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On Determination of the Source Term of a Modified KdV Equation

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ABSTRACT

We study an inverse problem to identify the source term depending on x of a modified KdV equation. In order to recover source term, we define an inverse problem subject to an overdetermination condition. We converted this inverse problem to an operator equation. The existence and uniqueness of this operator equation is investigated.

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INTRODUCTION

In this article, we investigate the inverse problem associated with the problem

$$u_t + au_{xxx} + d(t)u = f(x,t) = A(x)B(t), (x,t) \in Q = (0,T) \times (0,1) \tag{1}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0 \tag{2}$$

$$u(x,0) = u_0(x), x \in (0,1) \tag{3}$$

to determine $\{u, A\}$ where a is positive constant, d is continuous function defined in $[0,\infty)$.

The dynamics of small, finite perturbation in an inhomogeneous media are given by equation (1) [3]. Korteweg-de Vries (KdV) equation has great interest and there are many studies on it [1]-[4].

The inverse problem theory for differential equations is being developed to solve problems of mathematical physics. In the study of direct problems, the solution of the equation is derived by means of supplementary conditions.

In the case of inverse problems, the form of the equation is known but the equation is not known exactly. To determine the corresponding equation and its solution, some additional conditions (final overdetermination conditions) must be imposed.

Inverse problems for partial differential equations are extensively investigated. Inverse problems are classified according to the partial differential equations where they arise. The study of Prilepko and Orlovsky [5] is important for the systematic representation for elliptic, hyperbolic and parabolic inverse problems.

We are particularly interested in the inverse problems for determination of the source terms. Such problems have great interest [5, 6 ,7, 8, 9, 10, 11].

In the present study, we show the existence and uniqueness of the solution of the inverse problem to determine the part of the source term of a modified KdV equation. We call the determination of $u(x,t)$ when f is given in (1)-(3) as the direct problem. The determination of $\{u, A\}$, in (1)-(3) with the final overdetermination

$$u(x,t_0) = \alpha(x), t_0 \in (0,T), x \in (0,1) \tag{4}$$

and B is known, is called the inverse problem for recovering the source term A depending on x .

The corresponding direct problem (1)-(3), when $b \equiv 0$, a real constant, is a particular case of the problem, studied by Larkin [1] .

The paper is organized as follows: In section 2, we give some notations, definitions and results about the direct problem. The third section is devoted to derive an equivalent fixed point system for our inverse problem. In the last section, the existence and uniqueness of the fixed point of the system is proved.

PRELIMINARIES

In this section, we summarize the definition and results given in [1]. The usual notations of Sobolev spaces are used for the notations see [12]. For the properties of the solution of the problem (1)-(3), we refer the readers to [1].

Let us denote

$$d_+(t) = \max(0, d(t)), d_-(t) = d_+(t) - d(t)$$

$$(u,v) = \int_0^1 u(x,t)v(x,t)dx, \|u(t)\|^2 = (u,u).$$

The following theorem is proved in [1].

Theorem 1. If $a,b,d \in C[0,\infty); a(t) \geq a_0 > 0, \forall t \geq 0;$ $\sup_{t\geq 0}(|b(t)|+a(t)) < \infty, d_- \in L^1(0,\infty), u_0 \in L^2(0,1),$ $f \in L^1(0,\infty; L^2(0,1)), u_0 \in H^3(0,1) \cap H_0^2(0,1), U_{0x}(1) = 0,$ $f \in L^1(0,\infty; L^2(0,1)) \cap L^2(0,\infty; H^2(0,1) \cap H_0^1(0,1)), f_x(1,t) = 0$ for a. e. $t \geq 0$ then (1)-(3) has a unique solution $u = u(x,t)$ such that $u \in C(0,T; H^3(0,1) \cap H_0^1(0,1)) \cap L^2(0,T; H^4(0,1)),$ $u_t \in L^\infty(0,T; H_0^1(0,1)), u_x(1,t) = 0$ for all finite T.

DERIVATION OF A FIXED POINT SYSTEM

In this section, first we define our solution concept for the inverse problem (1)-(4) and construct an operator equation for the inverse problem. The equivalence of the operator equation and the inverse problem is proved.

Definition 1. We call the pair of functions $(u(x,t), A(x))$ as a solution of the inverse problem (1)-(4), if $u \in C(0,T; H^3(0,1) \cap H_0^1(0,1)) \cap L^2(0,T; H^4(0,1)), u_t \in L^\infty(0,T; H_0^1(0,1)), u_x(1,t) = 0$ for all finite T and $A \in L^2(0,1)$.

In order to find $A(x)$, first we replace t by t_0 in equation (1) to get

$$\begin{aligned} &u_t(x,t_0) + au_{xxx}(x,t_0) + d(t_0)u(x,t_0) = f(x,t_0) \\ &= A(x)B(t_0) \end{aligned}. \quad (5)$$

If we use equation (4) in (5), it turns to

$$u_t(x,t_0) + a\alpha'''(x) + d(t_0)\alpha(x) = A(x)B(t_0) \quad (6)$$

By solving the equation (6) for $A(x)$, we find

$$A(x) = \frac{u_t(x,t_0)}{B(t_0)} + \frac{a\alpha'''(x) + d(t_0)\alpha(x)}{B(t_0)}. \quad (7)$$

If we define

$$(LA)(x) : \frac{u_t(x,t_0)}{B(t_0)}, \Phi(x) = \frac{a\alpha'''(x) + d(t_0)\alpha(x)}{B(t_0)}.$$

The relation between A and u may be specified via

$L: L^2(0,1) \to L^2(0,1)$

with

$$A(x) = (LA)(x) + \Phi(x). \quad (8)$$

Theorem 2. If the problem (1)-(4) has a solution if and only if the operator equation (8) has a solution.

Proof. Assume that the problem (1)-(4) has a solution. Then, if we follow the steps given above, we derive the operator equation (8).

If the operator equation has a solution $A(x)$, we insert it in the equation (1). Since the problem (1)-(3) has a solution and it is unique by Theorem 1, we have to check whether this $u(x,t)$ satisfies equation (4).

To this end, we assume that

$$u(x,t_0) = \beta(x)$$

then we have

$$u_t(x,t_0) + a\beta'''(x) + d(t_0)\beta(x) = A(x)B(t_0). \quad (9)$$

If we subtract equation (6) from equation (9), we get

$$a(\beta''' - \alpha''') + d(t_0)(\beta - \alpha) = 0. \quad (10)$$

By denoting, $\beta - \alpha = y$, $d(t_0) = d$, equation (10) takes the form

$$ay''' + dy = 0. \quad (11)$$

The characteristic polynomial $p(r)$ of the differential equation (11) is

$$p(r) = ar^3 + d.$$

So, the general solution of the equation (11) is written in the following form

$$y(x) = ke^{(-d/a)^{(1/3)}x} + lxe^{(-d/a)^{(1/3)}x} + mx^2e^{(-d/a)^{(1/3)}x}. \quad (12)$$

Now, we use the conditions given in (3).

When $x=0$, then $y(0)=0$, it gives us that $k=0$. So, $y(x)$ is of the form

$$y(x)=lxe^{(-d/a)^{(1/3)}x}+mx^2e^{(-d/a)^{(1/3)}x}. \tag{13}$$

Since $u(1,t)=0$, then $y(1)=0$. If we use it in equation (13), we get $l=-m$. The new form of the general solution is

$$y(x)=lxe^{(-d/a)^{(1/3)}x}-lx^2e^{(-d/a)^{(1/3)}x}. \tag{14}$$

Taking into account that the value of $u_x(1,t)=0=y'(1)=0$, we can find the value of l as zero. Consequently, the problem (11) has only the solution $y\equiv 0$. It means $y(x)=\beta(x)-\alpha(x)\equiv 0$, hence $\beta(x)=\alpha(x),\forall x$. It proves that $u(x,t_0)$ satisfies (4).

EXISTENCE OF THE SOLUTION OF THE OPERATOR EQUATION

In this section, we study the existence and uniqueness of the fixed point of the derived operator equation (8).

Theorem 3. If

$$B(t)\in C^1(0,T), B(t_0)\neq 0, \frac{(|B'(t)|+e^{\gamma t}-e^{\gamma\xi}+ce^{\gamma\xi})^{1/2}}{B(t_0)}<1,$$

$b(t)\equiv 0, a\in R,$ then (8) has a unique fixed point.

Proof. For the proof, we estimate the norm of L. To this end, first we differentiate (1) with respect to t to find

$$\begin{aligned}&u_{tt}+au_{xxxt}+d'(t)u(x,t)+d(t)u_t(x,t)\\&=A(x)B'(t)\end{aligned}. \tag{15}$$

If we multiply (15) with u_t and integrate with respect to x from 0 to 1, we get

$$\begin{aligned}&\int_0^1 u_{tt}u_t dx+a\int_0^1 u_{xxxt}u_t dx+d'(t)\int_0^1 uu_t dx\\&+d(t)\int_0^1 u_tu_t dx=B'(t)\int_0^1 A(x)u_t dx.\end{aligned} \tag{16}$$

First integral in (16) is

$$\int_0^1 u_{tt}u_t dx=\frac{1}{2}\int_0^1\frac{d}{dt}(u_t)^2dx=\frac{1}{2}\frac{d}{dt}\|u_t\|^2. \tag{17}$$

If we use integration by parts for the second term in (16);

$$\begin{aligned}&\int_0^1 u_{xxxt}u_t dx=u_{xxt}(1,t)u_t(1,t)\\&-u_{xxt}(0,t)u_t(0,t)-\int_0^1 u_{xxt}u_{xt}dx.\end{aligned} \tag{18}$$

If we use (3) in (18), it becomes

$$\int_0^1 u_{xxxt}u_t dt=-\int_0^1 u_{xxt}u_{xt}dx. \tag{19}$$

Since $u_{xxt}u_{xt}=\frac{1}{2}\frac{d}{dx}(u_{xt})^2$, then (19) can be written as

$$\int_0^1 u_{xxxt}u_t dt=-\int_0^1 u_{xxt}u_{xt}dx=\frac{1}{2}(u_{xt}(0,t))^2. \tag{20}$$

The third term in (16) is

$$\int_0^1 uu_t dx=\frac{1}{2}\int_0^1\frac{d}{dt}(u)^2dx=\frac{1}{2}\frac{d}{dt}\|u\|^2. \tag{21}$$

By the definition of the norm, the forth term of (16) is

$$\int_0^1 u_tu_t dx=\|u_t\|^2. \tag{22}$$

By Cauchy's Inequality, the last term of (16) is estimated to be

$$\int_0^1 A(x)u_t dx\leq\frac{\|A\|^2}{2}+\frac{\|u_t\|^2}{2}. \tag{23}$$

With (17), (20), (21), (22) and (23), the equation (16) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\frac{1}{2}\frac{d'(t)d}{dt}\|u\|^2\\&+d(t)\|u_t\|^2\leq|B'(t)|\left(\frac{\|A\|^2}{2}+\frac{\|u_t\|^2}{2}\right).\end{aligned} \tag{24}$$

Rearranging (24), it takes the form

$$\frac{d}{dt}\|u_t\|^2+(d(t)-|B'(t)|)\|u_t\|^2\leq|B'(t)|\,\|A\| \tag{25}$$

Substituting $y := \|u_t\|^2, (d(t) - |B'(t)|) = \gamma$, then (25) becomes

$$y'(t) + \gamma y(t) \le D\|A\|^2, \text{where } D = \max_{t\in[0,T]} |B'(t)|. \quad (26)$$

Solving the inequality (26), we get

$$y(t) \le |B'(t)| \|A\|^2 (e^{\gamma t} - e^{\gamma\xi}) + y(\xi)e^{\gamma\xi}. \quad (27)$$

If we employ proposition 4 in [1], we get the inequality

$$y(t) \le |B'(t)| \|A\|^2 (e^{\gamma t} - e^{\gamma\xi}) + c\|A\|^2 e^{\gamma\xi}. \quad (28)$$

(28) shows that

$$\|u_t(t)\|^2 \le (|B'(t)| + e^{\gamma t} - e^{\gamma\xi} + ce^{\gamma\xi})\|A\|^2. \quad (29)$$

The norm of the operator   can be estimated by using (29) as

$$\|LA\| = \left\| \frac{u_t(x,t_0)}{B(t_0)} \right\| \le \frac{(|B'(t)| + e^{\gamma t} - e^{\gamma\xi} + ce^{\gamma\xi})^{1/2}}{B(t_0)} \|A\|. \quad (30)$$

If we impose the condition

$$\frac{(|B'(t)| + e^{\gamma t} - e^{\gamma\xi} + ce^{\gamma\xi})^{1/2}}{B(t_0)} < 1 \quad (31)$$

the operator L is contraction, so it has a unique fixed point [13].

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