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Research Article

Unrestricted Cesàro summability of d-dimensional Fourier series and Lebesgue points

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ABSTRACT. We generalize the classical Lebesgue's theorem to multi-dimensional functions. We prove that the Cesàro means of the Fourier series of the multi-dimensional function $f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ $(1 < p \le \infty)$ converge to f at each strong Lebesgue point.

Keywords: Cesàro summability, strong Hardy-Littlewood maximal function, strong Lebesgue points.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

In 1904, Fejér [3] investigated the arithmetic means of the partial sums of the trigonometric Fourier series of a one-dimensional function f, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means

$$\sigma_n f(x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{\imath kx}$$

converge to $(f(x-0)+f(x+0))/2$. Here, $\widehat{f}(k)$ denotes the k-th Fourier coefficient. One year later, Lebesgue [11] extended this theorem and obtained that every one-dimensional integrable function is Fejér summable at each Lebesgue point, thus almost everywhere. Some years later, M. Riesz [15] generalized this theorem for the Cesàro means of one-dimensional integrable functions (the definition can be found later).

The Cesàro summability is investigated in a great number of papers (see e.g. Gát [4, 5, 6], Goginava [7, 8, 9], Simon [17, 18], Nagy, Persson, Tephnadze and Wall [13, 14] and Weisz [19, 20]). In this short note, we generalize the result of Lebesgue and Riesz to this summability of multi-dimensional functions. We generalize the Lebesgue points and introduce the so called strong Lebesgue points. It is known that almost every point is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$. We introduce the strong Hardy-Littlewood maximal function $M_s f$ and show that the Cesàro means of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ can be estimated by $M_s f$ pointwise. Our main result is the following. If $M_s f(x)$ is finite and x is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x),$$

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where $\sigma_n^\alpha f$ denotes the n-th Cesàro means of the Fourier series of f. This implies the convergence of the Cesàro means almost everywhere as well as covers the one-dimensional results mentioned above. Note that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ with $1 < p \le \infty$. The results are not true for $L_1(\mathbb{T}^d)$ if $d > 1$. Similar theorems are known for the θ-means generated by a single function θ (see Feichtinger and Weisz [2] and the references therein). However, those results and proofs do not contain the results for Cesàro means. For the multi-dimensional Cesàro means, we need new ideas.

2. Strong maximal function and strong Lebesgue points

Let us fix $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d times. We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p \, d\lambda \right)^{1/p} \qquad (1 \le p < \infty),$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. We identify the torus $\mathbb{T}$ with $[-\pi, \pi]$. Set $\log^+ u := \max(0, \log u)$. For $k \in \mathbb{N}$ and $1 \le p < \infty$, a measurable function f is in the set $L_p(\log L)^k(\mathbb{T}^d)$ if

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{T}^d} |f|^p (\log^+ |f|)^k \, d\lambda \right)^{1/p} < \infty.$$

For $k = 0$, we get back the $L_p(\mathbb{T}^d)$ spaces. We have for all $k \in \mathbb{P}$ and $1 \le p < r \le \infty$ that

$$L_p(\mathbb{T}^d) \supset L_p(\log L)^{k-1}(\mathbb{T}^d) \supset L_p(\log L)^k(\mathbb{T}^d) \supset L_r(\mathbb{T}^d).$$

For $f \in L_1(\mathbb{T}^d)$, the strong Hardy-Littlewood maximal function is defined by

$$M_s f(x) := \sup_{h \in \mathbb{R}_+^d} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)| \, dt.$$

For $d > 1$, it is known that there is a function $f \in L_1(\mathbb{T}^d)$ such that $M_s f = \infty$ almost everywhere (see Jessen, Marcinkiewicz and Zygmund [10] and Saks [16]). Thus, in contrary to the one-dimensional case, M_s cannot be of weak type $(1,1)$ if $d > 1$. However, we know the following weak type inequality. If $f \in L(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\sup_{\rho > 0} \rho \lambda(M_s f > \rho) \le C + C \left\| |f| \left(\log^+ |f|\right)^{d-1} \right\|_1. \tag{2.1}$$

Moreover, for $1 < p \le \infty$, we have

$$\|M_s f\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)). \tag{2.2}$$

In this paper, the constants C and C_p may vary from line to line. If $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-t) \, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$. Here $h \to 0$ means that $h_j \to 0$ for all $j = 1, \ldots, d$. Note that this result does not hold for all $f \in L_1(\mathbb{T}^d)$ if $d > 1$ (see Jessen, Marcinkiewicz and Zygmund [10] and Saks [16]).

Motivated by this convergence result, a point $x \in \mathbb{T}^d$ is called a strong Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)| \, dt = 0.$$

Theorem 2.1. *Almost every point $x \in \mathbb{T}^d$ is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$.*

This is not true for $f \in L_1(\mathbb{T}^d)$ if $d > 1$. Note that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ for all $1 < p \leq \infty$. For the results of this section, see Chang and Fefferman [1], Zygmund [21] or Weisz [19, 20].

3. Rectangular Cesàro summability

For $\alpha \neq -1, -2, \dots$ and $n \in \mathbb{N}$, let

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Then $A_0^\alpha = 1$, $A_n^0 = 1$ and $A_n^1 = n+1$ $(n \in \mathbb{N})$. The k-th Fourier coefficient of a d-dimensional integrable function $f \in L_1(\mathbb{T}^d)$ is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath k \cdot x} \, dx \qquad (k \in \mathbb{Z}^d),$$

where $u \cdot x := \sum_{k=1}^d u_k x_k$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$. Since the Fourier series of f has bad convergence properties (see e.g. Weisz [20]), we consider the Cesàro summability.

Let $f \in L_1(\mathbb{T}^d)$, $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. The n-th rectangular Cesàro means $\sigma_n^\alpha f$ of the Fourier series of f and the Cesàro kernel K_n^α are introduced by

$$\sigma_n^\alpha f(x) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha \widehat{f}(k) e^{\imath k \cdot x}$$

and

$$K_n^\alpha(t) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha e^{\imath k \cdot t},$$

respectively. It is easy to see that

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\alpha(t) \, dt$$

and

$$K_n^\alpha = K_{n_1}^{\alpha_1} \otimes \cdots \otimes K_{n_d}^{\alpha_d},$$

where the functions $K_{n_i}^{\alpha_i}$ are the one-dimensional Cesàro kernels. The Cesàro means are also called (C, α)-means. If all $\alpha_i = 1$, then we get back the rectangular Fejér means. For the one-dimensional Cesàro kernels, it is known (see Zygmund [21]) that

$$K_n^\alpha(t) \leq C \min\left(n, \frac{1}{n^\alpha |t|^{\alpha+1}}\right) \tag{3.3}$$

and

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{T}} |K_n^\alpha| \, d\lambda \leq C,$$

where $n \in \mathbb{N}$, $0 < \alpha \leq 1$ and $t \in (-\pi, \pi)$.

4. Unrestricted convergence at Lebesgue points

Before proving the main results of this paper, we introduce the Herz space $E_\infty(\mathbb{R}^d)$ with the norm

$$\|f\|_{E_\infty} := \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} 2^{k_1+\ldots+k_d} \|f 1_{P_k}\|_\infty < \infty,$$

where

$$P_k := P_{k_1} \times \cdots \times P_{k_d} \qquad (k \in \mathbb{Z}^d)$$

and

$$P_i = \{x \in \mathbb{R} : 2^{i-1}\pi \le |x| < 2^i \pi\} \qquad (i \in \mathbb{Z}).$$

Obviously, $L_1(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d)$. First, we will estimate pointwise the maximal operator

$$\sigma_*^\alpha f := \sup_{n \in \mathbb{N}^d} |\sigma_n^\alpha f|$$

by the strong Hardy-Littlewood maximal function. To this end, we introduce the functions

$$h^{\alpha_j}(t) := \min\left\{1, |t|^{-\alpha_j-1}\right\} \qquad (t \in \mathbb{R})$$

and

$$h^\alpha := h^{\alpha_1} \otimes \cdots \otimes h^{\alpha_d}.$$

We get from (3.3) that

$$\text{(4.4)} \qquad \frac{1}{n_j} \left| \left(1_{(-\pi,\pi)} K_{n_j}^{\alpha_j}\right) \left(\frac{t}{n_j}\right) \right| \le \frac{C}{n_j} \min\left\{n_j, \frac{n_j}{|t|^{\alpha_j+1}}\right\} = C h^{\alpha_j}(t) \qquad (t \in \mathbb{R}).$$

It is easy to see that

$$\text{(4.5)} \qquad \|h^\alpha\|_{E_\infty(\mathbb{R}^d)} = \prod_{j=1}^d \|h^{\alpha_j}\|_{E_\infty(\mathbb{R})} \le C_\alpha.$$

Theorem 4.2. *Suppose that $0 < \alpha_j \le 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, then*

$$\sigma_*^\alpha f(x) \le C M_s f(x).$$

Proof. Observe that

$$\begin{aligned} |\sigma_n^\alpha f(x)| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} f(x-t) \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\, dt \right| \\ &= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t) \right| dt, \end{aligned}$$

where

$$P_{k_j}(n_j) := \{x \in \mathbb{R} : 2^{k_j-1}\pi/n_j \le |x| < 2^{k_j}\pi/n_j\} \qquad (j = 1, \ldots, d).$$

Then,

$$|\sigma_n^\alpha f(x)| \leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)|\, dt$$
$$\times \sup_{t \in P_{k_1}(n_1)\times\cdots\times P_{k_d}(n_d)} \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t) \right|$$
$$= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)|\, dt$$
$$\times \sup_{t \in P_{k_1}\times\cdots\times P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right) \right|. \tag{4.6}$$

Consequently, by (4.4),

$$|\sigma_n^\alpha f(x)| \leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} 2^{k_1+\ldots+k_d} M_s f(x) \sup_{t\in P_k} |h^\alpha(t)|$$
$$= C \left\| h^\alpha \right\|_{E_\infty(\mathbb{R}^d)} M_s f(x).$$

Inequality (4.5) finishes the proof. □

Inequalities (2.1) and (2.2) imply:

Corollary 4.1. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C + C \left\| f \right\|_{L_1(\log L)^{d-1}}.$$

If $1 < p \leq \infty$ and $f \in L_p(\mathbb{T}^d)$, then

$$\left\| \sigma_*^\alpha f \right\|_p \leq C_p \| f \|_p.$$

The usual density argument due to Marcinkiewicz and Zygmund [12] implies:

Corollary 4.2. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \quad a.e. \; .$$

In this paper, $n \to \infty$ means that $n_j \to \infty$ for all $j = 1, \ldots, d$. Now, we prove that the convergence in Corollary 4.2 holds at each strong Lebesgue point, whenever the corresponding strong Hardy-Littlewood maximal function is finite.

Theorem 4.3. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $M_s f(x)$ is finite and x is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x).$$

Proof. Let

$$G(u) := \int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t) - f(x)|\, dt \qquad (u \in \mathbb{R}_+^d).$$

Since x is a strong Lebesgue point of f, for all $\epsilon > 0$, we can find an integer $m \leq 0$ such that

$$\frac{G(u)}{\prod_{j=1}^d (2u_j)} \leq \epsilon \quad \text{if} \quad 0 < u_j \leq 2^m \pi, j = 1, \ldots, d. \tag{4.7}$$

Since

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^\alpha(t)\, dt = 1,$$

we have

$$|\sigma_n^\alpha f(x) - f(x)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t) \right| dt := A_1(x) + A_2(x),$$

where

$$A_1(x) := \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \times \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t) \right| dt$$

and

$$A_2(x) := \frac{1}{(2\pi)^d} \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \times \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t) \right| dt.$$

Here $\{\pi_1, \ldots, \pi_d\}$ is a permutation of $\{1, \ldots, d\}$ and $1 \leq j \leq d$. As in (4.6),

$$\begin{aligned} A_1(x) &\leq C \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \, dt \\ &\quad \times \sup_{t \in P_{k_1} \times \cdots \times P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right| \\ &\leq C \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} G\left(\frac{2^{k_1}\pi}{n_1}, \ldots, \frac{2^{k_d}\pi}{n_d} \right) \left(\prod_{j=1}^d n_j \right) \sup_{t \in P_k} |h^\alpha(t)| . \end{aligned}$$

Inequalities (4.7), (4.5) and $2^{k_j}/n_j \leq 2^m$ imply

$$A_1(x) \leq C\epsilon \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} 2^{k_1+\ldots+k_d} \sup_{t \in P_k} |h^\alpha(t)| \leq C\epsilon \left\| h^\alpha \right\|_{E_\infty(\mathbb{R}^d)} \leq C_\alpha \epsilon.$$

Similarly,

$$\begin{aligned} A_2(x) &\leq C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \\ &\quad \times \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \, dt \left(\prod_{j=1}^d n_j \right) \sup_{t \in P_k} |h^\alpha(t)| \\ &\leq C_p \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \\ &\quad \times 2^{k_1+\ldots+k_d} \sup_{t \in P_k} |h^\alpha(t)| \Big(M_s f(x) + |f(x)| \Big). \end{aligned}$$

Since $M_s f(x)$ and $f(x)$ are finite, the fact $\lfloor \log_2 n_{\pi_j} \rfloor \to \infty$ as $T \to \infty$ imply that $A_2(x) \to 0$ as $n \to \infty$. □

In the one-dimensional case, if x is a strong Lebesgue point, then $M_s f(x)$ is finite and $L_1(\log L)^{d-1}(\mathbb{T}^d) = L_1(\mathbb{T}^d)$, hence we get back the results due to Lebesgue [11] and Riesz [15] mentioned in the introduction. Recall that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ for $1 < p \leq \infty$ and $d > 1$. Since by Theorem 2.1 and (2.1) almost every point is a strong Lebesgue point and the strong maximal operator $M_s f$ is almost everywhere finite for $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, Theorem 4.3 implies Corollary 4.2. If f is continuous at a point x, then x is also a strong Lebesgue point. So we obtain:

Corollary 4.3. *Suppose that $0 < \alpha_j \leq 1$ for all $j = 1, \dots, d$. If $M_s f(x)$ is finite and $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ is continuous at a point x, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x).$$

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