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The Stability Analysis Depending Number of Predation in Discrete-Time Population Model and Allee Effect

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Abstract

In this paper, the stability analysis has been investigated by changing only the function $g(N_t)$ in the discrete-time population models as follows $N_{t+1} = N_t + \lambda N_t f(N_t) - N_t g(N_t) = F(\lambda, N_t)$. In fact, the authors Merdan and Duman have studied similar subject for different constants and function. In our case, this model has faster convergence and stable than the previous model under our assumptions on $g(N_t)$. We also observe that stability has been significantly increased in the model, when Allee effect is considered.

Keywords: Allee effect, stability analysis, discrete-time model.

Fark Zamanlı Popülasyon Modelinde Avcı Sayısına Bağlı Olarak Kararlılık Analizi ve Allee Etkisi

Özet

Bu çalışmada, sadece $g(N_t)$ fonksiyonunu değiştirerek, $N_{t+1} = N_t + \lambda N_t f(N_t) - N_t g(N_t) = F(\lambda, N_t)$ fark zamanlı popülasyon modelinin kararlılık analizi araştırıldı. Gerçekte yazarlar Merdan ve Duman, farklı sabit ve fonksiyonlar için benzer konuda çalıştılar. Bizim durumumuzda $g(N_t)$ üzerindeki varsayımlarımız altında bu model, önceki modelden daha hızlı kararlılığa ve yakınsamaya sahiptir. Biz aynı zamanda, Allee etkisi göz önüne alındığında modelin kararlılığının önemli bir şekilde arttığı gözlemledik.

Anahtar kelimeler: Allee etkisi, kararlılık analizi, fark zamanlı model.

Introduction

In recent years, the study of the stability in population dynamics have been an attractive search area. In these studies, most of the models have been constructed under the assumption of asexually reproducing organism. Such population dynamics models exhibit complexity in their movements. The general conclusion of these studies show that models are not structurally stable. This situation encouraged many scientists to modify the models and to develop better parameter estimation procedures [1-3].

According to the studies performed on different populations, sexually reproducing decreases the variations in the intensity of individual genotypes, and eventually increases the dynamic stability of the model. This situation is a conclusion of increasing parameter number [4].

Although Allee Effect may delay the stability in some population dynamics, it generally accelerates the stability of the models [5-10].

Merdan and Duman [5], have studied the stability of the following difference equation that involves predation effect at low population density,

$$N_{t+1} = N_t + \lambda N_t f(N_t) - N_t g(N_t) = F(\lambda, N_t) \tag{1}$$

and that equation with Allee Effect,

$$N_{t+1} = N_t + \lambda^* N_t \alpha(N_t) f(N_t) - N_t g(N_t) = F_\alpha(\lambda^*, N_t) \tag{2}$$

where N_t is the density at time t, λ is the per capita growth rate, and $f(N_t)$ is the function describing interactions among the individuals. They showed that when the population models given by Eq. (1) is subject to an Allee effect, stability of the equilibrium points increases. For f, g and α functions, we write,

f_1) $f(0) > 0$

f_2) for $\forall N \in (0,\infty)$, $f'(N) < 0$

f_3) $f(N)$ has at least one positive root.

g_1) for g(0)=0 and $\forall N \in (0,\infty)$, $0 < g(N) < \infty$

g_2) $\lim_{N\to\infty} g(N) = 0$

g_3) $g'(N_c) = 0$, $N_c \in (0,\infty)$ has only one critical value.

a_1) If N=0, then $\alpha(N) = 0$

a_2) for $N \in (0,\infty)$, $\alpha'(N) > 0$

a_3) $\lim_{N\to\infty} \alpha(N) = 1$

Now, we will give some theorems and results that is required later in this study.

Theorem 1.1: Assume that a positive equilibrium point N^* of Eq. (1) exist. Then it is locally stable if

$$\frac{-2}{N^* g(N^*)} < \frac{f'(N^*)}{f(N^*)} - \frac{g'(N^*)}{g(N)} < 0$$

holds [5].

Result 1.1: Let N^* be a positive equilibrium point of Eq. (1). Then we have three possible cases [5]:

i) If $N^* < N_c$, then the sufficient condition to make N^* unstable is,

$$\frac{f'(N^*)}{f(N^*)} - \frac{g'(N^*)}{g(N^*)} < \frac{-2}{N^* g(N^*)}$$

ii) If $N^* = N_c$, then it is unstable if and only if

$$\frac{f'(N^*)}{f(N^*)} < \frac{-2}{N^* g(N^*)}$$

iii) If $N^* > N_c$, then it is unstable if and only if

$$\frac{f'(N^*)}{f(N^*)} - \frac{g'(N^*)}{g(N^*)} < \frac{-2}{N^* g(N^*)} \quad \text{or} \quad \frac{f'(N^*)}{f(N^*)} > \frac{g'(N^*)}{g(N^*)}.$$

Theorem 1.2: Assume that a positive equilibrium point N^* of Eq. (2) exists. Then, it is locally stable if

$$\frac{-2}{N^* g(N^*)} < \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f'(N^*)}{f(N^*)} - \frac{g'(N^*)}{g(N^*)} < 0$$

satisfied [5].

Result 1.2: Let N^* be a positive equilibrium point of Eq. (1) satisfying $N^* \in (0, N_c]$, where g has only one critical point on $(0,\infty)$, that is, there is only one point $N_c \in (0,\infty)$ such that $g'(N_c) = 0$ [5]. Then, Theorem 1 and Corallary 1 (i)-(ii) imply that N^* is unstable if and only if $F'(\lambda, N^*) < -1$ [5]. On the other hand, since $\frac{\alpha'(N^*)}{\alpha(N^*)}$ is always positive, then we write $F'(\lambda, N^*) < F'_\alpha(\lambda^*, N^*)$ [5].

Result 1.3: The Allee effect in Eq. (2) increases the local stability of the equilibrium point N^* of Eq. (1) provided that $N^* \le N_c$ [5].

Main results

The main purpose of this study is to examine the stability of Eq. (1) and Eq. (2) with the choice of $g(N_t) = \frac{N_t}{1+N_t^2} + \frac{1}{N_t+1}$ for $N \in (0,\infty)$, K=1 and $f(N_t) = \frac{1-N_t}{K}$, whereas Merdan and Duman [5] examined the same problem by taking $g(N_t)$ as,

$$g(N_t) = \frac{N_t}{1+N_t^2}. \tag{3}$$

They obtained respectively following equations from Eq. (1) and Eq. (2) respectively,

$$N_{t+1} = N_t + \lambda N_t (1 - \frac{N_t}{K}) - \frac{N_t^2}{1+N_t^2} \tag{4}$$

$$N_{t+1} = N_t + \lambda^* \frac{N_t^2}{\alpha + N_t} (1 - \frac{N_t}{K}) - \frac{N_t^2}{1+N_t^2} \tag{5}$$

where $\lambda > 0$ ve $\lambda^* = \frac{\lambda}{\alpha(N^*)}$.

Now, if we consider the new form of $g(N_t)$ as,

$$g(N_t) = \frac{N_t}{1+N_t^2} + \frac{1}{N_t+1} \tag{6}$$

we obtained following results from Eq. (1) and Eq.(2) respectively.

$$N_{t+1} = N_t + \lambda N_t (1 - \frac{N_t}{K}) - \frac{N_t^2}{1 + N_t^2} - \frac{N_t}{N_t + 1} \tag{7}$$

$$N_{t+1} = N_t + \lambda^{**} \frac{N_t^2}{\alpha + N_t} (1 - \frac{N_t}{K}) - \frac{N_t^2}{1 + N_t^2} - \frac{N_t}{N_t + 1} \tag{8}$$

where $\lambda > 0$ and $\lambda^{**} = \frac{\lambda}{\alpha(N^{**})}$. The comparison of the results of the previous and the proposed models for $g(N_t)$ is shown in Figure 1.

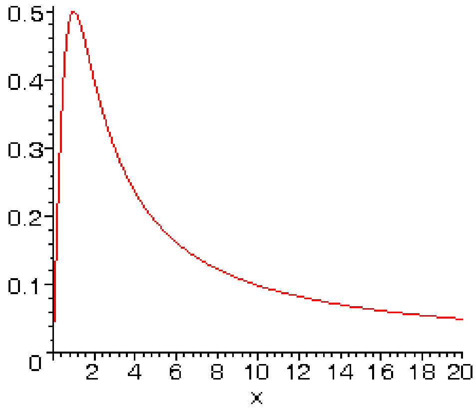

Figure 1.a. The graph of the function (3).

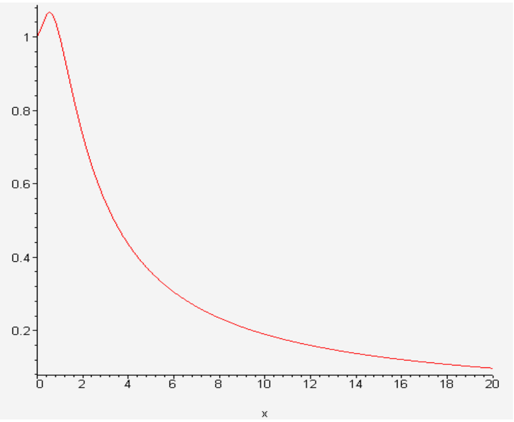

Figure 1.b. The graph of the function (6).

Population movement for a certain λ

By using Matlab package programming language, we can obtain Figure 2 from Eq. (7) and Eq. (8) respectively for $\lambda = 2.33$ and K=1. Then, results from Eq. (4) and Eq. (5) are shown in Figure 3 respectively.

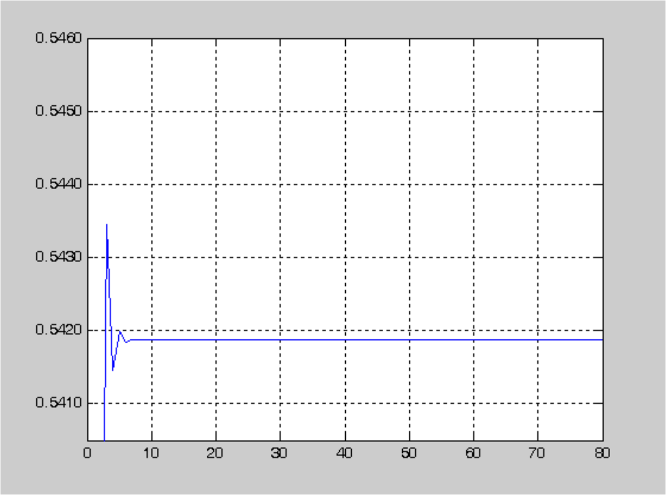

Figure 2.a. The graph of Model (7), for $\lambda = 2.33$ and $N_0 = 0.4$

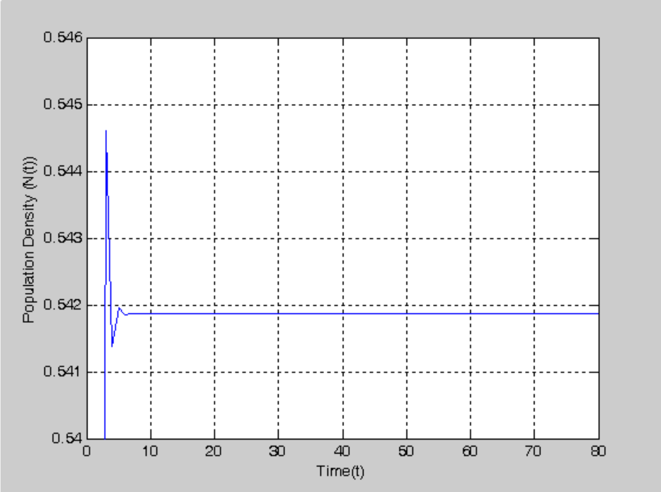

Figure 2.b. The graph of Model (8), for $\lambda^{**} = 2.5450$, $N_0 = 0.4$ and $\alpha = 0.05$

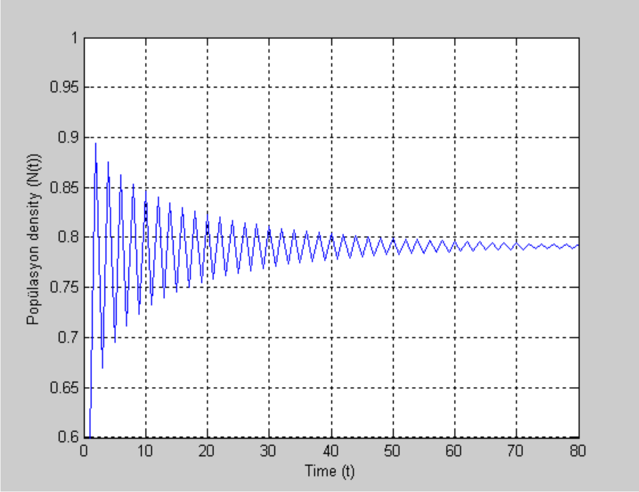

Figure 3.a. The graph of Model (4), for $\lambda \cong 2.33$ and $N_0 = 0.6$

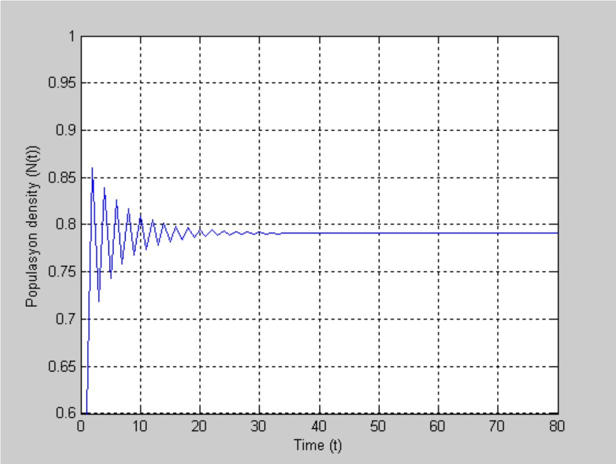

Figure 3.b. The graph of Model (5), for $\lambda^{*} = 2.9190$, $N_0 = 0.6$ and $\alpha = 0.2$.

Population movements for new model

For $\lambda = 2.5$ and K=1, we obtain respectively Figure 4.a and Figure 4.b from Eq. (7) and Eq. (8) and obtain respectively Figure 5.a and Figure 5.b from Eq. (4) and Eq. (5).

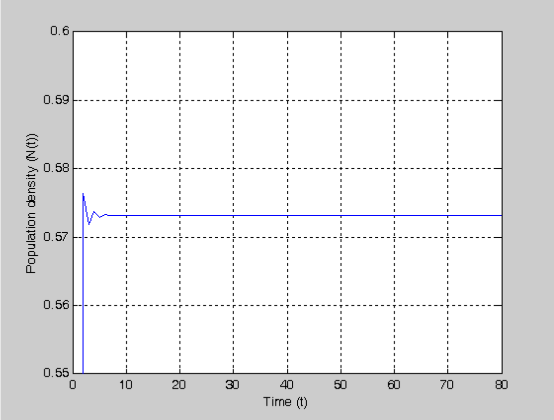

Figure 4.a. The graph of Model (7), for $\lambda = 2.5$ and $N_0 = 0.4$

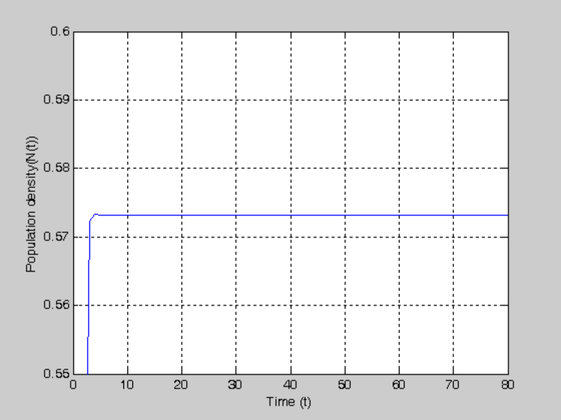

Figure 4.b. The graph of Model (8), for $\lambda^{**} = 2.37235$, $N_0 = 0.4$ and $\alpha = 0.2$

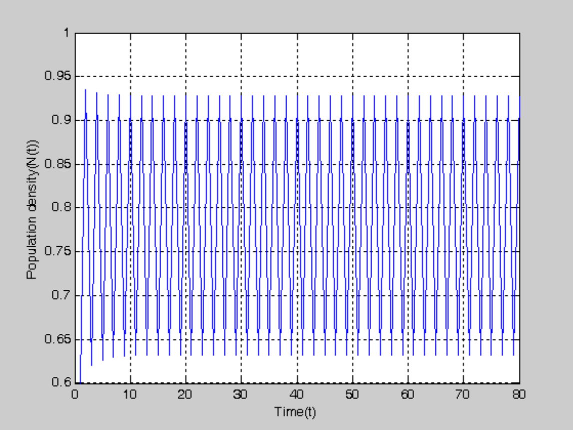

Figure 5.a. The graph of Model (4), for $\lambda = 2.5$ and $N_0 = 0.6$

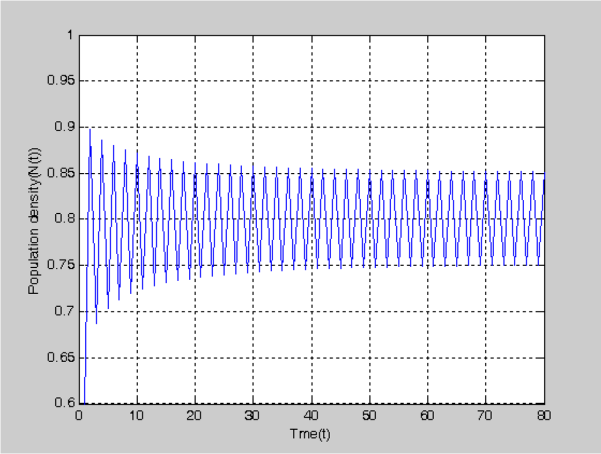

Figure 5.b. The graph of Model (5), for $\lambda^* = 3.3111$, $N_0 = 0.6$ and $\alpha = 0.2$

In the following table (see Table 1 below), it is seen that the comparison of the population movements for different parameters. In fact, while $s \rightarrow \infty$, the function (6) converges to the function (3). Therefore, it can be concluded that the behavior of the model (7) and (8) in our interest converges to that of (4) and (5).

Table 1. Equilibrium points values according to certain values of λ, α and s for N_0 initial conditions

$\frac{N_t}{N_t + s}$, (**s**=1,100,10^{14})	N_0	λ	α	λ^{**}	N^{**}
1	0.4	2.5	0.2	3.37235	0.57316
			0.4	4.24471	
			0.6	5.11706	
			0.8	5.98941	
100	0.4	2.5	0.2	3.12433	0.80086
			0.4	3.74865	
			0.6	4.37298	
			0.8	4.99730	
10^{14}	0.4	2.5	0.2	3.12140	0.80463
			0.4	3.74281	
			0.6	4.36420	
			0.8	4.98560	

N_0	α	λ	$\frac{N_t}{N_t+s}$, **(s=1,100,10^{14})**	λ^*	N^{**}	N^*
0.6	0.2	2.33	1	3.190	0.54187	0.7912
			100	2.922	0.78715	
			10^{14}	2.919	0.79116	
		2.43	1	3.297	0.56077	0.7993
			100	3.041	0.79543	
			10^{14}	3.038	0.79930	

N_0	α	λ	$\frac{N_t}{N_t+s}$, **(s=1,100,10^{14})**	λ^{**}	N^{**}
0.4	0.05	2.33	1	2.545	0.54187
			100	2.478	0.78715
			10^{14}	2.477	0.79116
		2.43	1	2.647	0.56077
			100	2.583	0.79543
			10^{14}	2.582	0.79930

Discussion

In this study, we investigated the stability of the model under a different choice of the function which increases the number of predators. We compared the stability of the models with different predator functions by the corresponding graphs. We also observed the behavior of the model with Allee effect. In addition to, we have also observed the changes in the model with low population density and with low Allee constant. It is interesting that even with low population density and high Allee effect, there is no extinction. However, it reaches to the equilibrium point faster.

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