Turk. J. Math. Comput. Sci.
13(1)(2021) 106–114
© MatDer
DOI : 10.47000/tjmcs.858793

Quaternionic and Dual Quaternionic Darboux Ruled Surfaces

Abdussamet Çalışkan

Fatsa Vocational School, Accounting and Tax Applications, Ordu University, Ordu.

Received: 12-01-2021 • Accepted: 03-05-2021

Abstract. In this paper, firstly the ruled surface drawn by the Darboux vector is expressed as a quaternion. Then, the spatial quaternionic definition of the striction curve is given and the integral invariants of the surface are calculated. Finally, the ruled surface which corresponds to a dual curve drawn by a dual Darboux vector is derived with the help of dual spatial quaternions and dual integral invariants of the ruled surface are obtained.

2010 AMS Classification: 53A04, 53A05

Keywords: Darboux vector, distribution parameter, dual spherical curve, dual angle of pitch, quaternion, spatial quaternion, ruled surface.

1. Introduction

Quaternions arose historically from Hamilton's essays to generalize complex numbers in some way that would apply to 3D space. He struggled for years attempting to make sense of an unsuccessful algebraic system containing one real and two imaginary parts. Hamilton had a brilliant stroke of imagination and invented in a single instant the idea of a three-part imaginary system that became the quaternion algebra [7, 9]. Quaternions are used in many scientific fields such as physic, quantum, camera, and robot kinematic. Shoemake takes the concept of the orientation frame for moving 3D objects and cameras and introduces quaternions to animators as a solution [10]. Quaternions are also used in curves and surfaces theory. The Serret-Frenet formulae for quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ are introduced by Bharathi and Nagaraj [3]. Şenyurt *et al.* calculate curvature and torsion of spatial quaternionic involute curve [13]. Recently, surfaces and ruled surfaces have been studied as quaternions. The invariants (shape operator, Gauss curvature, etc.) of surfaces are expressed quaternionically [1, 2]. Ruled surfaces are examined in both Euclidean space and dual space and some important results are given [4, 14]. Let α be a unit-speed curve. Then the three vector fields $\vec{t}(s), \vec{n}(s)$ and $\vec{b}(s)$ on the curve are unit vector fields that are mutually orthogonal at each point. We call $\vec{t}(s), \vec{n}(s)$ and $\vec{b}(s)$ the Frenet vectors on the curve. The Frenet formulas can be given

$$\vec{t}'(s) = \kappa(s)\vec{n}(s), \quad \vec{n}'(s) = -\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s), \quad \vec{b}'(s) = -\tau(s)\vec{n}(s)$$

where κ and τ the first and second curvature of unit speed curve, respectively [6].

Email address: abdussamet65@gmail.com (A. Çalışkan)

2. Preliminaries

Definition 2.1. A ruled surface in $\mathbb{R}^3$ is a surface that contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\vec{\varphi}(s,v) = \vec{\alpha}(s) + v\vec{x}(s). \tag{2.1}$$

where we call α the base curve, $\vec{x}$ the generator vector of ruled surface [6].

Dual numbers were introduced in the 19th century by W. K. Clifford. The set of dual numbers given by $ID = \{x + \varepsilon x^* : x, x^* \in IR, \varepsilon^2 = 0\}$. The set, $ID^3 = \{\vec{X} = \vec{x} + \varepsilon \vec{x}^* : \vec{x}, \vec{x}^* \in IR^3, \varepsilon^2 = 0\}$ meets the all real vector space axioms over the ring. The set is a module over the ring ID which is named ID - module or dual space. The vector $\vec{x}^*$ call vectorial moment of the vector $\vec{x}$ satisfying $\vec{x}^* = \vec{\alpha} \wedge \vec{x}$. If $\|\vec{X}\| = 1$, then the dual vector $\vec{X}$ is the dual point on the dual unit sphere. According to E.Study theorem, there exists a one-to-one transformation between the dual points on the unit dual sphere and the oriented lines in IR^3. A dual curve corresponds to a ruled surface. This dual curve is called the dual spherical image of the ruled surface [8].

The dual expression of ruled surface in (2.1) is

$$\vec{\varphi}(s,u) = \vec{x}(s) \wedge \vec{x}^*(s) + u\vec{x}(s)$$

where $\wedge$ is cross product.

Real quaternion is defined by [7, 9]

$$q = d + a\vec{e_1} + b\vec{e_2} + c\vec{e_3},\ \ a,b,c,d \in \mathbb{R}, \vec{e_1}, \vec{e_2}, \vec{e_3} \in \mathbb{R}^3.$$

The quaternion multiplication of q_1 and q_2 is given by [9]

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ &+ (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3. \end{aligned}$$

The quaternion inner product is defined as [3]

$$h(q_1,q_2) = \frac{1}{2}(q_1 \times \bar{q_2} + q_2 \times \bar{q_1}). \tag{2.2}$$

Let q be real quaternion. The quaternion satisfies $q + \bar{q} = 0$. In this case, the quaternion q is $q = a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$. This new expression is called spatial(pure) quaterninon and its set is denoted $\mathbb{Q}$ [3]. As a result, the multiplication of the two spatial quaternions is [3]

$$q_1 \times q_2 = -\langle q_1, q_2\rangle + q_1 \wedge q_2.$$

Definition 2.2. [3] Let $s \in I = [0,1]$ be the arc parameter along the smooth curve

$$\alpha : [0,1] \ \ \rightarrow \ \ \mathbb{Q},\ \ \alpha(s) = \sum_{n=1}^{3} \alpha_i(s)e_i.$$

This is called a spatial quaternionic curve.

Theorem 2.3. *[14] The angle of pitch and the pitch of the closed ruled surface, λ_x and l_x, are equal to the projection of the generator x on the Steiner rotation vector d and the Steiner translation vector V*

$$\begin{aligned} \lambda_x &= h(\vec{d}(s), \vec{x}(s)) \\ l_x &= h(\vec{V}(s), \vec{x}(s)). \end{aligned} \tag{2.3}$$

Let q and $q*$ be two real quaternions. Dual quaternion is denoted by [8]

$$Q = D + A\vec{e_1} + B\vec{e_2} + C\vec{e_3},\ \ D, A, B, C \in \mathbb{D}.$$

The symmetric dual-valued bilineer form H which is defined as [11]

$$H(P,Q) = \frac{1}{2}(P \times \overline{Q} + Q \times \overline{P}).$$

Let Q be dual quaternion. The dual quaternion satisfies $Q + \bar{Q} = 0$. In this case, the quaternion Q is $Q = A\vec{e_1} + B\vec{e_2} + C\vec{e_3}$. This new expression is called dual spatial quaterninon and its set is denoted $\mathbb{Q}_{\mathbb{D}}$ [11].

The dual quaternionic expression of the pitch of the ruled surface is given by [5]

$$L_X = h(\vec{d}^*(s), \vec{x}(s)) + h(\vec{d}(s), \vec{x}^*(s)). \tag{2.4}$$

The dual angle of pitch of the ruled surface is given as quaternionic [4]

$$\Lambda_x = -H(\vec{D}(s), \vec{X}(s)) \tag{2.5}$$

where $\vec{X}(s) = \vec{x}(s) + \varepsilon\vec{x}^*(s)$ and $\vec{D}(s) = \vec{d}(s) + \varepsilon\vec{d}^*(s)$.

The distribution parameter is defined as the ratio of the shortest distance between successive generators to the angle between successive generators. Without specifying the arc-parameter "s" of the curve for the sake of brevity, we calculate the integral invariants of the ruled surface. The distribution parameter is quaternionically given by [14]

$$P_x = \frac{h(\vec{x} \times \vec{x}', \vec{\alpha}')}{\mathbf{N}(\vec{x}')^2} = \frac{1}{2}\frac{\left((\vec{x} \times \vec{x}') \times \overline{\vec{\alpha}'} + \vec{\alpha}' \times \overline{(\vec{x} \times \vec{x}')}\right)}{\mathbf{N}(\vec{x}')^2}. \tag{2.6}$$

3. Main Results

3.1. **Quaternionic Darboux Ruled Surface.**

In this section, we first calculate the striction curve of the quaternionic ruled surface drawn by the Darboux vector. Then, if we investigate the distribution parameter of the surface, it is seen that the surface is developable. Finally, integral invariants (pitch, and angle of pitch) of the surface are obtained.

The Darboux vector is the angular velocity vector of the Frenet frame of a space curve [12]. According to spatial quaternion, the vector is expressed as

$$\vec{w}(s) = \vec{n}(s) \times \vec{n}'(s) = \tau(s)\vec{t}(s) + \kappa(s)\vec{b}(s). \tag{3.1}$$

The Steiner rotation and Steiner translation vectors are, respectively,

$$\vec{d}(s) = \oint \vec{w}(s)ds = \vec{t}(s)\oint \tau(s)ds + \vec{b}(s)\oint \kappa(s)ds, \;\; \vec{V}(s) = \oint \vec{t}(s)ds.$$

Quaternionic Darboux ruled surface is a surface swept out by straight line moving along spatial quaernionic curve α. The surface has a parametrization

$$\vec{\varphi}(s, v) = \vec{\alpha}(s) + v\vec{w}(s). \tag{3.2}$$

We call α the base curve and $\vec{w}$ the Darboux vector (generator vector).

A ruled surface has striction curve that other surfaces do not have. It has an important geometric meaning such that if there exists a common perpendicular to two constructive rulings, then the foot of the common perpendicular on the main ruling is called striction point and striction curve is the locus of these points. Now, by using quaternionic definition of the striction curve [14] and by taking into consideration the equation (2.2), we formula the striction curve belonging to φ

$$\begin{aligned} \vec{r}(s) &= \vec{\alpha}(s) - \frac{h(\vec{w}'(s), \vec{t}(s))}{\mathbf{N}(\vec{w}'(s))^2}\vec{w}(s) \\ &= \vec{\alpha}(s) - \frac{h(\tau'(s)\vec{t}(s) + \kappa'(s)\vec{b}(s), \vec{t}(s))}{\kappa'^2(s) + \tau'^2(s)}\vec{w}(s) \\ &= \vec{\alpha}(s) - \frac{\tau(s)\tau'(s)}{\kappa'^2(s) + \tau'^2(s)}\vec{t}(s) - \frac{\kappa(s)\tau'(s)}{\kappa'^2(s) + \tau'^2(s)}\vec{b}(s). \end{aligned} \tag{3.3}$$

Then the following proposition can be given:

Proposition 3.1. *The striction curve of the quaternionic ruled surface drawn by the Darboux vector is*

$$\vec{r}(s) = \vec{\alpha}(s) - \frac{\tau(s)\tau'(s)}{\kappa'^2(s) + \tau'^2(s)}\vec{t}(s) - \frac{\kappa(s)\tau'(s)}{\kappa'^2(s) + \tau'^2(s)}\vec{b}(s).$$

Theorem 3.2. *The quaternionic ruled surface drawn by the Darboux vector is developable.*

Proof. A ruled surface is developable if and only if its distribution parameter is zero. Based on the definition from the spatial quaternion and by using the equation (2.6), we have derived equation:

$$P_w = \frac{h(\vec{w} \times \vec{w}', \vec{\alpha}')}{\mathbf{N}(\vec{w}'(s))^2}.$$

Considering the numerator part in the above equation, we can write

$$\begin{aligned} h(\vec{w} \times \vec{w}', \vec{t}) &= \frac{1}{2}\Big((\vec{w} \times \vec{w}') \times \overline{\vec{t}} + \vec{t} \times \overline{(\vec{w} \times \vec{w}')}\Big) \\ &= \frac{1}{2}\Big(-(\vec{w} \times \vec{w}') \times \vec{t} + \vec{t} \times (\overline{\vec{w}'} \times \overline{\vec{w}})\Big) \\ &= \frac{1}{2}\Big(-\Big(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}\Big) \times \vec{t} + \vec{t} \times \Big(-\tau\tau' - \kappa\kappa' - (\kappa\tau' - \tau\kappa')\vec{n}\Big)\Big) \\ &= \frac{1}{2}\Big((-\tau\tau' - \kappa\kappa')\vec{t} + (\kappa\tau' - \tau\kappa')\vec{b} - (-\tau\tau' - \kappa\kappa')\vec{t} - (\kappa\tau' - \tau\kappa')\vec{b}\Big) \\ &= 0. \end{aligned}$$

Thus, $P_w = 0$ is found. Then, the quaternionic Darboux ruled surface is developable. □

Theorem 3.3. *The pitch, and the angle of the pitch quaternionic ruled surface drawn by the Darboux vector are*

$$\begin{cases} l_w = \tau \oint ds, \\ \lambda_w = \kappa \oint \kappa ds + \tau \oint \tau ds. \end{cases}$$

Proof. An orthogonal trajectory of quaternionic Darboux ruled surface is defined by differential equation

$$h(\vec{w}, d\vec{\varphi}) = 0 \Rightarrow h(\vec{w}, d\vec{\alpha} + \vec{w}dv + vd\vec{w}) = 0 \Rightarrow -h(\vec{w}, d\vec{\alpha}) = dv.$$

According to quaternionic inner product, the pitch of closed ruled surface is given by

$$l_x = -\oint_{(\alpha)} h(d\vec{\alpha}, \vec{x}) = \oint_{(\alpha)} dv.$$

By taking into consideration the above definition of the pitch, the pitch of the quaternionic Darboux ruled surface is

$$\begin{aligned} l_w &= h(\oint d\vec{\alpha}, \vec{w}) = h(\oint \vec{t} ds, \vec{w}) \\ &= \frac{1}{2}\Big(\vec{t} \oint ds \times \overline{\vec{w}} + \vec{w} \times \overline{\vec{t} \oint ds}\Big) \\ &= \frac{1}{2}\Big(\vec{t} \times (-\tau\vec{t} - \kappa\vec{b}) \oint ds - (\tau\vec{t} + \kappa\vec{b})) \times \vec{t} \oint ds \\ &= \tau \oint ds. \end{aligned}$$

From the equations (2.3), the angle of pitch of quaternionic Darboux ruled surface is found as

$$
\begin{aligned}
\lambda_w &= h(\vec{d},\vec{w}) = \frac{1}{2}\left(\vec{d}\times\overline{\vec{w}} + \vec{w}\times\overline{\vec{d}}\right)\\
&= \frac{1}{2}\Big(-(\vec{t}\oint\tau ds + \vec{b}\oint\kappa ds)\times\vec{w} + \vec{w}\times(-\vec{t}\oint\tau ds - \vec{b}\oint\kappa ds)\Big)\\
&= \frac{1}{2}\Big(-(\vec{t}\times\vec{w})\oint\tau ds - (\vec{b}\times\vec{w})\oint\kappa ds - (\vec{w}\times\vec{t})\oint\tau ds - (\vec{w}\times\vec{b})\oint\kappa ds)\Big)\\
&= \frac{1}{2}\Big(-(\vec{t}\times\vec{w})\oint\tau ds - (\vec{b}\times\vec{w})\oint\kappa ds - (\vec{w}\times\vec{t})\oint\tau ds - (\vec{w}\times\vec{b})\oint\kappa ds)\Big)\\
&= \frac{1}{2}\Big(-(\vec{t}\times(\tau\vec{t}+\kappa\vec{b}))\oint\tau ds - (\vec{b}\times(\tau\vec{t}+\kappa\vec{b}))\oint\kappa ds - ((\tau\vec{t}+\kappa\vec{b})\times\vec{t})\oint\tau ds\\
&\quad - ((\tau\vec{t}+\kappa\vec{b})\times\vec{b})\oint\kappa ds)\Big)\\
&= \frac{1}{2}\Big(-(-\tau-\kappa\vec{n})\oint\tau ds - (\tau\vec{n}-\kappa)\oint\kappa ds - (-\tau+\kappa\vec{n})\oint\tau ds - (-\tau\vec{n}-\kappa\vec{b})\oint\kappa ds)\Big)\\
&= \kappa\oint\kappa ds + \tau\oint\tau ds.
\end{aligned}
$$

□

Example 3.4. Consider quaternionic curve α defined by

$$\alpha(s) = \left(\frac{1}{\sqrt{2}}\cos^2(s), \frac{1}{\sqrt{2}}\sin^2(s), \frac{1}{2}\sin(2s)\right).$$

Frenet invariants belonging to the curve are

$$
\begin{aligned}
t(s) &= \frac{1}{\sqrt{1+\sin^2(s)}}\left(-\frac{\sqrt{2}}{2}\sin(2s), \frac{\sqrt{2}}{2}\sin(2s), \cos(2s)\right),\\
n(s) &= \frac{1}{\sqrt{1+\sin^2(s)}}\left(-\frac{\sqrt{2}}{2}\cos(2s), \frac{\sqrt{2}}{2}\cos(2s), -\sin(2s)\right),\\
b(s) &= \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right), \quad \kappa(s) = \frac{2}{\left(1+\sin^2(s)\right)^{\frac{3}{2}}}, \quad \tau(s) = 0.
\end{aligned}
$$

By considering the equation (3.1), the Darboux vector is written

$$\vec{w}(s) = \vec{n}(s)\times\vec{n}'(s) = \left(-\frac{\sqrt{2}}{\left(1+\sin^2(s)\right)^{\frac{3}{2}}}, -\frac{\sqrt{2}}{\left(1+\sin^2(s)\right)^{\frac{3}{2}}}, 0\right).$$

If α, the Darboux vector and quaternion inner product are substituted in the equation (3.3), base curve is to be striction curve. So, $\vec{r}(s) = \vec{\alpha}(s)$. If equation (3.2) is taken into account, then we obtain the closed ruled surface corresponding to the Darboux vector $\vec{w}$ as

$$\vec{\varphi}(s,v) = \left(\frac{1}{\sqrt{2}}\cos^2(s) - v\frac{\sqrt{2}}{\left(1+\sin^2(s)\right)^{\frac{3}{2}}}, \frac{1}{\sqrt{2}}\sin^2(s) - v\frac{\sqrt{2}}{\left(1+\sin^2(s)\right)^{\frac{3}{2}}}, \frac{1}{2}\sin(2s)\right).$$

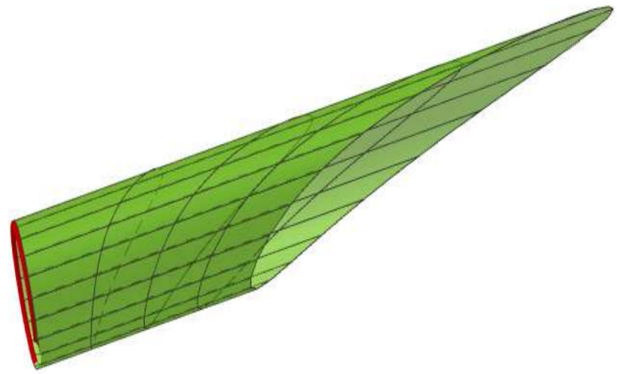

Figure 1. The green surface show the ruled surface and the red curve show the striction curve (base curve).

3.2. **Dual Quaternionic Darboux Ruled Surface.** In this section, the ruled surface which corresponds to a dual curve drawn by dual Darboux vector is derived with the help of dual spatial quaternions. Then, dual integral invariants of the surface are obtained as a quaternion.

A dual curve on a dual unit sphere corresponds to a ruled surface. The geometric location of dual vector $\vec{W}(s) = \vec{w}(s) + \varepsilon\vec{w^*}(s)$ draws dual curve on dual unit sphere.

The dual vector $\vec{D}(s) = \vec{d}(s) + \varepsilon\vec{d^*}(s)$ is called the dual Steiner vector [8]. By considering dual Darboux vector, we display

$$\begin{aligned}\vec{D}(s) &= \vec{d}(s) + \varepsilon\vec{d^*}(s) = \vec{t}(s)\oint\tau(s) + \vec{b}(s)\oint\kappa(s) + \varepsilon\Big(\vec{t^*}(s)\oint\tau(s) + \vec{t}(s)\oint\tau^*(s) \\ &\quad +\vec{b^*}(s)\oint\kappa(s) + \vec{b}(s)\oint\kappa^*(s)\Big).\end{aligned}$$

The parametric equation of dual quaternionic Darboux ruled surface (DQDR) corresponding to the dual curve is

$$\varphi_W(s,u) = \vec{w}(s)\times\vec{w}^*(s) + u\vec{w}(s), \vec{w}^*(s) = \vec{\alpha}(s)\times\vec{w}(s). \tag{3.4}$$

Without specifying the arc-parameter "s" of the curve for the sake of brevity, we calculate the integral invariants of DQDR.

Theorem 3.5. *The dual quaternionic Darboux ruled surface (DQDR) corresponding to the dual curve is developable.*

Proof. Based on the equations from (2.6) and (3.4), we have derived the distribution parameter:

$$\begin{aligned}P_W &= \frac{h(\vec{w}\times\vec{w'}, (\vec{w}\times\vec{w}^*)')}{\mathbf{N}(\vec{w'})^2} \\ &= \frac{h\left(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}, (\vec{w'}\times(\tau\vec{t^*} + \kappa\vec{b^*}) + \vec{w}\times(\tau\vec{t^*} + \kappa b^*)'\right)}{\mathbf{N}(\vec{w'})^2} \\ &= \frac{h\left(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}, (\vec{w'}\times(\tau\vec{t^*} + \kappa\vec{b^*}) + \vec{w}\times(\tau\vec{t^*} + \kappa\vec{b^*})'\right)}{\mathbf{N}(\vec{w'})^2}.\end{aligned}$$

Considering the numerator part in the above equation, we can write

$$\begin{aligned}
& h(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}, (\vec{w}^{*\prime} \times (\tau\vec{t}^* + \kappa\vec{b}^*) + \vec{w} \times (\tau\vec{t}^* + \kappa\vec{b}^*)')\\
= \; & h(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}, \tau\tau'(\vec{t} \times \vec{t}^*) + \tau'\kappa(\vec{t} \times \vec{b}^*) + \kappa'\tau(\vec{b} \times \vec{t}^*) + \kappa\kappa'(\vec{b} \times \vec{b}^*)\\
& + \tau'(\vec{w} \times \vec{t}^*) + \tau(\vec{w} \times \vec{t}^{*\prime}) + \kappa'(\vec{w} \times \vec{b}^*) + \kappa(\vec{w} \times \vec{b}^{*\prime}))\\
= \; & h(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}, -\tau^2\vec{t} + \kappa\tau^2(\vec{t} \times \vec{n}^*) - \kappa\tau\vec{b} + \kappa^2\tau(\vec{b} \times \vec{n}^*) - \kappa\tau\vec{b} - \kappa\tau^2(\vec{t} \times \vec{n}^*)\\
& + \kappa^2\vec{t} - \kappa^2\tau(\vec{b} \times \vec{n}^*))\\
= \; & h(-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}, (\kappa^2 - \tau^2)\vec{t} - 2\kappa\tau\vec{b})\\
= \; & \frac{1}{2}\Big(\big[-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}\big] \times \overline{\big[(\kappa^2 - \tau^2)\vec{t} - 2\kappa\tau\vec{b}\big]} + \big[(\kappa^2 - \tau^2)\vec{t} - 2\kappa\tau\vec{b}\big]\\
& \times \overline{\big[-\tau\tau' - \kappa\kappa' + (\kappa\tau' - \tau\kappa')\vec{n}\big]}\\
= \; & 0.
\end{aligned}$$

Thus, $P_W = 0$ is found. It is indicated that the distribution parameter of DQDR is developable. □

Theorem 3.6. *The pitch and dual angle of pitch of the DQDR corresponding to the dual curve are*

$$\begin{cases} L_w = \tau \oint \tau^* + \kappa \oint \kappa^*, \\ \Lambda_w = -\tau \oint \tau - \kappa \oint \kappa - \varepsilon\left(\tau \oint \tau^* + \kappa \oint \kappa^*\right). \end{cases}$$

Proof. By using the equation (2.4), the dual quaternionic expression of the pitch of the ruled surface in (3.4) is

$$\begin{aligned}
L_W \; = \; & h(\vec{d}, \vec{w}^*) + h(\vec{d}^*, \vec{w})\\
= \; & \frac{1}{2}\left(\vec{d} \times \overline{\vec{w}^*} + \vec{w}^*(s) \times \overline{\vec{d}}\right) + \frac{1}{2}\left(\vec{d}^* \times \overline{\vec{w}} + \vec{w} \times \overline{\vec{d}^*}\right)\\
= \; & \frac{1}{2}\Big(\big(\vec{t}\oint \tau + \vec{b}\oint \kappa\big) \times \overline{\vec{w}^*} + \vec{w}^* \times \overline{\big(\vec{t}\oint \tau + \vec{b}\oint \kappa\big)} + (\vec{t}^*\oint \tau + \vec{t}\oint \tau^* + \vec{b}^*\oint \kappa + \vec{b}\oint \kappa^*) \times \overline{\vec{w}}\\
& + \vec{w} \times \overline{\big(\vec{t}^*\oint \tau + \vec{t}\oint \tau^* + \vec{b}^*\oint \kappa + \vec{b}\oint \kappa^*\big)}\Big)\\
= \; & \frac{1}{2}\Big(\big(\vec{t}\oint \tau + \vec{b}\oint \kappa\big) \times (-\tau t^* - \kappa b^*) + (+\tau t^* + \kappa b^*) \times \big(-\vec{t}\oint \tau - \vec{b}\oint \kappa\big) + (\vec{t}^*\oint \tau + \vec{t}\oint \tau^*\\
& + \vec{b}^*\oint \kappa + \vec{b}\oint \kappa^*) \times (-\tau t - \kappa b) + (\tau t^* + \kappa b^*) \times \big(-\vec{t}^*\oint \tau - \vec{t}\oint \tau^* - \vec{b}^*\oint \kappa - \vec{b}\oint \kappa^*\big)\Big)\\
= \; & \tau \oint \tau^* + \kappa \oint \kappa^*.
\end{aligned}$$

By taking into consideretion the equation (2.5), we can obtain

$$\begin{aligned}
\Lambda_w & = -H(\vec{D}, \vec{W})\\
& = -\frac{1}{2}\left(\vec{D} \times \overline{\vec{W}} + \vec{W} \times \overline{\vec{D}}\right)\\
& = \frac{1}{2}\Big(\Big(\vec{t}\oint \tau + \vec{b}\oint \kappa + \varepsilon\vec{t}^*\oint \tau + \varepsilon\vec{t}\oint \tau^* + \varepsilon\vec{b}^*\oint \kappa + \varepsilon\vec{b}\oint \kappa^*\Big) \times (\vec{w} + \varepsilon\vec{w}^*)\\
& + (\vec{w} + \varepsilon\vec{w}^*) \times \Big(\vec{t}\oint \tau + \vec{b}\oint \kappa + \varepsilon\vec{t}^*\oint \tau + \varepsilon\vec{t}\oint \tau^* + \varepsilon\vec{b}^*\oint \kappa + \varepsilon\vec{b}\oint \kappa^*\Big)\\
& = -\tau \oint \tau - \kappa \oint \kappa - \varepsilon\left(\tau \oint \tau^* + \kappa \oint \kappa^*\right).
\end{aligned}$$

□

Example 3.7. Let $\alpha(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, s)$ be a circular helix curve. Then, we obtain Frenet invariants as follows:

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1), \ N(s) = (\cos s, \sin s, 0), \\ B(s) &= \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1), \\ \kappa(s) &= \frac{1}{\sqrt{2}}, \ \tau(s) = \frac{1}{\sqrt{2}}. \end{aligned}$$

The Darboux vector is given by

$$\vec{w}(s) = (0, 0, 1).$$

The vectorial moment of the Darboux vector is written as quaternionic as following:

$$\vec{w^*}(s) = \vec{\alpha}(s) \times \vec{w}(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 0).$$

Then, the dual Darboux vector is

$$\vec{W}(s) = \vec{w}(s) + \varepsilon\vec{w^*}(s) = \left(-\frac{\varepsilon}{\sqrt{2}}\sin s, \frac{\varepsilon}{\sqrt{2}}\cos s, 1\right).$$

Considering equation (3.4), we obtain closed ruled surface corresponding to the $(\widehat{w})$ dual curves as

$$\psi_w(s, v) = \vec{w} \times \vec{w^*} + v\vec{w} = \left(-\frac{1}{\sqrt{2}}\cos s, -\frac{1}{\sqrt{2}}\sin s, v\right).$$

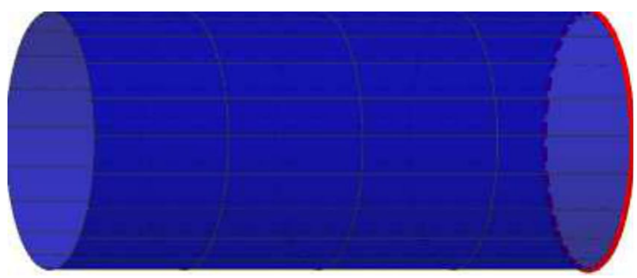

Figure 2. The blue surface show the ruled surface and the red curve $(\vec{w} \times \vec{w^*})$ show base curve.

4. Conclusion

It is known that the striction curve, distribution parameter, pitch, and angle of pitch are the invariants in the ruled surface. We quaternionically express the ruled surface drawn by the Darboux vector. These invariants are quaternionically calculated for the ruled surface. We examine the ruled surface in dual space. By using dual spatial quaternions, the integral invariants of the ruled surface and the relationships between the curvatures are obtained. Quaternionically demonstrated that surfaces are developable. In the future, the geometric properties that are examined with the help of the quaternions can be examined in the Lorentz and dual Lorentz space.

Acknowledgment

We would like to thank the reviewers for their detailed comments and suggestions for the manuscript.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] Aslan, S., Yaylı, Y., *Quaternionic shape operator*, Adv. Appl. Clifford Algebras, **27**(2017), 2921–2931.

[2] Babaarslan, M., Yaylı, Y., *A new approach to constant slope surfaces with quaternions*, ISRN Geom., **8**(2012), article ID 126358.

[3] Bharathi, K., Nagaraj, M., *Quaternion valued function of a real variable Serret-Frenet formulae*, Ind. J. P. Appl. Math., **18**(1987), 507–511.

[4] Çalışkan, A., Şenyurt, S., *The dual spatial quaternionic expression of ruled surfaces*, Thermal Science, **23**(1)(2019), 403–411.

[5] Çalışkan, A., Spatial Quaternionic Curves and Ruled Surfaces, Ph.D. Thesis, Ordu University, Ordu, Turkey, 2020.

[6] Do Carmo, M.P., Differential Geometry of Curves and Surfaces, Prentice-Hall, USA, 1976.

[7] Hamilton, W.R., Elements of Quaternions, New York, 1899.

[8] Hacısalihoğlu, H.H., Motion Geometry and Quaternions Theory (in Turkish), University of Gazi Press, Turkey, 1983.

[9] Hanson, A.J., Visualing Quaternions, Elsevier, USA, 2006.

[10] Shoemake, K., *Animating rotation with quaternion curves*, Siggraph Computer Graphics, **19**(1985), 245–254.

[11] Sivridağ, A.İ., Güneş, R., Keleş, S., *The Serret-Frenet formulae for dual quaternion-valued functions of a single real variable*, Mechanism and Machine Theory, **29**(5)(1994), 749-754.

[12] Stoker, J.J., Differential Geometry, Wiley-Interscience, New York, 1969.

[13] Şenyurt, S., Cevahir, C., Altun, Y., *On spatial quaternionic involute curve a new view*, Adv. Appl. Clifford Algebr., (2016), 1–10.

[14] Şenyurt, S., Çalışkan, A., *The quaternionic expression of ruled surfaces*, Filomat, **32**(16)(2018), 5753-5766.