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Subtraction Bialgebras

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Abstract

The notions of subtraction bialgebras, sub-subtraction bialgebras, biideals and complicated subtraction bialgebras are introduced, and related properties are investigated.

Keywords: Subtraction algebra, subtraction bialgebra, biideal.

Fark Bi-Cebirleri

Özet

Y. B. Jun, Y. H. Kim ve K. A. Oh, karmaşık fark cebiri kavramını tanımlayarak, bu kavram ile ilgili bazı özellikleri araştırdılar. Daha sonra Y. Çeven ve M. A. Öztürk, fark cebirleri ile ilgili bazı kavramları (alt fark cebiri, sınırlı fark cebiri, fark cebirlerinin birleşimi) tanımladı ve bazı özellikleri incelediler. Bu ve benzeri çalışmalar doğrultusunda bi-cebirsel yapılar dikkate alınarak, bu çalışmada fark bi-cebiri, alt fark bi-cebiri, bi-ideal ve karmaşık fark bi-cebiri kavramları tanımlandı ve bu kavramlarla ilgili özellikler incelendi.

Anahtar Kelimeler: Fark cebiri, fark bi-cebiri, bi-ideal.

Introduction

B. M. Schein [1] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [2]. He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [3] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun, H. S. Kim and E. H. Roh [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [5], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. In [6], Y. B. Jun, Y. H. Kim and K. A. Oh introduced the notion of complicated subtraction algebras and investigated some related properties. In [7], Y. Çeven and M. A. Öztürk introduced some additional concepts on subtraction algebras, so called sub-subtraction algebra, bounded subtraction algebra and union of subtraction algebras, and some properties are investigated. Bialgebraic structures, for example, bisemigroups, bigroups, bigroupoids, biloops, birings, bisemirings, binear-rings, etc., are discussed in [8]. In [9], Jun et al. established the structure of BCK/BCI bialgebras, and investigated some properties.

In this paper, considering bialgebra structures, we introduced the notions of subtraction bialgebras, sub-subtraction bialgebras, biideals and complicated subtraction bialgebras, and we give some properties of these structures.

1. Preliminaries

An algebra $(X;-)$ with a single binary operation "$-$" is called a subtraction algebra if for all $x, y, z \in X$ the following conditions hold:

(S1) $x-(y-x)=x$,
(S2) $x-(x-y)=y-(y-x)$,
(S3) $(x-y)-z=(x-z)-y$.

The subtraction determines an order relation on $X: a \leq b \Leftrightarrow a-b=0$, where $0=a-a$ is an element that doesn't depend on the choice of $a \in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [2], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to induced order. Here $a \wedge b = a-(a-b)$ and the complement of an element $b \in [0,a]$ is $a-b$.

In a subtraction algebra, the following statements are true [4, 10]:

(a1) $(x-y)-y=x-y$,

(a2) $x-0=x$ and $0-x=0$,

(a3) $(x-y)-x=0$,

(a4) $x-(x-y) \leq y$,

(a5) $(x-y)-(y-x) = x-y$,

(a6) $x-(x-(x-y)) = x-y$,

(a7) $(x-y)-(z-y) \leq x-z$,

(a8) $x \leq y$ if and only if $x = y-w$ for some $w \in X$,

(a9) $x \leq y$ implies $x-z \leq y-z$ and $z-y \leq z-x$ for all $z \in X$,

(a10) $x, y \leq z$ implies $x-y = x \wedge (z-y)$,

(a11) $(x \wedge y)-(x \wedge z) \leq x \wedge (y-z)$,

(a12) $(x-y)-z = (x-z)-(y-z)$.

Definition 1.1: [4] *A nonempty subset A of a subtraction algebra X is called an ideal of X if it satisfies*

(1) $0 \in A$,
(2) $(\forall x \in X)(\forall y \in A)(x-y \in A \Rightarrow x \in A)$.
We denote by $A \lhd X$.

Definition 1.2: [6] *Let X be a subtraction algebra. For any $a, b \in X$, let $G(a,b) = \{x \in X : x-a \leq b\}$. X is said to be complicated if for any $a, b \in X$ the set $G(a,b)$ has the greatest element.*

Note that $0, a, b \in G(a,b)$. The greatest element of $G(a,b)$ is denoted $a+b$.

Proposition 1.1. [7] *Let X be a subtraction algebra and I be a nonempty subset of X. Then I is an ideal of X if and only if $G(x,y) \subseteq I$ for all $x, y \in I$.*

2. Subtraction Bialgebras

Definition 2.1: *An algebra $X = (X, -, \ominus, 0)$ of type (2, 2, 0) is called a subtraction bialgebra if there exist two subsets X_1 and X_2 of X such that*

(*i*) $X = X_1 \cup X_2$,
(*ii*) $(X_1, -, 0)$ *is a subtraction algebra,*
(*iii*) $(X_2, \ominus, 0)$ *is a subtraction algebra.*
We denote by $X = X_1 \uplus X_2$.

Example 2.1: *Let $X = \{0, a, b, c, d, e\}$ and consider two proper subsets $X_1 = \{0, a, b\}$ and $X_2 = \{0, a, c, d, e\}$ of X together with Cayley tables respectively as follows:*

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

$\ominus$	0	a	c	d	e
0	0	0	0	0	0
a	a	0	a	a	a
c	c	c	0	c	c
d	d	d	d	0	d
e	e	e	e	e	0

Then $(X_1,-,0)$ *and* $(X_2,\ominus,0)$ *are subtraction algebras. Thus* $(X,-,\ominus,0)$ *is a subtraction bialgebra, i.e.,* $X = X_1 \uplus X_2$.

Example 2.2: *Let* $X = \{0,a,b,c,d,e,f,g\}$ *and consider two proper subsets* $X_1 = \{0,a,b,c\}$ *and* $X_2 = \{0,d,e,f,g\}$ *of* X *together with Cayley tables respectively as follows:*

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

$\ominus$	0	d	e	f	g
0	0	0	0	0	0
d	d	0	d	d	d
e	e	e	0	e	e
f	f	f	f	0	f
g	g	g	g	g	0

Then $(X_1,-,0)$ *and* $(X_2,\ominus,0)$ *are subtraction algebras. Thus* $(X,-,\ominus,0)$ *is a subtraction bialgebra, i.e.,* $X = X_1 \uplus X_2$.

Definition 2.2: *Let* $X = X_1 \uplus X_2$. *A subset* $A(\neq \varnothing)$ *of* X *is called a sub-subtraction bialgebra of* X *if there exist two subsets* A_1 *and* A_2 *of* X_1 *and* X_2, *respectively, such that*

(*i*) $A_1 \neq A_2$ *and* $A = A_1 \cup A_2$,
(*ii*) $(A_1,-,0)$ *is a sub-subtraction algebra of* $(X_1,-,0)$,
(*iii*) $(A_2,\ominus,0)$ *is a sub-subtraction algebra of* $(X_2,\ominus,0)$.

Example 2.3: *Let* X *be a subtraction bialgebra in Example 1, and let* $A_1 = \{0,a\}$ *and* $A_2 = \{0,c,d\}$. *In this case* $A_1 \neq A_2$ *and* A_1 *(resp.* A_2*) is a sub-subtraction algebra of* X_1 *(resp.* X_2*). Thus* $A = \{0,a,c,d\}$ *is sub-subtraction bialgebra of* X. *On the other hand, we can easly show that* $(A,\ominus,0)$ *is subtraction algebra. Also, note that* $A_3 = \{0,e\}$ *is a sub-*

subtraction algebra of X_2 *and* $A_1 \neq A_3$. *Thus* $B = \{0, a, e\}$ *is a sub-subtraction bialgebra of* X.

Theorem 2.1: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra and let* A *be a nonempty subset of* X. *Then* A *is a sub-subtraction bialgebra of* X *if and only if there exist two proper subsets* X_1 *and* X_2 *of* X *such that*

(*i*) $X = X_1 \cup X_2$, *where* $(X_1, -, 0)$ *and* $(X_2, \ominus, 0)$ *are subtraction algebras,*
(*ii*) $(A \cap X_1, -, 0)$ *is a sub-algebra of* $(X_1, -, 0)$,
(*iii*) $(A \cap X_2, \ominus, 0)$ *is a sub-algebra of* $(X_2, \ominus, 0)$.

Proof. Assume that A is a sub-subtraction bialgebra of X. Then $(A, -, \ominus, 0)$ is a subtraction bialgebra. Thus there exist two proper subsets A_1 and A_2 of A such that $A = A_1 \cup A_2$ and $(A_1, -, 0)$ and $(A_2, \ominus, 0)$ are subtraction algebras. Taking $A_1 = A \cap X_1$ and $A_2 = A \cap X_2$, we get that $(A_1, -, 0)$ and $(A_2, \ominus, 0)$ are sub-subtraction algebra of $(X_1, -, 0)$ and $(X_2, \ominus, 0)$ respectively.

Conversely, let A be a nonempty subset of a subtraction bialgebra $(X, -, \ominus, 0)$ satisfying conditions *(i)*, *(ii)* and *(iii)*. Hence

$$
\begin{aligned}
(A \cap X_1) \cup (A \cap X_2) &= ((A \cap X_1) \cup A) \cap ((A \cap X_1) \cup X_2) \\
&= ((A \cup A) \cap (X_1 \cup A)) \cap ((A \cup X_2) \cap (X_1 \cup X_2)) \\
&= (A \cap (A \cup X_1)) \cap ((A \cup X_2) \cap X) \\
&= A \cap (A \cup X_2) \text{ (since } A \subseteq A \cup X_1 \text{ and } A \cup X_2 \subseteq X \text{)} \\
&= A.
\end{aligned}
$$

The proof completes.

Definition 2.3: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra. A subset* $I (\neq \emptyset)$ *of* X *is called a biideal of* X *if there exist two subsets* I_1 *and* I_2 *of* X_1 *and* X_2, *respectively, such that* $I = I_1 \cup I_2$, $I_1 \lhd X_1$ *and* $I_2 \lhd X_2$.

Example 2.4: *Let* $X = \{0, a, b, c, x, y, z, t\}$ *and consider two proper subsets* $X_1 = \{0, a, b, c\}$ *and* $X_2 = \{0, x, y, z, t\}$ *of* X *together with Cayley tables respectively as follows:*

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$\ominus$	0	x	y	z	t
0	0	0	0	0	0
x	x	0	x	x	x
y	y	y	0	y	y
z	z	z	z	0	z
t	t	t	t	t	0

Then $X = X_1 \uplus X_2$, $I_1 = \{0,c\} \lhd X_1$ *and* $I_2 = \{0,z,t\} \lhd X_2$. *Therefore* $I = \{0,c,z,t\}$ *is a biideal of* X .

Example 2.5: *Let* X *be the subtraction bialgebra in Example 2.1, and let* $I_1 = \{0,b\} \lhd X_1$ *and* $I_2 = \{0,c,d,e\} \lhd X_2$. *Hence* $I = \{0,b,c,d,e\}$ *is a biideal of* X .

Example 2.6: *Let* $X = \{0,a,b,c,d,x,y\}$ *and consider two proper subsets* $X_1 = \{0,a,b,c,d\}$ *and* $X_2 = \{0,a,x,y\}$ *of* X *together with Cayley tables respectively as follows:*

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

$\ominus$	0	a	x	y
0	0	0	0	0
a	a	0	a	0
x	x	x	0	0
y	y	x	a	0

Then $X = X_1 \uplus X_2$, $I_1 = \{0,a\} \lhd X_1$ *and* $I_2 = \{0,x\} \lhd X_2$. *Hence* $I = \{0,a,x\}$ *is a biideal of* X . *But* $I = \{0,a,x\}$ *is not an ideal of* $(X_2,\ominus,0)$ *since* $y \ominus a = x \in I$ *and* $y \notin I$.

Theorem 2.2: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra. If* I *is a nonempty subset of* X *such that* $I \cap X_1 \lhd (X_1,-,0)$ *and* $I \cap X_2 \lhd (X_2,\ominus,0)$ *then* I *is a biideal of* X .

Proof. Taking $I_1 = I \cap X_1$ and $I_2 = I \cap X_2$ and hence

$$I_1 \cup I_2 = (I \cap X_1) \cup (I \cap X_2) = I \cap (X_1 \cup X_2) = I \cap X = I .$$

Thus I is a biideal of X .

Theorem 2.3: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra. Then any biideal of* X *is a sub-subtraction bialgebra of* X .

Proof. Straightforward.

Following example shows that the converse of Theorem 2.3 is not true.

Example 2.7: *Let* $X=\{0,a,b,c,d,e,f,g,x,y\}$ *and consider two proper subsets* $X_1=\{0,a,b,c,d,e,f,g\}$ *and* $X_2=\{0,a,x,y\}$ *of* X *together with Cayley tables respectively as follows:*

$-$	0	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0
a	a	0	a	a	a	a	a	a
b	b	b	0	b	b	b	b	b
c	c	c	c	0	c	c	c	c
d	d	d	d	d	0	d	d	d
e	e	e	e	e	e	0	e	e
f	f	f	f	f	f	f	0	f
g	g	g	g	g	g	g	g	0

$\ominus$	0	a	x	y
0	0	0	0	0
a	a	0	a	0
x	x	x	0	0
y	y	x	a	0

Then $X=X_1\uplus X_2$. *We say that* $A_1=\{0,a,b,c,d\}$ *and* $A_2=\{0,a,x\}$ *are sub-subtraction algebra of* X_1 *and* X_2, *respectively. Hence* $A=A_1\cup A_2=\{0,a,b,c,d,x\}$ *is a sub-subtraction bialgebra of* X. *However* A_1 *is an ideal of* X_1 *and* A_2 *is not an ideal of* X_2 *since* $y\ominus a=x\in A_2$ *and* $y\notin A_2$. *Hence* $A=\{0,a,b,c,d,x\}$ *is not an biideal of* X.

Let $X=X_1\uplus X_2$ be a subtraction bialgebra. Then we define the set $G(x,y)$ for any $x,y\in X$ in the following:

$$G(x,y)=\begin{cases} G_1(x,y) & , \quad x,y\in X_1\setminus X_2 \\ G_2(x,y) & , \quad x,y\in X_2\setminus X_1 \\ G_1(x,y)\cup G_2(x,y) & , \quad x,y\in X_1\cap X_2 \\ \varnothing & , \quad \text{in other cases} \end{cases}$$

Then we write the following definition.

Definition 2.4: *Let* $X=X_1\uplus X_2$ *be a subtraction bialgebra. Then* X *is called a complicated subtraction bialgebra if the nonempty set* $G(x,y)$ *for any* $x,y\in X$ *has the greatest element.*

Example 2.8: *Let* $X=\{0,a,b,c\}$ *and consider two subsets* $X_1=\{0,a\}$ *and* $X_2=\{0,a,b,c\}$ *of* X *together with Cayley tables respectively as follows:*

$-$	0	a
0	0	0
a	a	0

$\ominus$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $(X_1,-,0)$ *and* $(X_2,\ominus,0)$ *are subtraction algebras. Thus* $(X,-,\ominus,0)$ *is a subtraction bialgebra, i.e.,* $X = X_1 \uplus X_2$. *Then we obtain* $G_1(0,0)=\{0\}$, $G_1(0,a)=G_1(a,0)=\{0,a\}$, $G_1(a,a)=\{0,a\}$, *and also we have* $G_2(0,0)=\{0\}$, $G_2(0,a)=G_2(a,0)=\{0,a\}$, $G_2(0,b)=G_2(b,0)=\{0,b\}$, $G_2(0,c)=G_2(c,0)=\{0,a,b,c\}$, $G_2(a,a)=\{0,a\}$, $G_2(b,b)=\{0,b\}$, $G_2(c,c)=\{0,a,b,c\}$, $G_2(a,b)=G_2(b,a)=\{0,a,b,c\}$, $G_2(a,c)=G_2(c,a)=\{0,a,b,c\}$, $G_2(b,c)=G_2(c,b)=\{0,a,b,c\}$.

Therefore we can write all the sets $G(x,y)$ *for any* $x,y\in X$. *Some of them are in the following:*

$G(0,0)=G_1(0,0)\cup G_2(0,0)=\{0\}$, $G(0,a)=G_1(0,a)\cup G_2(0,a)=\{0,a\}$, $G(0,b)=\varnothing$, ... , $G(b,c)=G_2(b,c)=\{0,a,b,c\}$, $G(c,c)=G_2(c,c)=\{0,a,b,c\}$. *Thus* X *is a complicated subtraction bialgebra.*

Example 2.9: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra in Example 1. Then we have* $G_1(0,0)=\{0\}$, $G_1(0,a)=G_1(a,0)=\{0,a\}$, $G_1(a,a)=\{0,a\}$, $G_1(0,b)=G_1(b,0)=\{0,b\}$, $G_1(a,b)=G_1(b,a)=\{0,a,b\}$, $G_1(b,b)=\{0,b\}$ *and also we get* $G_2(0,0)=\{0\}$, $G_2(0,a)=G_2(a,0)=\{0,a\}$, $G_2(0,c)=G_2(c,0)=\{0,c\}$, $G_2(0,d)=G_2(d,0)=\{0,d\}$, $G_2(0,e)=G_2(e,0)=\{0,e\}$, $G_2(a,a)=\{0,a\}$, $G_2(c,c)=\{0,c\}$, $G_2(d,d)=\{0,d\}$, $G_2(e,e)=\{0,e\}$, $G_2(a,c)=G_2(c,a)=\{0,a,c\}$, $G_2(a,d)=G_2(d,a)=\{0,a,d\}$, $G_2(a,e)=G_2(e,a)=\{0,a,e\}$, $G_2(c,d)=G_2(d,c)=\{0,c,d\}$, $G_2(c,e)=G_2(e,c)=\{0,c,e\}$, $G_2(d,e)=G_2(e,d)=\{0,d,e\}$.

Therefore we can write all the sets $G(x,y)$ *for any* $x,y\in X$. *Some of them are in the following:*

$G(0,0)=G_1(0,0)\cup G_2(0,0)=\{0\}$, $G(0,a)=G_1(0,a)\cup G_2(0,a)=\{0,a\}$, $G(0,b)=\varnothing$, ... , $G(d,e)=G_2(d,e)=\{0,d,e\}$, $G(e,e)=G_2(e,e)=\{0,e\}$. *Since* $G(d,e)=G_2(d,e)=\{0,d,e\}$ *has not a greatest element,* X *is not a complicated subtraction bialgebra.*

Proposition 2.1: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra. If* X_1 *and* X_2 *are complicated subtraction algebras then* X *is a complicated bialgebra.*

Proof. Straightforward.

Theorem 2.4: *Let* $X = X_1 \uplus X_2$ *be a subtraction bialgebra and* $I(\neq \emptyset)$ *be subset of* X. *Then* I *is a biideal of* X *if and only if* $G(x,y) \subseteq I$ *for all* $x,y \in I$.

Proof. Let I be a biideal of X. Then there exist two proper subsets I_1 and I_2 of X_1 and X_2, respectively, such that $I = I_1 \cup I_2$ and $I_1 \lhd X_1$ and $I_2 \lhd X_2$. By Proposition 1.1, we have $G_1(x,y) \subseteq I_1$ for all $x,y \in I_1$ and $G_2(x,y) \subseteq I_2$ for all $x,y \in I_2$. Then we get that $G(x,y) \subseteq I$ for all $x,y \in I$.

Conversely, let $G(x,y) \subseteq I$ for all $x,y \in I$. Then since $X = X_1 \cup X_2$ and $I \subseteq X$, we write $I_1 = I \cap X_1$, $I_2 = I \cap X_2$ and $I = I_1 \cup I_2$. Hence by the definition of $G(x,y)$, we obtain $G(x,y) = G_1(x,y) \subseteq I_1$ for all $x,y \in I_1$ and $G(x,y) = G_2(x,y) \subseteq I_2$ for all $x,y \in I_2$. Hence using Proposition 1.1 again, we have $I_1 \lhd X_1$ and $I_2 \lhd X_2$. Then by Theorem 2.2, we have that I is a biideal of X.

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