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Resolvent Operator of the Matrix Schrödinger Equation on the Half-Line with Quasi-selfadjoint Potential

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Keywords

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Abstract

We obtain the resolvent operator of the matrix Schrödinger equation on the half-line with a quasi-selfadjoint matrix potential Q. We also assume each entry of Q is Lebesgue measurable on $(0,\infty)$ and Q has a finite first moment. We impose the general boundary condition at $x = 0$. This boundary value problem is not selfadjoint which makes it valuable and difficult in terms of the spectral analysis. Moreover, considering the most general boundary conditions generalizes many studies in the literature. We introduce the Jost matrix of this boundary value problem. We examine asymptotical and analytical properties of the Jost matrix in order to derive the resolvent operator and point spectrum. We use the quasi-selfadjointness of the matrix potential Q to obtain these properties. We show that the resolvent set consists of squares of the non-singular points of the Jost matrix in the upper complex plane. Moreover, we obtain the Green's function of this boundary value problem with the help of the Jost matrix. In the light of this main result, we show that the continuous spectrum is $[0,\infty)$ and the point spectrum consist of squares of the singular points of the Jost matrix in the upper complex plane. We also show that the set of spectral singularities is empty.

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1. INTRODUCTION

Self-adjoint operators defined on a Hilbert space correspond to physical observables in quantum mechanics. For this reason, one-dimensional Schrödinger equation

$$-y'' + q(x)y = k^2 y, \qquad x \in (0,\infty) \tag{1}$$

where q is a potential and k^2 is a spectral parameter has been studied in detail (Levitan & Sargsyan, 1975; Birman & Solomjak, 1987; Schmüdgen, 2012). It is well-known that if the potential q in Eq. (1) is real-valued and the boundary condition at $x = 0$ is given

$$\cos(t)\, y(0) + \sin(t)\, y'(0) = 0, \qquad t \in (0,\pi] \tag{2}$$

then the operator generated by the boundary value problem (1)-(2) is selfadjoint. On the other hand, if the potential q in Eq. (1) is complex-valued then the operator is not symmetric and hence not selfadjoint. There is a vast literature on the spectral properties of non-selfajoint differential operators in the last decades. This is because, non-selfajoint differential operators have many applications in quantum physics. In particular, non-

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selfadjoint operators are observed in physical systems which do not involve the conservation of energy law. Some selfadjoint problems also yields non-selfadjoint operators after separation of variables. One can consult for the first results about non-selfajoint differential operators to Naimark (1968) and for recent results and applications in quantum physics to Bagarello et. al., (2015) and Sjöstrand (2019).

There have been many attempts for the generalizations of the quantum mechanics. A recent attempt is to consider Hamiltonians which have an exact PT-symmetry instead of Hermitian Hamiltonians (Bender et. al., 2002; 2003; Bender, 2007). Later, Mostafazadeh (2010) replaced these Hamiltonians with pseudo-Hermitian Hamiltonians which are more general. Scholtz et al. (1992) introduced quasi-Hermitian Hamiltonians which constitute an important class of pseudo-Hermitian Hamiltonians. A quasi-Hermitian operator is a non-selfadjoint operator H in a Hilbert space such that $H^* = THT^{-1}$ where T is a positive, bounded operator which has a bounded inverse. A consistent quantum theory awaits to be built for quasi-selfadjoint Hamiltonians.

The matrix Schrödinger equation on the half-line is given by

$$-y'' + Q(x)y = k^2 y, \qquad x \in (0, \infty) \tag{3}$$

where k^2 is a spectral parameter and Q is an $n \times n$ matrix potential. Equation (3) has several applications in scattering problems in quantum mechanics and especially in quantum graphs (Berkolaiko & Kuchment, 2013). To be more precise, Equation (3) corresponds to a quantum star graph with n edges of infinite length and describes the behavior of n connected very thin quantum wires forming a one-vertex graph with open ends. This model can be useful in designing elementary gates in quantum computing and nanotubes for microscopic electronic devices, where, for example, strings of atoms may form a star-shaped graph. There is a substantial interest on quantum graphs recently. In particular, Kottos & Smilansky (1997) revealed that quantum graphs enable us to model quantum chaos. After that, there has been a burst of interest in quantum graphs. In the study of quantum graphs, general boundary conditions are encountered most. That's the reason we consider the most general boundary conditions on the contrary to previous studies (Olgun & Coskun, 2010; Arpat & Mutlu, 2015).

In the selfadjoint case, Aktosun et al. (2011); Weder (2017); Aktosun & Weder (2013; 2018; 2020) studied Equation (3) together with general selfadjoint boundary conditions at $x = 0$ where Q is selfadjoint, Lebesgue measurable on (0,∞) and

$$\int_0^\infty (1+t)\|Q(t)\|dt < \infty \tag{4}$$

holds for any matrix norm $\|,\|$. Note that general selfadjoint boundary condition at the origin has been given in several equivalent forms (Aktosun et al., 2011; Aktosun & Weder, 2013; 2018; 2020; Weder, 2017). One of these forms is

$$My(0) + \mathrm{N}y'(0) = 0 \tag{5}$$

where M and N are $n \times n$ constant matrices (not dependent on k) which satisfy

$$MN^* = NM^* \tag{6}$$

$$rank(M|N) = n$$

where "*" denotes the adjoint of a matrix (Aktosun et al., 2011). Aktosun et al. (2011) investigated the small-energy analysis and later, Aktosun & Weder (2013) studied the high-energy analysis for the above boundary value problem (3)-(5). Equation (3) where $Q(x)$ is a non-selfadjoint, completely continuous operator in an infinite dimensional separable Hilbert space for each $x \in (0, \infty)$ is called Schrödinger's operator equation. This equation is the generalization of the Equation (3) to the infinite dimension. Gasymov et al. (1967) studied the point spectrum of Schrödinger's operator equation with selfadjoint, completely continuous operator coefficients.

For the non-selfadjoint case, Equation (3) where $Q \neq Q^*$ together with the Dirichlet boundary condition was studied in Olgun & Coskun (2010) and conditions involving spectral parameter was studied in Yokus & Coskun (2018; 2019). Eigenvalues and spectral singularities correspond to the singular points of the Jost matrix (Olgun & Coskun, 2010). Furthermore, these results are generalized to infinite dimensional case by considering Schrödinger's operator equation, namely, Equation (3) with $Q(x)$ is a non-selfadjoint, completely continuous operator in an infinite dimensional separable Hilbert space for each $x \in (0, \infty)$ (Bairamov et al., 2017). In this infinite dimensional case, eigenvalues and spectral singularities correspond to the singular points of the Jost operator which is an operator-valued function. Hence new methods are used from operator theory. Recently, Mutlu & Kır Arpat (2020) considered Schrödinger's operator equation on the real line and examined its spectral properties.

In this study, we consider Equation (3) where

- $n \times n$ matrix potential $Q(x)$ is Lebesgue measurable on $(0, \infty)$ for every $x \in (0, \infty)$,
- Q satisfies the condition (4),
- Q is quasi-selfadjoint i.e. $Q^*(x) = PQ(x)P^{-1}$ for each $x \in (0, \infty)$ where P is a Hermitian positive-definite $n \times n$ matrix.

We consider general boundary conditions at the origin which can be stated

$$-B^*y(0) + A^*y'(0) = 0 \tag{7}$$

where A and B are $n \times n$ constant matrices (not dependent on k) such that

$$B^*P^{-1}A = A^*P^{-1}B \tag{8}$$

$$rank(A|B) = n \tag{9}$$

Note that the condition (9) can be exchanged with (Aktosun et al., 2011)

$$A^*A + B^*B > 0 \tag{10}$$

The equation (9) is required to ensure that there are correct number of independent boundary conditions, and the equation (7) reduces to (6) when $P = I_n$. Therefore, this paper generalizes the studies Aktosun et al. (2011); Aktosun & Weder (2013; 2018; 2020); Weder (2017). Moreover, this study complements the study Olgun & Coskun (2010) in which only the Dirichlet condition at origin is considered. Note that the introduction of the general boundary condition (7) completely changes the structure of the Jost matrix. Clearly the boundary value problem under consideration is not selfadjoint. Let us denote this non-selfadjoint operator by L hereafter.

This paper consists of 5 sections. In section 2 we present the methods applied in the spectral analysis of L. We state certain auxiliary results related to the spectral properties of L in Section 3. We present our results in Section 4. In particular, we obtain the Jost matrix and investigate its properties. Then, we derive the resolvent of L and obtain the point spectrum, continuous spectrum and spectral singularities of L. Finally, we present some concluding remarks in Section 5.

Here we introduce the notations used throughout this paper. We will write $M > 0$ to indicate that the matrix M is positive-definite. Let us denote the upper-half complex plane by H, its closure by $\overline{H} = H \cup \mathbb{R}$, the complex conjugate of a complex number z by $\bar{z}$ and $n \times n$ unit matrix by I_n. Let $L_2(\mathbb{R}_+, \mathbb{C}^n)$ denote the Hilbert space of complex-valued vector functions $f(x) = \big(f_1(x),\ f_2(x), \dots, f_n(x)\big)$ $(0 < x < \infty)$ such that each component of f lies in $L_2(0, \infty)$.

2. MATERIAL AND METHOD

In scattering theory, the Jost function of equation (1) is the Wronskian of the regular solution (a solution of equation (1) which satisfies $y(0,k)=0,\ y'(0,k)=1$) and the Jost solution (a solution of equation (1) which satisfies the asymptotic relation $f(x,k)=e^{ikx}[1+o(1)],\ x\to\infty$). The eigenvalues of the operator can be obtained as zeros of the Jost function (called the dispersion relation). Jost function is also used in the construction of the Green's function of the operator. Analogously, we can define the Jost function which is now an $n\times n$ matrix $J(k)$ and call it the Jost matrix of the matrix Schrödinger equation. In this case the dispersion relation becomes $det J(k)=0$ (Agranovic & Marchenko, 1965).

It is well known that the Jost matrix is very fundamental for spectral analysis of matrix Schrödinger equations (Agranovic & Marchenko, 1965). The Jost matrix provides the dispersion relation of matrix Schrödinger operators. For this reason, we define the Jost matrix of the boundary value problem generated by (3) and (7)-(9). In order to guarantee the existence of Jost matrix we use the quasi-selfadjointness of the potential Q. Then, we investigate the analytic and asymptotic behaviors of the Jost matrix. Subsequently, we use these properties in order to construct the Green's function and the resolvent operator of L. Using the structure of the Green's function, we deduce that the continuous spectrum is filling the positive half axis and the eigenvalues correspond to the squares of the singular points in the upper complex plane of the Jost matrix. We also prove that there are not any spectral singularities.

3. PRELIMINARIES

In this section, we outline relevant properties of certain solutions of Equation (3) including the Jost solution. For the proofs of the very well-known results presented in these sections we refer the interested reader to Agranovic & Marchenko (1965); Aktosun et al. (2011).

The domain of L consists of vector functions f from $L_2(\mathbb{R}_+,\mathbb{C}^n)$ such that

- f has absolutely continuous derivative f' on every interval $[0,a]$, $(0<a<\infty)$,
- $Lf := -f''+Qf\in L_2(\mathbb{R}_+,\mathbb{C}^n)$,
- The boundary condition (7) holds.

The Jost solution $F(x,k)$ of Equation (3) is $n\times n$ matrix function which satisfies the asymptotic relations

$$F(x,k)=e^{ikx}[I_n+o(1)],\qquad F_x(x,k)=ike^{ikx}[I_n+o(1)],\qquad x\to\infty,\qquad k\in\overline{H}\backslash\{0\}$$

The Jost solution has the representation

$$F(x,k)=e^{ikx}I_n+\int_x^\infty e^{ikt}K(x,t)dt\,,\qquad k\in\overline{H}\backslash\{0\}$$

where

$$\|K(x,t)\|\le c\int_{\frac{x+t}{2}}^{\infty}\|Q(u)\|du\,,\ \ k\in\overline{H}\backslash\{0\}$$

where $c>0$ is a constant (Agranovic & Marchenko, 1965). $F(x,k)$ and $F_x(x,k)$ are analytic functions of k in H and continuous in $\overline{H}$ for every fixed x. (Agranovic & Marchenko, 1965).

There exists a matrix solution $G(x,k)$ of Equation (3) which satisfies the asymptotic relations

$$G(x,k)=e^{-ikx}[I_n+o(1)],\qquad G_x(x,k)=-ike^{-ikx}[I_n+o(1)],\qquad x\to\infty,\qquad k\in\overline{H}\backslash\{0\}$$

Similarly, $G(x,k)$ and $G_x(x,k)$ are analytic functions of k in H and continuous in $\overline{H}$ for every x. (Agranovic & Marchenko, 1965). It is also well-known that for $k \in \overline{H}\backslash\{0\}$, every vector solution to Equation (3) can be expressed

$$u(x,k) = F(x,k)\alpha + G(x,k)\beta$$

where $\alpha, \beta \epsilon \mathbb{C}^n$ are constant vectors (Agranovic & Marchenko, 1965).

It is well-known that there exist $n \times n$ matrix solutions which satisfy given specific initial conditions at a finite number. Therefore, there exists a matrix solution $R(x,k)$ of Equation (3) which satisfies the initial conditions

$$R(0,k) = P^{-1}A, \qquad R'(0,k) = P^{-1}B \tag{11}$$

where P is the Hermitian, positive-definite $n \times n$ matrix appearing in Equation (8) and A and B are $n \times n$ matrices appearing in Equation (7) (Aktosun et al., 2011). For each fixed x, $R(x,k)$ is entire and for these reason $R(x,k)$ is called the regular solution of Equation (3) (Aktosun et al., 2011).

4. RESULTS

4.1 The Jost Matrix

In this subsection we present the Jost matrix of the Equation (3) where quasi-selfadjoint matrix potential Q is Lebesgue measurable on (0,∞) and satisfies condition (4) and then examine some of its properties.

Let U and V be $n \times n$ matrix solutions of Equation (3). Let the Wronskian of U and V is defined by

$$[U,V] \coloneqq UV' - U'V$$

Taking the matrix adjoint of Equation (3) and using the fact that Q is quasi-selfadjoint we obtain the adjoint equation

$$-z'' + zPQ(x)P^{-1} = \left(\bar{k}\right)^2 z, \qquad x \in (0,\infty) \tag{12}$$

For a real value of k, it is obvious that if $y(x,k)$ is a matrix solution of (3), then $y(x,k)^*$ and $y(x,-k)^*$ are solutions of (12). Moreover if $y(x,k)$ has an analytic extension for each fixed x as a function of k from the real line to the upper-half complex plane, then $y(x,-k)^*$ also has an analytic extension from the real line to the upper-half complex plane and that extension coincides with $y\left(x,-\bar{k}\right)^*$ for $k \in H$. Therefore, $F(x,-k)^*$ and $F_x(x,-k)^*$ have analytic extensions from the real line to the upper-half complex plane which are $F\left(x,-\bar{k}\right)^*$ and $F_x\left(x,-\bar{k}\right)^*$ for $k \in H$ respectively.

Let $U(x,k)$ and $V(x,k)$ be $n \times n$ matrix solutions of Equation (3). It easily follows by a direct computation that the Wronskians $[V(x,k)^*, PU(x,k)] = [V(x,k)^*P, U(x,k)]$ and $[U(x,k)^*, PV(x,k)] = [U(x,k)^*P, V(x,k)]$ are independent of x for real values of k. Furthermore if $U(x,k)$ and $V(x,k)$ have analytic extensions from the real line to the upper-half complex plane, then the Wronskians $\left[V\left(x,-\bar{k}\right)^*, PU(x,k)\right]$ and $\left[V\left(x,\bar{k}\right)^*, PU(x,k)\right]$ are independent of x for $k \in \overline{H}$. As a result, we can obtain various equalities by evaluating the Wronskian at $x=0$ or $x \to \infty$. For example

$$[F(x,\pm k)^*, PF(x,\pm k)] = \pm 2ikP, \qquad k \in \mathbb{R} \tag{13}$$

$$\left[F\left(x,-\bar{k}\right)^*, PF(x,k)\right] = 0, \qquad k \in \overline{H} \tag{14}$$

The Jost matrix is defined by

$$J(k) := \left[F(x,-\bar{k})^{*}, PR(x,k)\right], \qquad k \in \bar{H} \tag{15}$$

where $R(x,k)$ is the regular solution described by (11) and $F(x,k)$ is the Jost solution. Note that the Wronskian (15) is independent of x. Evaluating the Wronskian at $x = 0$ and imposing the initial value conditions (11) we have

$$J(k) = F(0,-\bar{k})^{*}B - F'(0,-\bar{k})^{*}A, \qquad k \in \bar{H} \tag{16}$$

It easily follows from (16) and the fact that $F(0,-\bar{k})^{*}$ and $F_x(0,-\bar{k})^{*}$ are analytic in H that the Jost matrix is analytic in H and continuous in $\bar{H}$.

Theorem 1 The Jost matrix $J(k)$ has an inverse for each real nonzero k.

Proof. The condition (10) and $P > 0$ yields

$$B^*P^{-2}B + A^*A > 0$$

and hence there exists a unique matrix $M > 0$ given by

$$M := (B^*P^{-2}B + A^*A)^{1/2}$$

Since $M > 0$ it follows $M^* = M$, M is invertible and thus

$$M^{*-1}(B^*P^{-2}B + A^*A)M^{-1} = I_n$$

Let us define

$$C = \begin{pmatrix} P^{-1}B & A \\ A & -P^{-1}B \end{pmatrix}, \qquad H = C\begin{pmatrix} M^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}$$

Using (8) we have $H^*H = I_{2n}$ and H is unitary. Therefore $HH^* = I_{2n}$ yielding

$$P^{-1}BM^{-2}B^*P^{-1} + AM^{-2}A^* = I_n \tag{17}$$

$$AM^{-2}B^*P^{-1} - P^{-1}BM^{-2}A^* = 0 \tag{18}$$

Let

$$S(k) := F'(0,-k)^*P^{-1}BM^{-2} + F(0,-k)^*PAM^{-2}, \qquad k \in \mathbb{R} \tag{19}$$

Using (13), (16), (17) and (19) we get

$$J(k)S(k)^* - S(k)J(k)^* = [F(x,-k)^*, PF(x,-k)] = -2ikP \tag{20}$$

Suppose that $J(k)$ does not have an inverse for a real nonzero k_0. There exists a nonzero vector $w \in \mathbb{C}^n$ such that $w^*J(k_0) = J(k_0)w = 0$. This implies together with (20) and $P > 0$ that

$$0 = w^*J(k_0)S(k_0)^*w - w^*S(k_0)J(k_0)^*w = -2ik_0w^*Pw \neq 0$$

which gives a contradiction. Therefore, $J(k)$ is invertible for each real nonzero k.

4.2 Resolvent Operator

In this subsection we derive the resolvent operator $R_k(L) \coloneqq (L - k^2 I)$ of L where I denotes the identity operator. Later, we obtain the point spectrum, continuous spectrum and the set of spectral singularities of L.

Theorem 2 The resolvent set $\rho(L)$ of L is

$$\rho(L) = \{k^2 :\ k \in H, \qquad detJ(k) \neq 0\}$$

and the resolvent operator $R_k(L) = (L - k^2 I)$ is defined by

$$R_k(L)g(x) = \int_0^\infty K(x,t;k)g(t)dt$$

where $g \in L_2(\mathbb{R}_+, \mathbb{C}^n)$ and

$$K(x,t;k) = \begin{cases} F(x,k)(J(k)^*)^{-1} R\left(t,\bar{k}\right)^* P, & 0 \leq t \leq x, \\ R(x,k)J(k)^{-1} F\left(t,-\bar{k}\right)^* P, & x < t < \infty. \end{cases}$$

Proof. We must solve

$$-y'' + Q(x)y - k^2 y = g(x), \qquad x \in (0,\infty) \tag{21}$$

where $y, g \in L_2(\mathbb{R}_+, \mathbb{C}^n)$ and the vector-valued function $y(x,k)$ satisfies (7). Note that a vector solution of the homogeneous part of (21) can be expressed

$$y(x,k) = R(x,k)\alpha + F(x,k)\beta$$

where α and β are constant vectors in and $\mathbb{C}^n$ and $k \in H$. Suppose that $detJ(k) \neq 0$ for $k \in H$. Using variation of parameters method we try to find the general solution of (21) in the following form

$$y(x,k) = R(x,k)a(x) + F(x,k)b(x) \tag{22}$$

such that $a(x)$ and $b(x)$ are vector functions taking values in $\mathbb{C}^n$. Differentiating (22) with respect to x we get

$$y'(x,k) = R'(x,k)a(x) + R(x,k)a'(x) + F'(x,k)b(x) + F(x,k)b'(x) \tag{23}$$

Suppose that

$$R(x,k)a'(x) + F(x,k)b'(x) = 0 \tag{24}$$

Differentiating (23) with respect to x we have

$$y''(x,k) = R''(x,k)a(x) + R'(x,k)a'(x) + F''(x,k)b(x) + F'(x,k)b'(x) \tag{25}$$

Plugging (25) and (22) in (21) yields

$$R'(x,k)a'(x) + F'(x,k)b'(x) = -g(x) \tag{26}$$

Multiplying (24) with $R'(x,\bar{k})^* P$ and (26) with $R(x,\bar{k})^* P$ from left and subtracting two equalities yields

$$\left[R(x,\bar{k})^*, PR(x,k)\right]a'(x) + \left[R(x,\bar{k})^*, PF(x,k)\right]b'(x) = -R(x,\bar{k})^* Pg(x) \tag{27}$$

Since the Wronskian $\left[R(x,\bar{k})^*, PR(x,k)\right]$ is independent of x evaluating at $x = 0$ and using (8) we have

$$\left[R(x,\bar{k})^*, PR(x,k)\right] = R(0,\bar{k})^* PR'(0,k) - R'(0,\bar{k})^* PR(0,k)$$

$$= (P^{-1}A)^* PP^{-1}B - (P^{-1}B)^* PP^{-1}A = 0$$

Furthermore, the Wronskian $\left[R(x,\bar{k})^*, PF(x,k)\right]$ is independent of x and using (11)

$$\left[R(x,\bar{k})^*, PF(x,k)\right] = R(0,\bar{k})^* PF'(0,k) - R'(0,\bar{k})^* PF(0,k)$$

$$= (P^{-1}A)^* PF'(0,k) - (P^{-1}B)^* PF(0,k) = A^* F'(0,k) - B^* F(0,k) \tag{28}$$

and also

$$\left(A^* F'(0,k) - B^* F(0,k)\right)^* = F'(0,k)^* A - F(0,k)^* B = F'(0,-\bar{k})^* A - F(0,-\bar{k})^* B = -J(k) \tag{29}$$

From (28) and (29) we have

$$\left[R(x,\bar{k})^*, PF(x,k)\right] = -J(k)^*$$

Since $J(k)$ is invertible, $J(k)^*$ is also invertible and from (27) we have

$$b'(x) = (J(k)^*)^{-1} R(x,\bar{k})^* Pg(x)$$

and hence

$$b(x) = \int_0^x (J(k)^*)^{-1} R(t,\bar{k})^* Pg(t)dt + \beta \tag{30}$$

for some constant vector β.

If we multiply equation (24) with $F'(x,-\bar{k})^* P$ from left and equation (26) with $F(x,-\bar{k})^* P$ from left and then subtract two resulting equalities we have

$$\left[F(x,-\bar{k})^*, PR(x,k)\right]a'(x) + \left[F(x,-\bar{k})^*, PF(x,k)\right]b'(x) = -F(x,\bar{k})^* Pg(x)$$

From (14) and (15) it follows

$$J(k)a'(x) = -F(x,-\bar{k})^* Pg(x) \tag{31}$$

Since $J(k)$ is invertible we have from (31) that

$$a'(x) = -\left(J(k)\right)^{-1} F(x,-\bar{k})^* Pg(x)$$

and hence

$$a(x) = \alpha + \int_x^\infty \big(J(k)\big)^{-1} F\big(t, -\bar{k}\big)^* Pg(t)\, dt \tag{32}$$

for some constant vector α.

Plugging (30) and (32) in (22)

$$y(x,k) = R(x,k)\alpha + R(x,k)\int_x^\infty \big(J(k)\big)^{-1} F\big(t,-\bar{k}\big)^* Pg(t)\, dt + F(x,k)\int_0^x (J(k)^*)^{-1} R\big(t,\bar{k}\big)^* Pg(t) dt + F(x,k)\beta$$

Since the vector function $y(x,k)$ should lie in $L_2(\mathbb{R}_+, \mathbb{C}^n)$ it follows that $\alpha = 0$. We have

$$y'(x,k) = R'(x,k)a(x) + F'(x,k)b(x) \tag{33}$$

Evaluating (33) at $x = 0$ yields

$$y'(0,k) = P^{-1}B\int_0^\infty \big(J(k)\big)^{-1} F\big(t,-\bar{k}\big)^* Pg(t)\, dt + F'(0,k)\beta$$

Imposing the boundary condition (7)

$$-B^*\left[P^{-1}A\int_0^\infty \big(J(k)\big)^{-1} F\big(t,-\bar{k}\big)^* Pg(t)\, dt + F(0,k)\beta\right] + A^*\left[P^{-1}B\int_0^\infty \big(J(k)\big)^{-1} F\big(t,-\bar{k}\big)^* Pg(t)\, dt + F'(0,k)\beta\right] = 0$$

Using (8) the last equation simplifies to

$$[A^*F'(0,k) - B^*\, F(0,k)]\beta = 0$$

and using (29) we have $J(k)^*\beta = 0$. Since $J(k)$ is invertible this gives $\beta = 0$. Finally, we find

$$y(x,k) = R(x,k)\int_x^\infty \big(J(k)\big)^{-1} F\big(t,-\bar{k}\big)^* Pg(t)\, dt + F(x,k)\int_0^x (J(k)^*)^{-1} R\big(t,\bar{k}\big)^* Pg(t) dt$$

Corollary 1 Let us denote the continuous spectrum, point spectrum and the set of spectral singularities of L by $\sigma_c(L)$, $\sigma_d(L)$ and $\sigma_{ss}(L)$ respectively. Then, it follows

$$\sigma_c(L) = [0,\infty)$$

$$\sigma_d(L) = \{k^2 \colon\ k \in H, \qquad detJ(k) = 0\}$$

$$\sigma_{ss}(L) = \emptyset$$

Proof. It follows from Theorem 2 that k^2 is an eigenvalue of L iff $detJ(k) = 0$ for $k \in H$ i.e. the resolvent operator doesn't exist. It is easy to show similarly to the scalar case that $\sigma_c(L) = [0,\infty)$ (Naimark, 1968). Spectral singularities are the poles of the kernel of the resolvent and are also in the continuous spectrum. Therefore, Theorem 1 and Theorem 2 implies that $\sigma_{ss}(L) = \emptyset$.

5. CONCLUSIONS

The present paper is aimed at complementing the studies Olgun & Coskun (2010); Bairamov et al. (2017) regarding the matrix Schrödinger equation with non-selfadjoint operator coefficients on the half-line together with Dirichlet boundary condition at $x = 0$ and generalizing the studies Aktosun et al. (2011); Weder (2017); Aktosun & Weder (2013; 2018; 2020) regarding the selfadjoint matrix Schrödinger operator generated by (3)-(5). We consider the matrix Schrödinger equation on the half-line with the quasi-selfadjoint matrix potential together with (7)-(9) which will contribute to the understanding of recently introduced pseudo-Hermitian quantum mechanics by Mostafazadeh (2010).

Even though the matrix Schrödinger operator L with quasi-selfadjoint potential is non-selfadjoint, there aren't any spectral singularities, and the structure of the spectrum is the same as the selfadjoint matrix Schrödinger operator i.e. $Q = Q^*$. However, there may be complex eigenvalues of L.

The particular case $A = 0$ implies that B is invertible from (9) and thus yielding the Dirichlet boundary condition at $x = 0$. Therefore, the boundary value problem under investigation generalizes Olgun & Coskun (2010). On the other hand, the particular case $P = I$ in the quasi-selfadjointness relation $Q^*(x) = PQ(x)P^{-1}$ implies that $Q^* = Q$ and thus one obtains the selfadjoint matrix Schrödinger operator studied in Agranovic & Marchenko (1965); Aktosun et al. (2011); Weder (2017); Aktosun & Weder (2013; 2018; 2020). Therefore, our study also generalizes these studies.

The boundary value problem investigated in this paper is intimately related to non-compact quantum star graphs which have many applications in scattering problems in quantum mechanics. As for quantum graphs, consideration of most general boundary conditions rather than just Dirichlet boundary conditions is more relevant. For this reason, most general boundary conditions are imposed in our study. As a result, the results obtained in this paper will be useful for studying spectral properties of such quantum graphs.

CONFLICT OF INTEREST

There is no conflict of interest in this research article.

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