta_n = \sqrt{\max_{0\le\varkappa\le\frac{n+a}{n+b}} \tilde{A}_n((t-\varkappa)^2;\varkappa)} = \sqrt{\frac{V_n^2}{2U_n}} = \frac{1}{\sqrt{2}}\frac{V_n}{\sqrt{U_n}},$$

where $\beta = 1$, $\alpha = p = 0$, $a = b$ is taken again,

$$\delta_n = \sqrt{\max_{0\le\varkappa\le\frac{n+a}{n+b}} \tilde{A}_n((t-\varkappa)^2;\varkappa)} = \sqrt{\max_{0\le\varkappa\le\frac{n+a}{n+b}} B_n((t-\varkappa)^2;\varkappa)} = \frac{1}{2\sqrt{n}}.$$

This is the same result as in the classical Bernstein polynomial.

**Theorem 2.3.** *$h \in C[0,1]$ and $h$ are limited to the whole real axis, then*

$$\lim_{n\to\infty} \left\|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right\|_C = 0.$$

*Proof.* From Lemma 2.1

$$\lim_{n\to\infty} \left\|\tilde{A}_n(1;\varkappa) - 1\right\|_C = \lim_{n\to\infty} \max_{0\le\varkappa\le\frac{n+a}{n+b}} \left|\tilde{A}_n(1;\varkappa) - 1\right| = 0,$$

$$\begin{aligned}
\lim_{n\to\infty} \left\|\tilde{A}_n(t;\varkappa) - \varkappa\right\|_C &= \lim_{n\to\infty} \max_{0\le\varkappa\le\frac{n+a}{n+b}} \left|\tilde{A}_n(t;\varkappa) - \varkappa\right| \\
&= \lim_{n\to\infty} \max_{0\le\varkappa\le\frac{n+a}{n+b}} \left|\left(\frac{\beta p}{n} - \frac{\alpha v}{n+p+v}\right)\varkappa + \alpha\frac{n+a}{n+b}\frac{u}{n+p+v}\right| \\
&= \lim_{n\to\infty} \max_{0\le\varkappa\le\frac{n+a}{n+b}} \left|\frac{\beta p}{n} - \alpha\frac{u-v}{n+p+v}\right| \frac{n+a}{n+b} = 0,
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \left\|\tilde{A}_n(t^2;\varkappa) - \varkappa^2\right\|_C &= \lim_{n\to\infty} \max_{0\le\varkappa\le\frac{n+a}{n+b}} \left|\tilde{A}_n(t^2;\varkappa) - \varkappa^2\right| \\
&= \lim_{n\to\infty} \max_{0\le\varkappa\le\frac{n+a}{n+b}} \left|\left(\zeta_n + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n\right)(1+\zeta_n)\right)\varkappa^2 \right. \\
&\quad \left. + \left(\alpha^2\theta_n^2 - 2\alpha\theta_n + 2\alpha u n\vartheta_n + 1\right)\frac{\eta_n}{n}\varkappa + \kappa_n\right| \\
&= 0,
\end{aligned}$$

as the conditions of the Korovkin theorem [2, 11] are met, the theorem is proven. □

## 3. Approximation of $\tilde{A}_n(h;\varkappa)$ Operators

Now, the $\tilde{A}_n(h;\varkappa)$ operator's rate of approximation to the function $h$ will be calculated with the help of the continuity module defined and characterized below:

$$\omega(h;\delta) = \sup_{\substack{\varkappa,t\in\left[0,\frac{n+a}{n+b}\right] \\ |\varkappa-t|\le\delta}} |h(t) - h(\varkappa)|.$$

Best known and most frequently used properties of the continuity module:

$$\omega(h;\delta) \ge 0,$$

$$|h(t) - h(\varkappa)| \le \omega(h;\delta)\left(1 + \frac{|\varkappa - t|}{\delta}\right),$$

$$\omega(h;|\varkappa - t|) \ge |h(t) - h(\varkappa)|.$$

**Theorem 3.1.** *If $h \in C[0,1]$ and $h$ are limited on the whole real axis, then*

$$\left\|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right\|_C \le \left(1 + \frac{\sqrt{2}}{2}\right)\omega\left(h;\frac{V_n}{\sqrt{U_n}}\right).$$

*Proof.* From among the properties of the Lemma 2.2, equation (2.1) and the continuity module,

$$\begin{aligned}
\left|\tilde{A}_n(h;\varkappa)-h(\varkappa)\right| &= \left|\tilde{A}_n(h(t)-h(\varkappa);\varkappa)\right| \\
&\leq \tilde{A}_n(|h(t)-h(\varkappa)|\,;\varkappa) \\
&\leq \tilde{A}_n(\omega\,(h;|t-\varkappa|)\,;\varkappa) \\
&\leq \tilde{A}_n\left(\omega\,(h;\delta_n)\left(1+\frac{|\varkappa-t|}{\delta}\right);\varkappa\right) \\
&= \tilde{A}_n\left(\left(1+\frac{|\varkappa-t|}{\delta}\right);\varkappa\right)\omega\,(h;\delta_n)
\end{aligned}$$

if the linearity of the operator $\tilde{A}_n$ and the Cauchy-Schwarz-Bunyakovsky inequation are used,

$$\leq\left(\tilde{A}_n\,(1;\varkappa)+\frac{1}{\delta_n}\sqrt{\tilde{A}_n((t-\varkappa)^2;\varkappa)}\right)\omega\,(h;\delta_n)\,.$$

If the maximum values are taken from both sides of the equation on the $\left[0,\frac{n+a}{n+b}\right]$ range and (2.3) is used,

$$\left\|\tilde{A}_n(h;\varkappa)-h(\varkappa)\right\|_C\leq\left(1+\frac{1}{\delta_n}\frac{1}{\sqrt{2}}\frac{V_n}{\sqrt{U_n}}\right)\omega\,(h;\delta_n)\,.$$

Again, here, when $\delta_n=\frac{1}{\sqrt{2}}\frac{V_n}{\sqrt{U_n}}$ is chosen, the proof is completed, where $U_n$ and $V_n$ are described as an outcome of Lemma 2.2. □

**Theorem 3.2.** *If* $h$, $[0,1]$ *on a range of* $0<\alpha\leq 1$ *belongs with the class* $Lip_M\alpha$*, i.e.*

$$|h(t)-h(\varkappa)|\leq M\,|t-\varkappa|^{\alpha}\,.$$

*If the Lipschitz condition is met, then*

$$\left\|\tilde{A}_n(h;\varkappa)-h(\varkappa)\right\|_C\leq M\left(\frac{V_n^2}{2U_n}\right)^{\frac{\alpha}{2}}\,.$$

*Proof.*

$$\begin{aligned}
\left|\tilde{A}_n(h;\varkappa)-h(\varkappa)\right| &= \left|\tilde{A}_n(h(t)-h(\varkappa);\varkappa)\right| \\
&\leq \tilde{A}_n(|h(t)-h(\varkappa)|\,;\varkappa) \\
&\leq \tilde{A}_n(M\,|t-\varkappa|^{\alpha}\,;\varkappa)
\end{aligned}$$

if the $\tilde{A}_n$ linearity and the inequality of Cauchy-Schwarz-Bunyakovsky are used

$$\leq M\tilde{A}_n((t-\varkappa)^2;\varkappa)^{\frac{\alpha}{2}}.$$

If the maximum values are taken from both sides of the equation on the $\left[0,\frac{n+a}{n+b}\right]$ range and (2.3) is used,

$$\left\|\tilde{A}_n(h;\varkappa)-h(\varkappa)\right\|_C\leq M\left(\frac{V_n^2}{2U_n}\right)^{\frac{\alpha}{2}}$$

and the proof is completed. □

## 4. Some Plots

In this section, we will demonstrate that the operator we defined with the help of graphs and a table of numerical values (2.2) has a better approach than the operator defined by Bernstein [4], Schurer [15], Stancu [16], Deo [6], and Izgi [7], for $a\leq b$.

**Example 4.1.** Figure 1 shows the $\tilde{A}_n(h;\varkappa)$ operator is approach to the $h(\varkappa)=\frac{\varkappa\sin\pi\varkappa}{1+\varkappa^2}$ (black) function for the values $n=5(pink)$, $n=15$ (blue), $n=25$(green), $n=50$ (yellow), and $n=100$ (red).

**Example 4.2.** Figure 2 shows the $\tilde{A}_n(h;\varkappa)$ operator is approximation to the $h(\varkappa)=\varkappa^3\sin 2\pi\varkappa$ (black) function for the values $n=50$ (blue), $n=100$ (red), and $n=200$ (green).

Figure 1. $\tilde{A}_n(h;\varkappa)$ Operator is approximation to the function $h(\varkappa) = \frac{\varkappa \sin \pi\varkappa}{1+\varkappa^2}$ for different n values

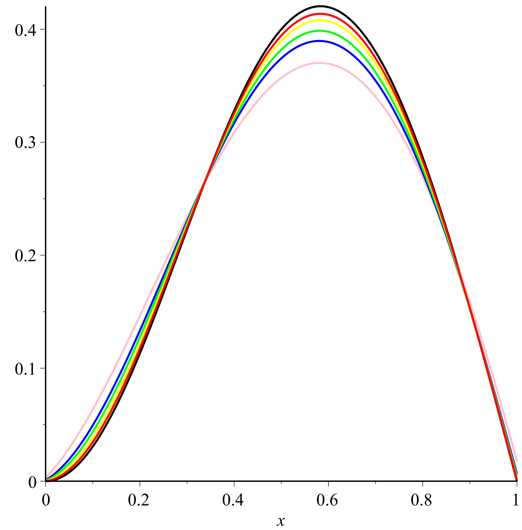
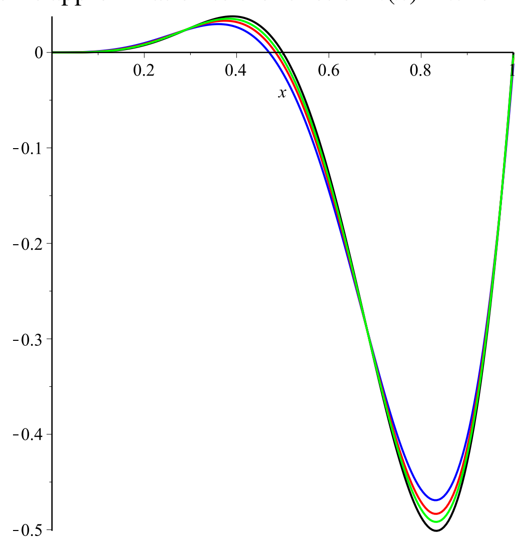



Figure 2. $\tilde{A}_n(h;\varkappa)$ Operator is approximation to the function $h(\varkappa) = \varkappa^3 \sin 2\pi\varkappa$ for different n values




**Example 4.3.** Figure 3 shows how the $\tilde{A}_n(h;\varkappa)$ (red), Schurer (green), Doe (brown), Stancu (blue), Izgi (yellow), and classical Bernstein (pink) operators approach the $h(\varkappa) = \frac{\varkappa \sin \pi\varkappa}{1+\varkappa^2}$ (black) function for the value n=5.

Table 1 shows the numerical values obtained with the maximum value of the statement $\left|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right|$, in order to examine how the $\tilde{A}_n(h;\varkappa)$ operator approximation the function $h(\varkappa) = \varkappa^3 \sin 2\pi\varkappa$ for different $n$, $\varkappa$ values and $a \leq b$.

The definition Izgi [7] provided to compare the approaches of different operators can be given as it comprises statements that can be simplified as the numerator and the denominator. $L_n$ and $T_n$ are operators defined in the same range:

Figure 3. Comparing the operator with the Bernstein, Stancu, Schurer, Izgi, and Doe operators for $n = 5$ value

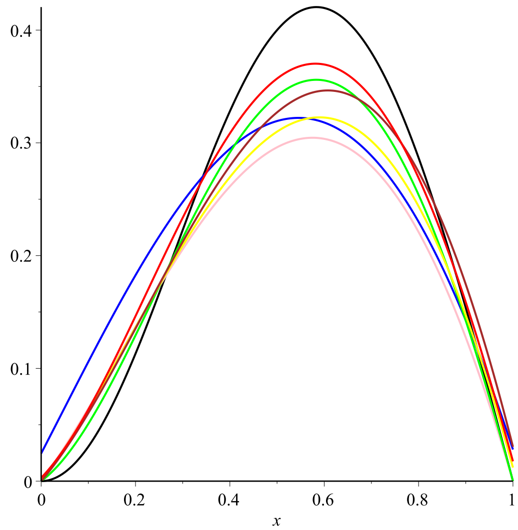



Table 1. Error margins between the $\tilde{A}_n(h;\varkappa)$ operator and $h(\varkappa) = \varkappa^3 \sin 2\pi\varkappa$.

| $n$ | $\varkappa = 0.2$ | $\varkappa = 0.4$ | $\varkappa = 0.6$ | $\varkappa = 0.8$ |
|---|---|---|---|---|
| 50 | 0.002186129458 | 0.01105454068 | 0.0176784371 | 0.0289053429 |
| 100 | 0.001231535748 | 0.00578340652 | 0.0099801896 | 0.0158938469 |
| 250 | 0.000529728394 | 0.00237235555 | 0.0043169198 | 0.0067480187 |
| 500 | 0.000271386963 | 0.00119560429 | 0.0022176598 | 0.0034433621 |
| 900 | 0.000152411674 | 0.00066650362 | 0.0012471474 | 0.0019304993 |

$$\lim_{n\to\infty} \frac{\sup_{a\le\varkappa\le b} |L_n(h;\varkappa) - h(\varkappa)|}{\sup_{a\le\varkappa\le b} |T_n(h;\varkappa) - h(\varkappa)|} = \begin{cases} 0, & T_n,\ \textit{faster} \\ \infty, & L_n,\ \textit{faster} \\ c(\text{constant}), & \textit{equally fast} \end{cases}$$

Based on this definition, it is possible to examine the rate of approximation of the operators defined by Bernstein [4], Schurer [15], Stancu [16], Deo [6], and Izgi [7] and the $\tilde{A}_n(h;\varkappa)$ operator to the function $h(\varkappa) = \frac{\varkappa \sin \pi\varkappa}{1+\varkappa^2}$.

As shown in Figure 3, the operators will be compared in Table 2 for $a \le b$ and different $n$ values in order to use the $\varkappa = 0.6$ point where the difference is seen more clearly. First of all, $E_1$ for Bernstein [4], $E_2$ for Schurer [15], $E_3$ for Stancu [16], $E_4$ for Deo [6] and $E_5$ for Izgi [7] ;

$$E_1 = \frac{\sup_{a\le\varkappa\le b} \left|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right|}{\sup_{a\le\varkappa\le b} |B_n(h;\varkappa) - h(\varkappa)|}, \quad E_2 = \frac{\sup_{a\le\varkappa\le b} \left|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right|}{\sup_{a\le\varkappa\le b} |Sr_n(h;\varkappa) - h(\varkappa)|},$$

$$E_3 = \frac{\sup_{a\le\varkappa\le b} \left|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right|}{\sup_{a\le\varkappa\le b} |St_n(h;\varkappa) - h(\varkappa)|}, \quad E_4 = \frac{\sup_{a\le\varkappa\le b} \left|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right|}{\sup_{a\le\varkappa\le b} |D_n(h;\varkappa) - h(\varkappa)|},$$

$$E_5 = \frac{\sup_{a\le\varkappa\le b} \left|\tilde{A}_n(h;\varkappa) - h(\varkappa)\right|}{\sup_{a\le\varkappa\le b} |Z_n(h;\varkappa) - h(\varkappa)|}$$

defined.

Table 2. Rate of errors between the operator $\tilde{A}_n(h;\varkappa)$ and other operators.

| $n$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---|---|---|---|---|---|
| 50 | 0.029833790240 | 0.029833790240 | 0.50781478720 | 0.029833790010 | 0.39025536560 |
| 100 | 0.016082703360 | 0.016082702880 | 0.30950848060 | 0.016082700060 | 0.25906284560 |
| 250 | 0.006745264924 | 0.006745265402 | 0.13877664440 | 0.006745264908 | 0.12734821250 |
| 500 | 0.003427741602 | 0.003427742078 | 0.07199830767 | 0.003427741598 | 0.06877087011 |
| 900 | 0.001918176732 | 0.001918176017 | 0.04065166956 | 0.001918176703 | 0.03959891401 |

Table 2 shows that our operator $\tilde{A}_n(h;\varkappa)$ approaches the $h(\varkappa) = \frac{\varkappa \sin \pi\varkappa}{1+\varkappa^2}$ function better than the operators of Bernstein [4], Schurer [15], Stancu [16], Deo [6], and Izgi [7].

## 5. Conclusions

As a result, the new generalization Bernstein operators that we have defined displays a better approximation than the more studied Bernstein [4], Schurer [15], Stancu [16], Deo [6], and Izgi [7] operators. We have clearly shown this with numerical value tables and graphs.

## Data Availability

All data generated or analysed during this study are included in this published article (and its supplementary information files).

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## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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