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Pure Extending Objects

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Abstract

In this paper we introduce two new concepts, namely, pure extending objects and $\mathscr{K}$-nonsingular objects and then, we prove that any pair of subisomorphic $\mathscr{K}$-nonsingular objects in a finitely accessible additive category with kernels $\mathscr{A}$ are isomorphic to each other if and only if for any object Y and any pure extending $\mathscr{K}$-nonsingular object X, if X and Y are subisomorphic to each other then $X \cong Y$.

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1. Introduction

It is well known that every finitely accessible additive category $\mathscr{A}$ has an associated Grothendieck functor category $\mathscr{E}$ consisting all contravariant functors from $\mathscr{A}_0$ to the category of abelian groups. A contravariant functor in $\mathscr{E}$ is *flat* if it is direct limit of finitely generated projective objects. Yoneda's lemma induces an equivalence between $\mathscr{A}$ and the subcategory of flat objects of $\mathscr{E}$. By this equivalence, f is a pure monomorphism in $\mathscr{A}$ if and only if it is a monomorphism in $\mathscr{E}$ (see [4]).
The motivation of this paper comes from the comprehensive work by Dehghani and Rizvi [5] on isomorphic modules which are mutually subisomorphic. They ask also when any pair of subisomorphic extending modules are isomorphic to each other and they prove that, any pair of nonsingular subisomorphic R-modules are isomorphic to each other if and only if for any R-module Y and any nonsingular extending R-module X, if X and Y are subisomorphic to each other then $X \cong Y$. ([5, Theorem 2.19]). The present paper considers a extension of an extending module/object, namely pure extending objects, to the finitely accessible additive categories and then, for these object we extend [5, Theorem 2.19] to the finitely accessible additive categories.
Throughout $\mathscr{A}$ will denote a finitely accessible additive category.

2. Pure extending objects

A module M is called extending if and only if every submodule is essential in a direct summand of M ([6]). As it stated in [6], every injective module is extending but class of extending modules retains many of its desirable properties.
Let A, A' and A'' be objects in $\mathscr{A}$. A pure monomorphism $p : A \to A'$ is said to be *pure essential* if whenever $f : A' \to A''$ is a morphism such that fp is a pure monomorphism, then f also must be a pure monomorphism . A non-zero object in $\mathscr{A}$ is *pure uniform* if all non-zero subobjects are pure essential ([2]).

Definition 2.1. Let M be an object in $\mathscr{A}$. M is called *pure extending* if every pure subobject is pure essential in a direct summand of M.

As it stated in [3], every object of a Grothendieck category $\mathscr{E}$ has an injective envelope, so every injective object of $\mathscr{E}$ is extending by [3, Corollary 5.2]. We shall give some examples in finitely accessible additive categories: Every pure-injective object is pure extending. Clearly every pure uniform object is pure extending and every indecomposable pure extending object is pure uniform.

Proposition 2.2. *Let A be an object of $\mathscr{A}$. If A is extending in $\mathscr{E}$, then it is pure extending in $\mathscr{A}$.*

Proof. Let S be a pure subobject of A. There exists a direct summand D of such that S is essential in D, since A is extending in $\mathscr{E}$. Notice that D is flat as well. Hence S is pure essential subobject of D in the category $\mathscr{A}$. □

Proposition 2.3. *Let A be an object of $\mathscr{A}$. If A is pure extending then any direct summand of A is also pure extending.*

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Proof. Let D be a direct summand of A and let P be a pure subobject of D. Clearly P is a pure subobject of A. Then, by hypothesis, there exists a direct summand D' of A such that P is pure essential in D'. On the other hand, D and D' are flat objects in the Grothendieck category $\mathscr{E}$ and their intersection D'' is flat in $\mathscr{E}$. Hence D'' is an object of $\mathscr{A}$. Therefore P is pure essential in D''.
Now consider the following morphisms

$$P \xrightarrow{f} D'' \xrightarrow{g} D' \xrightarrow{h} D$$

where f and g are pure essential monomorphisms and $\psi = hgf$ is a pure monomorphism. Since gh is pure essential, h is a pure monomorphism. Thus D' is a direct summand of D. This completes the proof. □

3. $\mathscr{K}$-nonsingular objects

Following [3], [5] and [8], we introduce a new concept which extends the notion of nonsingular modules/objects to finitely accessible additive categories.

Definition 3.1. Let $\mathscr{A}$ be a finitely accessible additive category with kernels and let A be an object of $\mathscr{A}$. A is called *$\mathscr{K}$-nonsingular* if for any $\varphi \in S = End_{\mathscr{A}}(A)$, $Ker\varphi$ is pure essential in A implies $\varphi = 0$.

Lemma 3.2. *Let $\mathscr{A}$ be a finitely accessible additive category with kernels and let A be a $\mathscr{K}$-nonsingular object of $\mathscr{A}$. Then, any direct summand of A is $\mathscr{K}$-nonsingular.*

Proof. Suppose that A is $\mathscr{K}$-nonsingular and write $A = A' \oplus A''$. Let $f : A' \to A'$ be a morphism such that $Kerf$ is pure essential in A'. Consider the morphism $\varphi = f \oplus 0 : A' \oplus A'' \to A' \oplus A''$. Therefore $Ker\varphi = Kerf \oplus A''$ is pure essential in A. Since A is $\mathscr{K}$-nonsingular by hypothesis, $\varphi = 0$ and so $f = 0$. □

Lemma 3.3. *Let $\mathscr{A}$ be a finitely accessible additive category with kernels and let A be a pure injective $\mathscr{K}$-nonsingular object of $\mathscr{A}$. Then, any pure essential subobject of A is $\mathscr{K}$-nonsingular.*

Proof. Assume that A is $\mathscr{K}$-nonsingular, $f : B \to A$ is a pure essential monomorphism and B is not $\mathscr{K}$-nonsingular. Let $\varphi_B : B \to B$ be a non-zero morphism such that $Ker\varphi_B$ pure essential in B and let $\varphi_A : A \to A$ be its extension. Notice that $Ker\varphi_B$ is pure subobject of $Ker\varphi_A$ and $Ker\varphi_B$ is pure essential in A, since composite of pure essential morphisms is pure essential. But since A is $\mathscr{K}$-nonsingular, $Ker\varphi_B$ can not be pure essential. □

It can be seen from [8, Example 2.19] that the converse of Lemma 3.3 is not true in general.
Following [5], the objects X and Y in a finitely accessible additive category $\mathscr{A}$ are called *subisomorphic* to each other whenever X is isomorphic to a subobject of Y and Y is isomorphic to a subobject of X. Now we are ready to give our main result which is an extension of [5, Theorem 2.19]:

Theorem 3.4. *Let $\mathscr{A}$ be a finitely accessible additive category with kernels. The following statements are equivalent:*
i)For any object Y and any pure extending $\mathscr{K}$-nonsingular object X, if X and Y are subisomorphic to each other then $X \cong Y$.
ii) Any pair of $\mathscr{K}$-nonsingular subisomorphic objects are isomorphic to each other.

Proof. Assume that Z is any $\mathscr{K}$-nonsingular object. Then $Y = PE(Z)^{\mathbb{N}} \oplus Z$ and $X = PE(Z)^{\mathbb{N}}$ are subisomorphic $\mathscr{K}$-nonsingular objects where $PE(Z)$ denotes the pure injective envelope of Z (see [7]). X is pure extending, since $X = PE(Z)$ is pure injective. Hence, If (i) holds then $X \cong Y$. The converse is clear, since Y is also $\mathscr{K}$-nonsingular. □

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