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Selfadjoint Singular Quasi-Differential Operators of First Order

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ABSTRACT

In this work, using the Calkin-Gorbachuk method firstly all selfadjoint extensions of the minimal operator generated by first order linear singular quasi-differential expression in the weighted Hilbert space of vector-functions on right semi-axis have been described. Lastly, the structure of the spectrum set of these extensions has been investigated.

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Selfadjoint operator; Quasi-differential operator; Spectrum.

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INTRODUCTION

In the first years of previous century, J. von Neumann [13] and M. H. Stone [12] investigated the theory of selfadjoint extensions of linear densely defined closed symmetric operators in a Hilbert spaces. Applications to scalar linear even order symmetric differential operators and description of all selfadjoint extensions in terms of boundary conditions were done by I. M. Glazman in his seminal work [5] and by M. A. Naimark [10] in his book. In this sense the famous Glazman-Krein-Naimark (or Everitt-Krein-Glazman-Naimark) Theorem in mathematical literature should be noted. In mathematical literature there is another so-called Calkin-Gorbachuk method (see [6], [11]).

Our motivation in this paper originates from the interesting researches of W. N. Everitt, L. Markus, A. Zettl, J. Sun, D. O'Regan, R. Agarwal [2], [3], [4], [14] in scalar cases. Throughout this paper A. Zettl's and J. Suns's view about these topics is to be taken into consideration in [14]: A selfadjoint ordinary differential operator in Hilbert space is generated by two things:

(1) a symmetric (formally selfadjoint) differential expression;

(2) a boundary condition which determined selfadjoint differential operators.

And also for a given selfadjoint differential operator, a basic question is: What is its spectrum?

In this work, in Section 3 the representation of all selfadjoint extensions of a symmetric quasi-differential operator, generated by first order symmetric quasi-differential expression in the weighted Hilbert space of vector-functions defined at the right semi-axis in terms of boundary conditions have been described. In Section 4, the structure of spectrum of these selfadjoint extensions is investigated.

STATEMENT OF THE PROBLEM

In the weighted Hilbert space $L_w^2(H,(a,\infty))$ where H is a separable Hilbert space and $\alpha \in \mathbb{R}$, we will consider the following quasi-differential expression given by

$$l(u) = i\frac{\alpha(t)}{w(t)}(\alpha u)'(t) + Au(t).$$

Here

(1) $\alpha, w : (a,\infty) \to (0,\infty)$,
(2) $\alpha, w \in C(a,\infty)$,
(3) $\int_a^\infty \frac{w(s)}{\alpha^2(s)} ds < \infty$,
(4) $A : D(A) \subset H \to H$ is a selfadjoint operator.

In this case, since

$$\left(l(u),v\right)_{L^2_w(H,(a,\infty))}$$
$$\int_a^\infty \left(i\frac{\alpha(t)}{w(t)}(\alpha u)'(t), v(t)\right)_H w(t)dt + (Au,v)_{L^2_w(H,(a,\infty))}$$
$$= i\int_a^\infty \left((\alpha u)'(t),(\alpha v)(t)\right)_H dt + (u,Av)_{L^2_w(H,(a,\infty))}$$
$$= i\left[(\alpha u,\alpha v)(\infty) - (\alpha u,\alpha v)(a)\right]$$
$$-i\int_a^\infty \left((\alpha u)(t),(\alpha v)'(t)\right)_H dt + (u,Av)_{L^2_w(H,(a,\infty))}$$
$$= \int_a^\infty \left(u(t), i\frac{\alpha(t)}{w(t)}(\alpha v)'(t)\right)_H w(t)dt + (u,Av)_{L^2_w(H,(a,\infty))}$$
$$= \left(u,l^+(v)\right)_{L^2_w(H,(a,\infty))}$$
$$= \left(u,l(v)\right)_{L^2_w(H,(a,\infty))},$$

then the differential-operator expression $l(.)$ is formally symmetric.

The minimal L_0 and maximal L operators corresponding to differential-operator expression in $L^2_w(H,(a,\infty))$ can be defined by using the classical techniques (see [7]).

On the other hand one can easily see that

$$D(L) = \left\{u \in L^2_w(H,(a,\infty)) : l(u) \in L^2_w(H,(a,\infty))\right\},$$
$$D(L_0) = \left\{u \in D(L) : (\alpha u)(a) = (\alpha u)(\infty) = 0\right\}.$$

DESCRIPTION OF SELFADJOINT EXTENSIONS

In this section using the Calkin-Gorbachuk method we will investigate the general representation of all selfadjoint extensions of the minimal operator L_0.

First, let us prove the following assertion.

Lemma 3.1 The deficiency indices of the minimal operator L_0 in $L^2_w(H,(a,\infty))$ are in form

$$\left(n_+(L_0), n_-(L_0)\right) = (\dim H, \dim H).$$

Proof. For the simplicity of calculations it will be taken A=0. It is clear that the general solutions of differential equations

$$i\frac{\alpha(t)}{w(t)}(\alpha u_\pm)'(t) \pm iu_\pm(t) = 0, t > a$$

can be given as

$$u_\pm(t) = \frac{1}{\alpha(t)} exp\left(\mp\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right) f, f \in H, t > a.$$

From these representations we have

$$\|u_+\|^2_{L^2_w(H,(a,\infty))}$$
$$= \int_a^\infty w(t)\|u_+(t)\|^2_H dt$$
$$= \int_a^\infty \left\| \frac{1}{\alpha(t)} exp\left(-\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right) f\right\|^2_H w(t)dt$$
$$= \int_a^\infty \frac{w(t)}{\alpha^2(t)} exp\left(-2\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right)dt \|f\|^2_H$$
$$= \int_a^\infty exp\left(-2\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right) d\left(\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right)\|f\|^2_H$$
$$= \frac{1}{2}\left(1 - exp\left(-2\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right)\|f\|^2_H < \infty.$$

Consequently, $n_+(L_0) = \dim\ker(L + iE) = \dim H$.

On the other hand, it is clear that for any $f \in H$, one can obtain

$$\| u_- \|^2_{L^2_w(H,(a,\infty))}$$
$$= \int_a^\infty w(t) \| u_-(t) \|^2_H dt$$
$$= \int_a^\infty \frac{w(t)}{\alpha^2(t)} exp\left(2\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right)dt \| f \|^2_H$$
$$= \int_a^\infty exp\left(2\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right) d\left(\int_a^t \frac{w(s)}{\alpha^2(s)}ds\right) \| f \|^2_H$$
$$= \frac{1}{2}\left(exp\left(2\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right) - 1\right) \| f \|^2_H < \infty.$$

Consequently, $n_-(L_0) = \dim\ker(L - iE) = \dim H$. This completes the proof.

As a result, the minimal operator L_0 has at least one selfadjoint extension (see [6]).

In order to describe these extensions we need to obtain the space of boundary values.

Definition 3.2 [6] Let $\mathcal{H}$ be any Hilbert space and $S : D(S) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$ having equal finite or infinite deficiency indices. A triplet $(\mathbf{H}, \gamma_1, \gamma_2)$ where $\mathbf{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathbf{H}$, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$\left(S^* f, g\right)_{\mathcal{H}} - \left(f, S^* g\right)_{\mathcal{H}}$$
$$= \left(\gamma_1(f), \gamma_2(g)\right)_{\mathbf{H}} - \left(\gamma_2(f), \gamma_1(g)\right)_{\mathbf{H}}$$

while for any $F_1, F_2 \in \boldsymbol{H}$, there exists an element $f \in D(S^*)$

such that $\gamma_1(f)=F_1$ and $\gamma_2(f)=F_2$.

Lemma 3.3 The triplet (H,γ_1,γ_2),

$$\gamma_1 : D(L)\to H,\gamma_1(u)=\frac{1}{\sqrt{2}}\big((\alpha u)(\infty)-(\alpha u)(a)\big),$$
$$\gamma_2 : D(L)\to H,\gamma_2(u)=\frac{1}{i\sqrt{2}}\big((\alpha u)(\infty)+(\alpha u)(a)\big)$$
$$,u\in D(L)$$

is a space of boundary values of the minimal operator L_0 in $L_w^2(H,(\mathrm{a},\infty))$.

Proof. For any $u,v\in D(L)$, ,

$$\begin{aligned}
&(Lu,v)_{L_w^2(H,(a,\infty))}-(u,Lv)_{L_w^2(H,(a,\infty))}\\
&=\left(i\frac{\alpha}{w}(\alpha u)^{'}+Au,v\right)_{L_w^2(H,(a,\infty))}\\
&\quad-\left(u,i\frac{\alpha}{w}(\alpha v)^{'}+Av\right)_{L_w^2(H,(a,\infty))}\\
&=\left(i\frac{\alpha}{w}(\alpha u)^{'},v\right)_{L_w^2(H,(a,\infty))}-\left(u,i\frac{\alpha}{w}(\alpha v)^{'}\right)_{L_w^2(H,(a,\infty))}\\
&=\int_a^\infty\left(i\frac{\alpha(t)}{w(t)}(\alpha u)^{'}(t),v(t)\right)_H w(t)dt\\
&\quad-\int_a^\infty\left(u(t),i\frac{\alpha(t)}{w(t)}(\alpha v)^{'}(t)\right)_H w(t)dt\\
&=i\left[\int_a^\infty\big((\alpha u)^{'}(t),(\alpha v)(t)\big)_H dt+\int_a^\infty\big((\alpha u)(t),(\alpha v)^{'}(t)\big)_H dt\right]\\
&=i\int_a^\infty\big((\alpha u)(t),(\alpha v)(t)\big)'_H dt\\
&=i\Big[\big((\alpha u)(\infty),(\alpha v)(\infty)\big)_H-\big((\alpha u)(a),(\alpha v)(a)\big)_H\Big]\\
&=\big(\gamma_1(u),\gamma_2(v)\big)_H-\big(\gamma_2(u),\gamma_1(v)\big)_{\boldsymbol{H}}.
\end{aligned}$$

Now for any given element $f,g\in H$ one can find the function $u\in D(L)$ such that

$$\gamma_1(u)=\frac{1}{\sqrt{2}}\big((\alpha u)(\infty)-(\alpha u)(a)\big)=f,$$
$$\gamma_2(u)=\frac{1}{i\sqrt{2}}\big((\alpha u)(\infty)+(\alpha u)(a)\big)=g.$$

From this it is obtained that $(\alpha u)(\infty)=(ig+f)/\sqrt{2}$ and $(\alpha u)(a)=(ig-f)/\sqrt{2}$.

If we choose the function u(.) as below

$$u(t)=\frac{1}{\alpha(t)}\left(1-e^{a-t}\right)(ig+f)/\sqrt{2}$$
$$+\frac{1}{\alpha(t)}e^{a-t}(ig-f)/\sqrt{2}$$

then it is clear that $u\in D(L)$ and $\gamma_1(u)=f,\gamma_2(u)=g$.

The following result can be established by using the method given in [6].

Theorem 3.4 If $\tilde{L}$ is a selfadjoint extension of the minimal operator L_0 in $L_w^2(H,(a,\infty))$, then it is generated by the differential-operator expression $l(.)$ and boundary condition

$$(\alpha u)(a)=W(\alpha u)(\infty),$$

where $W:H\to H$ is an unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\tilde{L}$, i.e. $\tilde{L}=L_W$ and vice versa.

Proof. It is known that all selfadjoint extension of the minimal operator L_0 are described by the differential-operator expression $l(.)$ with boundary condition

$$(V-E)\gamma_1(u)+i(V+E)\gamma_2(u)=0,$$

where $V:H\to H$ is an unitary operator. Therefore from Lemma 3.3 we obtain

$$(V-E)\big((\alpha u)(\infty)-(\alpha u)(a)\big)$$
$$+(V+E)\big((\alpha u)(\infty)+(\alpha u)(a)\big)=0,u\in D(\tilde{L}).$$

From this, it is implies that

$$(\alpha u)(a)=-V(\alpha u)(\infty).$$

Choosing W=-V in last boundary condition, we have

$$(\alpha u)(a)=W(\alpha u)(\infty).$$

THE SPECTRUM OF THE SELFADJOINT EXTENSIONS

In this section the structure of the spectrum set of the selfadjoint extensions of the minimal operator L_0 in $L_w^2(H,(a,\infty))$ will be examined.

Theorem 4.1 The spectrum of any selfadjoint extension L_W is in form

$$\sigma(L_W)=\left\{\lambda\in\mathbb{R}:\lambda=\left(\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)^{-1}(arg\mu+2n\pi)\right.$$
$$\left.\mu\in\sigma\left(Wexp\left(iA\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)\right),n\in\mathbb{Z}\right\}.$$

Proof. Consider the following problem to spectrum of the extension L_W, i.e.

$$l(u)=\lambda u+f,u,f\in L_w^2(H,(a,\infty)),\lambda\in\mathbb{R},$$
$$(\alpha u)(a)=W(\alpha u)(\infty),$$

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that is,

$$i\frac{\alpha(t)}{w(t)}(\alpha u)'(t)+Au(t)=\lambda u(t)+f(t), t>a,$$
$$(\alpha u)(a)=W(\alpha u)(\infty).$$

The general solution of the last differential equation,

$$(\alpha u)'(t)=i\frac{w(t)}{\alpha^2(t)}(A-\lambda E)(\alpha u)(t)-i\frac{w(t)}{\alpha(t)}f(t)$$

is in form

$$u(t;\lambda)=\frac{1}{\alpha(t)}exp\left(i(A-\lambda E)\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)f_\lambda$$
$$+\frac{i}{\alpha(t)}\int_t^\infty exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds,$$
$$f_\lambda\in H, t>a.$$

In this case

$$\left\|\frac{1}{\alpha(t)}exp\left(i(A-\lambda E)\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)f_\lambda\right\|^2_{L^2_w(H,(a,\infty))}$$
$$=\int_a^\infty\frac{w(t)}{\alpha^2(t)}dt\|f_\lambda\|^2_H<\infty$$

and

$$\left\|\frac{i}{\alpha(t)}\int_t^\infty exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds\right\|^2_{L^2_w(H,(a,\infty))}$$
$$=\int_a^\infty\frac{1}{\alpha^2(t)}\left\|\int_t^\infty exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds\right\|^2_H w(t)dt$$
$$\le\int_a^\infty\frac{1}{\alpha^2(t)}\left[\int_t^\infty\left\|exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\right\|_H\left\|\frac{w(s)}{\alpha(s)}f(s)\right\|_H ds\right]^2 w(t)dt$$
$$\le\int_a^\infty\frac{1}{\alpha^2(t)}\left(\int_t^\infty\frac{\sqrt{w(s)}}{\alpha(s)}\sqrt{w(s)}\|f(s)\|_H ds\right)^2 w(t)dt$$
$$\le\int_a^\infty\frac{1}{\alpha^2(t)}\left(\int_t^\infty\frac{w(s)}{\alpha^2(s)}ds\right)\left(\int_t^\infty w(s)\|f(s)\|^2_H ds\right)w(t)dt$$
$$\le\int_a^\infty\frac{w(t)}{\alpha^2(t)}dt\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\int_a^\infty w(s)\|f(s)\|^2_H ds$$
$$\le\left(\int_a^\infty\frac{w(t)}{\alpha^2(t)}dt\right)^2\|f\|^2_{L^2_w(H,(a,\infty))}<\infty.$$

Hence, $u(.,\lambda)\in L^2_w(H,(a,\infty))$ for $\lambda\in\mathbb{R}$.

From this and boundary condition, we have

$$\left(E-Wexp\left(i(A-\lambda E)\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)\right)f_\lambda$$
$$=-i\int_a^\infty exp\left(i(A-\lambda E)\int_s^a\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds.$$

Therefore in order to $\lambda\in\sigma(L_W)$ the necessary and sufficient condition is

$$exp\left(i\lambda\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)=\mu\in\sigma\left(Wexp\left(iA\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)\right).$$

Since the operator $Wexp\left(iA\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)$ is an isometric operator, then $|\mu|=1$.

Consequently,

$$\lambda\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds=arg\mu+2n\pi, n\in\mathbb{Z}.$$

On the other hand since $\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds>0$, then

$$\lambda=\left(\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)^{-1}(arg\mu+2n\pi),$$
$$\mu\in\sigma\left(Wexp\left(iA\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)\right), n\in\mathbb{Z}.$$

This completes the proof.

Remark 4.2 Note that the similar problems in different singular multipoint cases in the corresponding direct sums of Hilbert spaces of vector-functions have been investigated in [1], [8], [9].

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