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# Kenmotsu Manifolds with Generalized Tanaka-Webster Connection

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## Abstract

The object of the present paper is to study generalized Tanaka-Webster connection on a Kenmotsu manifold. Some conditions for $\varphi$-conformally flat, $\varphi$-conharmonically flat, $\varphi$-concircularly flat, $\varphi$-projectively flat, $\varphi$- $W_2$ flat and $\varphi$-pseudo projectively flat Kenmotsu manifolds with respect to generalized Tanaka-Webster connection are obtained.

*Keywords*: Kenmotsu Manifold, Einstein Manifold, Curvature Tensor, Tanaka-Webster Connection.

## Genelleştirilmiş Tanaka-Webster Konneksiyonlu Kenmotsu Manifoldlar

## Özet

Bu çalışmada bir Kenmotsu manifold üzerinde genelleştirilmiş Tanaka-Webster konneksiyonu çalışıldı. Genelleştirilmiş Tanaka-Webster konneksiyonuna sahip $\varphi$ -conformally flat, $\varphi$ -conharmonically flat, $\varphi$ -concircularly flat, $\varphi$ -projectively flat, $\varphi$ -$W_2$ flat ve $\varphi$ -pseudo projectively flat Kenmotsu manifoldlar için bazı şartlar elde edildi.

*Anahtar Kelimeler*: Kenmotsu Manifold, Einstein Manifold, Eğrilik Tensörü, Tanaka-Webster Konneksiyon.

## 1. Introduction

In [10], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such manifolds, the sectional curvature of plane sections containing $\xi$ is a constant $c$ and it was proved that they can be divided into three classes [10]:

$(i)$ Homogeneous normal contact Riemannian manifolds with $c > 0$,

$(ii)$ Global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$,

$(iii)$ A warped product space $\mathbb{R} \times_f \mathbb{C}$ if $c < 0$.

It is known that the manifolds of class ($i$) are characterized by admitting a Sasakian structure. The differential geometric properties of the manifolds of class ($iii$) investigated by Kenmotsu [5] and the obtained structure is now known as Kenmotsu structure. In general, these structures are not Sasakian [5]. Kenmotsu manifolds have been studied by many authors such as De and Pathak [2], Jun, De and Pathak [4], Özgür and De [6], Yıldız and De [14], Yıldız, De and Acet [15] and many others.

On the other hand, the Tanaka-Webster connection [9,12] is the canonical of fine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. Tanno [11] defined the generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

In this paper, Kenmotsu manifolds with generalized Tanaka-Webster connection are studied. Section 2 is devoted to some basic definitions. In section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to generalized Tanaka-Webster connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to the generalized Tanaka-Webster connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In section 4, conformal curvature tensor of generalized Tanaka-Webster connection is studied. In section 5, it is proved that a $\varphi$-conharmonically flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection is an $\eta$-Einstein manifold. Section 6 and 7, contain some results for $\varphi$-concircularly flat and $\varphi$-projectively flat Kenmotsu manifolds with generalized Tanaka-Webster connection, respectively. In section 8, we study $\varphi$- $W_2$ flat Kenmotsu manifolds with respect to generalized Tanaka-Webster connection. In the last section, we show that

$\varphi$-pseudo projectively flat Kenmotsu manifolds with respect to generalized Tanaka-Webster is an $\eta$-Einstein manifold.

**2. Preliminaries**

We recall some general definitions and basic formulas for late use.

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact Riemannian manifold, where $\varphi$ is a $(1,1)-$tensor field, $\xi$ is the structure vector field, $\eta$ is a $1-$form and $g$ is the Riemannian metric. It is well known that the $(\varphi, \xi, \eta, g)$ structure satisfies the conditions [1]

$$\varphi\xi = 0, \eta(\varphi X) = 0, \eta(\xi) = 1 \tag{1}$$

$$\varphi^2 X = -X + \eta(X)\xi \tag{2}$$

$$g(X, \xi) = \eta(X) \tag{3}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{4}$$

for any vector field $X$ and $Y$ on $M$. Moreover, if

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X \tag{5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$

where $\nabla$ denotes Levi-Civita connection on $M$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold.

In this case, it is well known that [5]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{7}$$

$$S(X, \xi) = -2n\eta(X), \tag{8}$$

where $S$ denotes the Ricci tensor. From (7), we can easily see that

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X \tag{9}$$

$$R(X, \xi)\xi = \eta(X)\xi - X. \tag{10}$$

Since $S(X, Y) = g(QX, Y)$, we have

$$S(\varphi X, \varphi Y) = g(Q\varphi X, \varphi Y),$$

where $Q$ is the Ricci operator.

Using the properties (2) and (8), we get

$$S(\varphi X, \varphi Y) = S(X, Y) + (2n)\eta(X)\eta(Y), \tag{11}$$

by virtue of $g(X, \varphi Y) = -g(\varphi X, Y)$ and $Q\varphi = \varphi Q$. Also we have

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{12}$$

A Kenmotsu manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{13}$$

for any vector fields $X$ and $Y$, where $a$ and $b$ are functions on $M$.

The generalized Tanaka-Webster connection [11] $\bar{\nabla}$ for a contact metric manifold $M$ is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)Y \cdot \xi - \eta(Y)\nabla_X \xi + \eta(X)\varphi Y, \tag{14}$$

for all vector fields $X$ and $Y$, where $\nabla$ is Levi-Civita connection on $M$.

By using (6), the generalized Tanaka-Webster connection $\bar{\nabla}$ for a Kenmotsu manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X + \eta(X)\varphi Y, \tag{15}$$

for all vector fields $X$ and $Y$.

**3. Curvature Tensor**

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold. The curvature tensor $\bar{R}$ of $M$ with respect to the generalized Tanaka-Webster connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{16}$$

Then, in a Kenmotsu manifold, we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y, \tag{17}$$

where $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{X,Y]} Z$ is the curvature tensor of $M$ with respect to Levi-Civita connection $\nabla$.

**Theorem 3.1** *In a Kenmotsu manifold, Riemannian curvature tensor with respect to the generalized Tanaka-Webster connection* $\bar{\nabla}$ *has following properties*

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0 \tag{18}$$

$$\bar{R}(X,Y,Z,V) + \bar{R}(Y,X,Z,V) = 0 \tag{19}$$

$$\bar{R}(X,Y,Z,V) + \bar{R}(X,Y,V,Z) = 0 \tag{20}$$

$$\bar{R}(X,Y,Z,V) - \bar{R}(Z,V,X,Y) = 0, \tag{21}$$

*where* $\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$.

The Ricci tensor $\bar{S}$ and the scalar curvature $\bar{\tau}$ of the manifold $M$ with respect to the generalized Tanaka-Webster connection $\bar{\nabla}$ are defined by

$$\bar{S}(X,Y) = \sum_{i=1}^{n} g(\bar{R}(e_i,X)Y,e_i) \quad + \sum_{i=1}^{n} g(\bar{R}(\varphi e_i,X)Y,\varphi e_i)$$

$$+g(\bar{R}(\xi,X)Y,\xi) \tag{22}$$

$$\bar{\tau} = \sum_{i=1}^{n} \bar{S}(e_i,e_i) \quad + \sum_{i=1}^{n} \bar{S}(\varphi e_i,\varphi e_i) \quad + \bar{S}(\xi,\xi), \tag{23}$$

respectively, where $\{e_i, \varphi e_i, \xi\}$, $(i = 1,2,\ldots,n)$, is an orthonormal $\varphi$-basis of $M$.

**Lemma 3.1** *Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with the generalized Tanaka-Webster connection $\bar{\nabla}$. Then, we have*

$$\bar{R}(X,Y)\xi = \bar{R}(\xi,X)Y = \bar{R}(\xi,X)\xi = 0 \tag{24}$$

$$\bar{S}(X,\xi) = 0, \tag{25}$$

*for all $X,Y,Z \in TM$.*

Moreover, on a (2n+1)-dimensional Kenmotsu manifolod M, we have

$$\bar{S}(X,Y) = S(X,Y) + 2ng(X,Y) \tag{26}$$

$$\bar{\tau} = \tau + 4n^2 + 2n, \tag{27}$$

where $S$ and $\tau$ denote the Ricci tensor and scalar curvature of Levi-Civita connection $\nabla$, respectively. From (26), it is obvious that $\bar{S}$ is symmetric.

## 4. $\varphi$-Conformally Flat Kenmotsu Manifold With Generalized Tanaka-Webster Connection

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. The conformal curvature tensor [13] of $M$ is defined by

$$\bar{C}(X,Y)V = \bar{R}(X,Y)V - \frac{1}{2n-1}\begin{pmatrix}\bar{S}(Y,V)X - \bar{S}(X,V)Y \\ +g(Y,V)\bar{Q}X - g(X,V)\bar{Q}Y\end{pmatrix}$$

$$+\frac{\bar{\tau}}{2n(2n-1)}(g(Y,V)X - g(X,V)Y). \tag{28}$$

By using (17), (26) and (27) in (28), we obtain

$$\bar{C}(X,Y)V = R(X,Y)V + g(Y,V)X - g(X,V)Y \tag{29}$$
$$-\frac{1}{2n-1}\begin{pmatrix} S(Y,V)X + 2ng(Y,V)X \\ -S(X,V)Y - 2ng(X,V)Y \\ +g(Y,V)QX + 2ng(Y,V)X \\ -g(X,V)QY - 2ng(X,V)Y \end{pmatrix}$$
$$+\frac{\tau+4n^2+2n}{2n(2n-1)}(g(Y,V)X - g(X,V)Y).$$

**Definition 4.1** *A differentiable manifold $M$ satisfying the condition*

$$\varphi^2\bar{C}(\varphi X,\varphi Y)\varphi U = 0, \tag{30}$$

*is called $\varphi$-conformally flat.*

It can be easily seen that $\varphi^2\bar{C}(\varphi X,\varphi Y)\varphi U = 0$ holds if and only if

$$g(\bar{C}(\varphi X,\varphi Y)\varphi U,\varphi V) = 0, \tag{31}$$

for any $X,Y,U,V \in TM$.

In view of (28), $\varphi$-conformally flatness means that

$$g(\bar{R}(\varphi X,\varphi Y)\varphi U,\varphi V) = \frac{1}{2n-1}\begin{pmatrix} \bar{S}(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -\bar{S}(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ +\bar{S}(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -\bar{S}(\varphi Y,\varphi V)g(\varphi X,\varphi U) \end{pmatrix}$$
$$-\frac{\bar{\tau}}{2n(2n-1)}\begin{pmatrix} g(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -g(\varphi X,\varphi U)g(\varphi Y,\varphi V) \end{pmatrix}. \tag{32}$$

Using (17), (26) and (27), from (32) we have

$$g(R(\varphi X,\varphi Y)\varphi U,\varphi V) + g(\varphi Y,\varphi U)g(\varphi X,\varphi V) - g(\varphi X,\varphi U)g(\varphi Y,\varphi V)$$
$$= \frac{1}{2n-1}\begin{pmatrix} S(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ +2ng(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -S(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ -2ng(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ +S(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ +2ng(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -S(\varphi Y,\varphi V)g(\varphi X,\varphi U) \\ -2ng(\varphi Y,\varphi V)g(\varphi X,\varphi U) \end{pmatrix}$$

$$-\frac{\tau+4n^2+2n}{2n(2n-1)}\begin{pmatrix} g(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -g(\varphi X,\varphi U)g(\varphi Y,\varphi V) \end{pmatrix}. \tag{33}$$

Choosing $\{e_i, \varphi e_i, \xi\}$ as an orthonormal $\varphi$-basis of $M$ and contraction of (33) with respect to $X$ and $V$ we obtain

$$S(\varphi Y,\varphi U)+2ng(\varphi Y,\varphi U)=\frac{1}{2n-1}\begin{pmatrix}(2n-2)S(\varphi Y,\varphi U)\\+(8n^2+\tau-2n)g(\varphi Y,\varphi U)\end{pmatrix}$$
$$-\frac{\tau+4n^2+2n}{2n(2n-1)}((2n-1)g(\varphi Y,\varphi U)), \quad (34)$$

for any vector fields $Y$ and $U$ on $M$. From equations (4) and (11), we get

$$S(Y,U)=\left(\frac{\tau+2n}{2n}\right)g(Y,U)-\left(\frac{\tau+4n^2+2n}{2n}\right)\eta(Y)\eta(U),$$

which implies that $M$ is an $\eta$-Einstein manifold.

Therefore, we have the following.

**Theorem 4.1** *Let $M$ be a $(2n+1)$-dimensional $\varphi$-conformally flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Then $M$ is an $\eta$-Einstein manifold.*

## 5. $\varphi$-Conharmonically Flat Kenmotsu Manifold With Generalized Tanaka-Webster Connection

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. The conharmonic curvature tensor [3] of $M$ is defined by

$$\bar{K}(X,Y)V=\bar{R}(X,Y)V-\frac{1}{2n-1}\begin{pmatrix}\bar{S}(Y,V)X-\bar{S}(X,V)Y\\+g(Y,V)\bar{Q}X-g(X,V)\bar{Q}Y\end{pmatrix}. \quad (35)$$

By using (17), (26) and (27), we obtain from (35)

$$\bar{K}(X,Y)V=R(X,Y)V+g(Y,V)X-g(X,V)Y$$
$$-\frac{1}{2n-1}\begin{pmatrix}S(Y,V)X+2ng(Y,V)X\\-S(X,V)Y-2ng(X,V)Y\\+g(Y,V)QX+2ng(Y,V)X\\-g(X,V)QY-2ng(X,V)Y\end{pmatrix}. \quad (36)$$

**Definition 5.1** *A differentiable manifold M satisfying the condition*

$$\varphi^2\bar{K}(\varphi X,\varphi Y)\varphi U=0, \quad (37)$$

*is called $\varphi$-conharmonically flat.*

It can be easily seen that $\varphi^2\bar{K}(\varphi X,\varphi Y)\varphi U = 0$ holds if and only if

$$g(\bar{K}(\varphi X,\varphi Y)\varphi U,\varphi V) = 0, \tag{38}$$

for any $X, Y, U, V \in TM$.

If $M$ is a $(2n+1)$-dimensional $\varphi$-conharmonically flat Kenmotsu manifold then we have

$$g(\bar{R}(\varphi X,\varphi Y)\varphi U,\varphi V) = \frac{1}{2n-1}\begin{pmatrix} \bar{S}(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -\bar{S}(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ +\bar{S}(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -\bar{S}(\varphi Y,\varphi V)g(\varphi X,\varphi U) \end{pmatrix}, \tag{39}$$

in view of (35). By using (17), (26) and (27) in (39), we have

$$g(R(\varphi X,\varphi Y)\varphi U,\varphi V) + g(\varphi Y,\varphi U)g(\varphi X,\varphi V) - g(\varphi X,\varphi U)g(\varphi Y,\varphi V)$$

$$= \frac{1}{2n-1}\begin{pmatrix} S(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ +2ng(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -S(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ -2ng(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ +S(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ +2ng(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -S(\varphi Y,\varphi V)g(\varphi X,\varphi U) \\ -2ng(\varphi Y,\varphi V)g(\varphi X,\varphi U) \end{pmatrix}. \tag{40}$$

Since $\{e_i, \varphi e_i, \xi\}$ is an orthonormal basis of vector fields on $M$, a suitable contraction of (40) with respect to $X$ and $V$ gives

$$S(\varphi Y,\varphi U) + 2ng(\varphi Y,\varphi U) = \frac{1}{2n-1}\begin{pmatrix} (2n-2)S(\varphi Y,\varphi U) \\ +(8n^2+\tau-2n)g(\varphi Y,\varphi U) \end{pmatrix}, \tag{41}$$

for any vector fields $Y$ and $U$ on $M$.

From equations (4) and (11), we get

$$S(Y,U) = (\tau+4n^2)g(Y,U) - (\tau+4n^2+2n)\eta(Y)\eta(U),$$

which implies that $M$ is an $\eta$-Einstein manifold.

Hence, we have the following.

**Theorem 5.1** *Let $M$ be a $(2n+1)$-dimensional $\varphi$-conharmonically flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Then $M$ is an $\eta$-Einstein manifold.*

## 6. $\varphi$ -Concircularly Flat Kenmotsu Manifold With Generalized Tanaka-Webster Connection

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. The concircular curvature tensor [13] of $M$ is defined by;

$$\bar{Z}(X,Y)V = \bar{R}(X,Y)V - \frac{\bar{\tau}}{2n(2n-1)}(g(Y,V)X - g(X,V)Y). \tag{42}$$

From (17), (27) and (42), we get

$$\begin{aligned}\bar{Z}(X,Y)V = R(X,Y)V + g(Y,V)X - g(X,V)Y \\ - \frac{\tau+4n^2+2n}{2n(2n-1)}(g(Y,V)X - g(X,V)Y).\end{aligned} \tag{43}$$

**Definition 6.1** *A differentiable manifold $M$ satisfying the condition*

$$\varphi^2\bar{Z}(\varphi X,\varphi Y)\varphi U = 0, \tag{44}$$

*is called $\varphi$-concircularly flat.*

It is obvious that (44) holds if and only if

$$g(\bar{Z}(\varphi X,\varphi Y)\varphi U,\varphi V) = 0, \tag{45}$$

for any $X,Y,U,V \in TM$.

On a $(2n+1)$-dimensional $\varphi$-concircularly flat Kenmotsu manifold, we obtain

$$g(\bar{R}(\varphi X,\varphi Y)\varphi U,\varphi V) = \frac{\tau+4n^2+2n}{2n(2n-1)}\begin{pmatrix} g(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -g(\varphi X,\varphi U)g(\varphi Y,\varphi V)\end{pmatrix}, \tag{46}$$

by virtue of (42). Using (17) in the last equation above, we have

$$\begin{aligned}g(R(\varphi X,\varphi Y)\varphi U,\varphi V) + g(\varphi Y,\varphi U)g(\varphi X,\varphi V) - g(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ = \frac{\tau+4n^2+2n}{2n(2n-1)}\begin{pmatrix} g(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -g(\varphi X,\varphi U)g(\varphi Y,\varphi V)\end{pmatrix}.\end{aligned} \tag{47}$$

Taking into account the orthonormal $\varphi$-basis $\{e_i,\varphi e_i,\xi\}$ of $M$ and contraction of (47) gives

$$S(\varphi Y,\varphi U) + 2ng(\varphi Y,\varphi U) = \frac{\tau+4n^2+2n}{2n(2n-1)}((2n-1)g(\varphi Y,\varphi U)), \tag{48}$$

for any vector fields $Y$ and $U$ on $M$.

From (4) and (11), we get

$$S(Y,U) = \left(\frac{\tau+2n}{2n}\right) g(Y,U) - \left(\frac{\tau+4n^2+2n}{2n}\right) \eta(Y)\eta(U),$$

which implies that $M$ is an $\eta$-Einstein manifold.

Therefore we have the following.

**Theorem 6.1** *Let $M$ be a $(2n+1)$-dimensional $\varphi$-concircularly flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Then $M$ is an $\eta$-Einstein manifold.*

## 7. $\varphi$-Projectively Flat Kenmotsu Manifold With Generalized Tanaka-Webster Connection

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. The projective curvature tensor [13] of $M$ is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}(\bar{S}(Y,Z)X - \bar{S}(X,Z)Y). \tag{49}$$

By using (17) and (26), from (49) we obtain

$$\bar{P}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y - \frac{1}{2n}\begin{pmatrix} S(Y,Z)X + 2ng(Y,Z)X \\ -S(X,Z)Y - 2ng(X,Z)Y \end{pmatrix}. \tag{50}$$

**Definition 7.1** *A differentiable manifold M satisfying the condition*

$$\varphi^2\bar{P}(\varphi X,\varphi Y)\varphi U = 0, \tag{51}$$

*is called $\varphi$-projectively flat*.

One can easily see that $\varphi^2\bar{P}(\varphi X,\varphi Y)\varphi U = 0$ holds if and only if

$$g(\bar{P}(\varphi X,\varphi Y)\varphi U,\varphi V) = 0, \tag{52}$$

for any $X,Y,U,V \in TM$.

In view of (49), on a $(2n+1)$-dimensional $\varphi$-projectively flat Kenmotsu manifold, we have

$$g(\bar{R}(\varphi X,\varphi Y)\varphi U,\varphi V) = \frac{1}{2n}\begin{pmatrix} \bar{S}(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -\bar{S}(\varphi X,\varphi U)g(\varphi Y,\varphi V) \end{pmatrix}. \tag{53}$$

Then from (53), we have

$$g(R(\varphi X,\varphi Y)\varphi U,\varphi V)+g(\varphi Y,\varphi U)g(\varphi X,\varphi V)-g(\varphi X,\varphi U)g(\varphi Y,\varphi V)$$
$$=\frac{1}{2n}\begin{pmatrix} S(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ +2ng(\varphi Y,\varphi U)g(\varphi X,\varphi V) \\ -S(\varphi X,\varphi U)g(\varphi Y,\varphi V) \\ -2ng(\varphi X,\varphi U)g(\varphi Y,\varphi V) \end{pmatrix}, \tag{54}$$

by virtue of (17) and (26).

Choosing $\{e_i,\varphi e_i,\xi\}$ as an orthonormal $\varphi$-basis of $M$ and so by suitable contraction of (54) with respect to $X$ and $V$ we obtain

$$S(\varphi Y,\varphi U)+2ng(\varphi Y,\varphi U)=\frac{1}{2n}\begin{pmatrix}(2n-1)S(\varphi Y,\varphi U) \\ +(4n^2-2n)g(\varphi Y,\varphi U)\end{pmatrix}, \tag{55}$$

for any vector fields $Y$ and $U$ on $M$.
From equations (4) and (11), we get

$$S(Y,U)=-2ng(Y,U),$$

which implies that $M$ is an Einstein manifold.
Hence, we have the following.

**Theorem 7.1** *Let $M$ be a $(2n+1)$-dimensional $\varphi$-projectively flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Then $M$ is an Einstein manifold.*

## 8. $\varphi$-W $_2$ Flat Kenmotsu Manifold With Generalized Tanaka-Webster Connection

In [7] Pokhariyal and Mishra have introduced new tensor fields, called $W_2$ and $E$-tensor field, in a Riemannian manifold and study their properties.
The curvature tensor $W_2$ is defined by

$$W_2(X,Y,Z,V)=R(X,Y,Z,V)+\frac{1}{n-1}(g(X,Z)S(Y,V)-g(Y,Z)S(X,V)),$$

where $S$ is a Ricci tensor of type $(0,2)$.
Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. The $\overline{W}_2$-curvature tensor of $M$ is defined by

$$\overline{W}_2(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{2n}(g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y). \tag{56}$$

By using (17) and (27) in the last equation above we obtain

$$\bar{W}_2(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y - \frac{1}{2n}\begin{pmatrix} g(Y,Z)QX + 2ng(Y,Z)X \\ -g(X,Z)QY - 2ng(X,Z)Y \end{pmatrix}. \tag{57}$$

**Definition 8.1** *A differentiable manifold* $M$ *satisfying the condition*

$$\varphi^2\bar{W}_2(\varphi X,\varphi Y)\varphi U = 0, \tag{58}$$

*is called* $\varphi$-$W_2$ *flat.*

It can be easily seen that $\varphi^2\bar{W}_2(\varphi X,\varphi Y)\varphi U = 0$ holds if and only if

$$g(\bar{W}_2(\varphi X,\varphi Y)\varphi U,\varphi V) = 0, \tag{59}$$

for any $X,Y,U,V \in TM$.

In view of (56), $\varphi$-$W_2$ flatness on a $(2n+1)$-dimensional Kenmotsu manifold means that

$$g(\bar{R}(\varphi X,\varphi Y)\varphi U,\varphi V) = \frac{1}{2n}\begin{pmatrix} \bar{S}(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -\bar{S}(\varphi Y,\varphi V)g(\varphi X,\varphi U) \end{pmatrix}. \tag{60}$$

Then we have

$$g(R(\varphi X,\varphi Y)\varphi U,\varphi V) + g(\varphi Y,\varphi U)g(\varphi X,\varphi V) - g(\varphi X,\varphi U)g(\varphi Y,\varphi V)$$
$$= \frac{1}{2n}\begin{pmatrix} S(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ +2ng(\varphi X,\varphi V)g(\varphi Y,\varphi U) \\ -S(\varphi Y,\varphi V)g(\varphi X,\varphi U) \\ -2ng(\varphi Y,\varphi V)g(\varphi X,\varphi U) \end{pmatrix}, \tag{61}$$

via (17), (26) and (60).

Let $\{e_i,\varphi e_i,\xi\}$ be an orthonormal $\varphi$-basis of $M$. If we contract (61) with respect to $X$ and $V$ we get

$$S(\varphi Y,\varphi U) + 2ng(\varphi Y,\varphi U) = \frac{1}{2n}\begin{pmatrix} (\tau + 4n^2)g(\varphi Y,\varphi U) \\ -S(\varphi Y,\varphi U) \end{pmatrix}, \tag{62}$$

for any vector fields $Y$ and $U$ on $M$.

From equations (4) and (11), then we get

$$S(Y,U) = \left(\frac{\tau}{2n+1}\right)g(Y,U) - \left(\frac{\tau+4n^2+2n}{2n+1}\right)\eta(Y)\eta(U),$$

which implies that $M$ is an $\eta$-Einstein manifold.

Therefore, we have the following.

**Theorem 8.1** *Let* $M$ *be a* $(2n+1)$*-dimensional* $\varphi$-$W_2$ *flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Then* $M$ *is an* $\eta$*-Einstein manifold.*

## 9. $\varphi$-Pseudo Projectively Flat Kenmotsu Manifold with Generalized Tanaka-Webster Connection

Prasad [8] defined and studied a tensor field $\bar{P}$ on a Riemannian manifold of dimension $n$, which includes projective curvature tensor $P$. This tensor field $\bar{P}$ is known as pseudo-projective curvature tensor.

In this section, we study pseudo-projective curvature tensor in a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection $\bar{\nabla}$ and we denote this curvature tensor with $\bar{P}\bar{P}$.

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. The pseudo-projective curvature tensor $\bar{P}\bar{P}$ of $M$ with generalized Tanaka-Webster connection $\bar{\nabla}$ is defined by

$$\bar{P}\bar{P}(X,Y)V = a\bar{R}(X,Y)V + b(\bar{S}(Y,V)X - \bar{S}(X,V)Y) - \frac{\bar{\tau}}{(2n+1)}\left(\frac{a}{2n} + b\right)(g(Y,V)X - g(X,V)Y), \tag{63}$$

where $a$ and $b$ are constants such that $a,\ b \neq 0$.

If $a = 1$ and $b = \frac{1}{2n+2}$ then $(63)$ takes the form

$$\bar{P}\bar{P}(X,Y)V = \bar{R}(X,Y)V + \frac{1}{2n+2}(\bar{S}(Y,V)X - \bar{S}(X,V)Y) - \frac{\bar{\tau}}{(2n+2)n}(g(Y,V)X - g(X,V)Y). \tag{64}$$

By using $(17)$, $(26)$ and $(27)$ in $(64)$, we get

$$\bar{P}\bar{P}(X,Y)V = R(X,Y)V + g(Y,V)X - g(X,V)Y + \frac{1}{2n+2}\begin{pmatrix} S(Y,V)X + 2ng(Y,V)X \\ -S(X,V)Y - 2ng(X,V)Y \end{pmatrix} - \frac{\tau + 4n^2 + 2n}{(2n+2)n}(g(Y,V)X - g(X,V)Y). \tag{65}$$

**Definition 9.1** *A differentiable manifold M satisfying the condition*

$$\varphi^2\bar{P}\bar{P}(\varphi X, \varphi Y)\varphi U = 0, \tag{66}$$

*is called $\varphi$-pseudo projectively flat.*

It can be easily seen that $\varphi^2 \bar{P}\bar{P}(\varphi X, \varphi Y)\varphi U = 0$ holds if and only if

$$g(\bar{P}\bar{P}(\varphi X, \varphi Y)\varphi U, \varphi V) = 0, \tag{67}$$

for any $X, Y, U, V \in TM$.

One can easily see that on a $\varphi$-pseudo projectively flat Kenmotsu manifold,

$$g(\bar{R}(\varphi X, \varphi Y)\varphi U, \varphi V) = \frac{1}{2n+2}\begin{pmatrix} \bar{S}(\varphi X, \varphi U)g(\varphi Y, \varphi V) \\ -\bar{S}(\varphi Y, \varphi U)g(\varphi X, \varphi V) \end{pmatrix} + \frac{\bar{\tau}}{(2n+2)n}\begin{pmatrix} g(\varphi Y, \varphi U)g(\varphi X, \varphi V) \\ -g(\varphi X, \varphi U)g(\varphi Y, \varphi V) \end{pmatrix}, \tag{68}$$

holds, in view of (63). Using equations (17), (26) and (27) in (68), we have

$$\begin{aligned} &g(R(\varphi X, \varphi Y)\varphi U, \varphi V) + g(\varphi Y, \varphi U)g(\varphi X, \varphi V) - g(\varphi X, \varphi U)g(\varphi Y, \varphi V) \\ &= \frac{1}{2n+2}\begin{pmatrix} S(\varphi X, \varphi U)g(\varphi Y, \varphi V) \\ +2ng(\varphi X, \varphi U)g(\varphi Y, \varphi V) \\ -S(\varphi Y, \varphi U)g(\varphi X, \varphi V) \\ -2ng(\varphi Y, \varphi U)g(\varphi X, \varphi V) \end{pmatrix} \\ &+ \frac{\tau + 4n^2 + 2n}{(2n+2)n}(g(\varphi Y, \varphi U)g(\varphi X, \varphi V) - g(\varphi X, \varphi U)g(\varphi Y, \varphi V)). \end{aligned} \tag{69}$$

Choosing $\{e_i, \varphi e_i, \xi\}$ as an orthonormal basis of vector fields in $M$ and contracting (69), we obtain

$$S(\varphi Y, \varphi U) + 2ng(\varphi Y, \varphi U) = \frac{1}{2n+2}\begin{pmatrix} (1-2n)S(\varphi Y, \varphi U) \\ -(4n^2 - 2n)g(\varphi Y, \varphi U) \end{pmatrix} + \frac{\tau + 4n^2 + 2n}{(2n+2)n}((2n-1)g(\varphi Y, \varphi U)), \tag{70}$$

for any vector fields $Y$ and $U$ on $M$.

From equations (4) and (11), we get

$$S(Y, U) = \left(\frac{\tau(2n-1) - 2n(n+1)}{4n^2 + n}\right)g(Y, U) - \left(\frac{\tau(2n-1) + 2n(4n^2-1)}{4n^2 + n}\right)\eta(Y)\eta(U),$$

which implies that $M$ is an $\eta$-Einstein manifold.

Therefore, we have the following.

**Theorem 9.1** *Let $M$ be a $(2n+1)$-dimensional $\varphi$-pseudo projectively flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection. Then $M$ is an $\eta$-Einstein manifold.*

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