Hittite Journal of Science and Engineering, 2018, 5 (3) 179-183
ISSN NUMBER: 2148-4171
DOI: 10.17350/HJSE19030000088

Exact Solutions of (n+1)-Dimensional Space-Time Fractional Zakharov-Kuznetsov Equation

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ABSTRACT

In this article, we study the (n+1)-dimensional space time fractional Zakharov-Kuznetsov equation for calculating the exact solutions. For this purpose fractional derivative is used in the form of modified Riemann-Liouville derivatives. Complex fractional transformation is applied for transforming the nonlinear partial differential equation into another nonlinear ordinary differential equation. Exact solutions are obtained by using modified simple equation method and (1/G')-expansion method. Obtained solutions are new and may be of significant importance in the field of plasma physics to investigate the waves in the magnetized plasma and in the dust plasma.

Article History:
Received: 2017/07/26
Accepted: 2017/12/10
Online: 2018/04/27

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Keywords:
Modified Riemann-Liouville derivatives; Complex fractional transformation; Modified simple equation method; (1/G')-expansion method; Zakharov-Kuznetsov equation.

INTRODUCTION

Nonlinear partial differential equations (NPDEs) are very useful to model many real world problems in science and engineering. Finding the exact solution of such equations is an important area of research. Fractional differential equations (FDE's) are also getting the attention of the researchers in the recent years. Many real world problems are modelled via FDE's in fluid dynamics. Exact solutions of such models play an important role in the mathematical sciences [1-6].

Exact solutions for a variety of FDE's are computed by researchers with different techniques including (G'/G)-expansion method [7], lie group analysis method [8], new extended trial equation method [9], first integral method [5], exp-function method [10], generalized Kudryashov method [2] and many others are suggested for obtaining the exact solutions of FDE's.

In mathematical physics, Zakharov-Kuznetsov (ZK) equation is used to describe the nonlinear development of ion-acoustic waves in magnetized plasma [11]. It is comprised of cold ions and hot isothermal electrons in the presence of a uniform magnetic field. It is also known as the generalizations of the well-known classical KdV equation. In this article, we are interested to calculate the exact solutions of (1+n)-dimensional fractional ZK equation of the following form

$$D_t^{\alpha}u + \alpha u D_{(x_1)}^{\alpha}u + D_{(x_1x_1)}^{2\alpha}u + D_{(x_2x_2)}^{2\alpha}u + D_{(x_3x_3)}^{2\alpha}u + \ldots + D_{(x_nx_n)}^{2\alpha} = 0, \tag{1}$$

where $0 < \alpha \leq 1$ and a is any arbitrary constant.

The article is arranged as follows. Modified Riemann-Liouville derivative of order α is defined in section 2 together with its properties. In section 3, complex fractional transformation is applied to convert the nonlinear PDE into an ODE and then exact solutions are obtained with MSE method and (1/G')-expansion method. Results are provided in section 4. References are given in the end of the article.

MODIFIED RIEMANN-LIOUVILLE DERIVATIVE AND ITS PROPERTIES

The modified Riemann-Liouville derivative of order α for a continuous function is defined as follows [3].

$$D^{\alpha}_{(x_1)}h(x_1)$$
$$= \begin{cases} \left\{ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx_1} \displaystyle\int_0^{x_1} (x_1-\tau)^{-\alpha} \left(h(\tau)-h(0)\right) d\tau \right., \; 0<\alpha<1 \\ \left(h^n(x_1)\right)^{\alpha-n}, \qquad n \le \alpha < n+1, \quad n \ge 1 \end{cases} \quad (2)$$

where $h: R \to R$, $x_1 = h(x_1)$ denotes a continuous function not necessarily first order differential. Here we have mentioned only following important properties from literature.

(1) If $h: R \to R$, is a continuous function, then its fractional derivative in the form of integral with respect to $(dx_1)^{\alpha}$

$$D^{\alpha}_{(x_1)}h(x_1) = \left\{ \frac{1}{\Gamma(\alpha)} \frac{d}{dx_1} \int_0^{x_1} (x_1-\tau)^{\alpha-1} h(\tau) d\tau \right.$$
$$= \frac{1}{\Gamma(1+\alpha)} \frac{d}{dx_1} \int_0^{x_1} h(\tau)(d\tau)^{\alpha} \quad , \quad 0<\alpha<1.$$

(2) For any constant k, the fractional derivative is:

$$D^{\alpha}_{(x_1)}(k) = 0.$$

(3) For the functions $g(x_1)$ and $h(x_1)$ and the constants a and b, fractional derivative for their linear combination is:

$$D^{\alpha}_{(x_1)}(\alpha g(x_1) + b(h(x_1))) = \alpha D^{\alpha}_{(x_1)}(g(x_1)) + b D^{\alpha}_{(x_1)}(h(x_1))$$

(4) For $h(x_1)=(x_1)^p$ the fractional derivative will be

$$D^{\alpha}_{(x_1)}((x_1)^p) = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)}(x_1)^{(p-\alpha)}.$$

EXACT SOLUTIONS

To find the exact solution of Eq. (1), we transform the fractional ZK equation in to another nonlinear ODE by applying the following complex fractional transformation

$$\xi = \frac{(x_1)^{\alpha}}{\Gamma(1+\alpha)} + \frac{(x_2)^{\alpha}}{\Gamma(1+\alpha)} + \frac{(x_3)^{\alpha}}{\Gamma(1+\alpha)} + \ldots + \frac{(x_n)^{\alpha}}{\Gamma(1+\alpha)} - \frac{w(t)^{\alpha}}{\Gamma(1+\alpha)}. \quad (3)$$

This results in the following ODE

$$-wU' + aUU' + nU'' = 0, \quad (4)$$

where $U = u(\xi)$ and $U' = \dfrac{du}{d\xi}$. Integrating Eq. (4) with respect to ξ taking constant of integration zero, it results

$$-wU + \frac{1}{2}aU^2 + nU' = 0. \quad (5)$$

Exact Solutions With Modified Simple Equation Method

Here we have utilized the MSE method [4] to find the exact solutions. Applying the equation balance method to achieve the value *m=1*. Therefore, the solution will take the form as

$$U(\xi) = k_0 + k_1\left(\frac{\varphi'}{\varphi}\right), \quad (6)$$

where k_0, k_1 are the constant to be determined such that $k_1 \neq 0$. Now we are in need to find the values of U^2 and U' from Eq. (6). i.e

$$\left.\begin{aligned} U' &= k_1\left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) \\ U^2 &= k_0^2 + 2k_0k_1\left(\frac{\varphi'}{\varphi}\right) + k_1^2\left(\frac{\varphi'}{\varphi}\right)^2 \end{aligned}\right\} \quad (7)$$

Putting the above values in Eq. (5) and compare the different powers of φ equal to zero, we have the following system of equations

$$-wk_0 + \frac{1}{2}ak_0^2 = 0, \quad (8)$$

$$nk_1\varphi'' + k_1(ak_0 - w)\varphi' = 0, \quad (9)$$

$$\left(\frac{1}{2}ak_1^2 - nk_1\right)\varphi'^2 = 0. \quad (10)$$

Using Eq. (8), it gives two values of the constant k_0 as below:

$$k_0 = 0, \quad \frac{2w}{a}. \quad (11)$$

As $k_1 \neq 0$, so Eq. (10) yields the following value of k_1 :

$$k_1 = \frac{2n}{a}. \quad (12)$$

Now we have to find the value of φ . For this we have the following two cases.

Case 1: When $k_0 = 0$, Eq. (9) gives the following result

$$(n\varphi'' - w\varphi')k_1 = 0, \qquad \frac{\varphi''}{\varphi'} = \frac{w}{n}. \quad (13)$$

Integrating w.r.t. ξ , we have

$$\ln(\varphi') + \ln(c) = \frac{w}{n}\xi,$$
$$\varphi' = c_1 e^{\frac{w}{n}\xi}. \tag{14}$$

where c_1 is the constant of integration. Integrating again in Eq. (14), it results,

$$\varphi = \frac{nc_1}{w} e^{\frac{w}{n}\xi} + c_2 , \tag{15}$$

where c_1 and c_2 are constants of integration $c_1 = \frac{1}{c}$. Hence the exact solution given in Eq. (6) will takes the form as

$$U(\xi) = k_0 + k_1 \left(\frac{c_1 e^{\frac{w}{n}\xi}}{\frac{nc_1}{w} e^{\frac{w}{n}\xi} + c_2} \right). \tag{16}$$

After substituting the values of k_0, k_1 in Eq. (16) and simplifying, we obtain

$$U_1(\xi) = \frac{2nw}{a}\left(\frac{c_1\left(\cosh\left(\frac{w}{2n}\xi\right) + \sinh\left(\frac{w}{2n}\xi\right)\right)}{(nc_1 + wc_2)\cosh\left(\frac{w}{2n}\xi\right) + (nc_1 - wc_2)\sinh\left(\frac{w}{2n}\xi\right)} \right). \tag{17}$$

Case 1 (a): If we put $c_1 = \frac{wc_2}{n}$, solution given in Eq. (17) will take the form as

$$U_2(\xi) = \frac{w}{a}\left(1 + \tanh\left(\frac{w}{2n}\xi\right)\right). \tag{18}$$

Case 1 (b): Similarly, when $c_1 = -\frac{wc_2}{n}$, solution will take the form as

$$U_3(\xi) = \frac{w}{a}\left(1 + \coth\left(\frac{w}{2n}\xi\right)\right). \tag{19}$$

Case 2: When $k_0 = \frac{2w}{a}$, Eq. (6) yields

$$U(\xi) = k_0 + k_1 \left(\frac{c_1 e^{-\frac{w}{n}\xi}}{\frac{nc_1}{w} e^{-\frac{w}{n}\xi} + c_2} \right). \tag{20}$$

Substituting the values of k_0, k_1 in Eq. (20) and simplifying, we have

$$U_4(\xi) = \frac{2w}{a} + \frac{2nw}{a}\left(\frac{c_1\left(\cosh\left(\frac{w}{2n}\xi\right) - \sinh\left(\frac{w}{2n}\xi\right)\right)}{(nc_1 + wc_2)\cosh\left(\frac{w}{2n}\xi\right) + (-nc_1 + wc_2)\sinh\left(\frac{w}{2n}\xi\right)} \right). \tag{21}$$

Case 2 (a): If we put $c_1 = \frac{wc_2}{n}$, Eq. (21) will take the following form

$$U_5(\xi) = \frac{2w}{a} + \frac{w}{a}\left(1 - \tanh\left(\frac{w}{2n}\xi\right)\right). \tag{22}$$

Case 2 (b): Similarly, if we put $c_1 = -\frac{wc_2}{n}$, Eq. (21) will be

$$U_6(\xi) = \frac{2w}{a}\frac{w}{a}\left(\coth\left(\frac{w}{2n}\xi\right) - 1\right), \tag{23}$$

where ξ is given in Eq. (3).

Exact Solution With (1/G')-Expansion Method

Here we have used the (1/G')-expansion method [12] for calculating the exact solutions. Balancing the terms U' and U^2 in Eq. (3.3), it yields M=1. Hence the solution will take the form as

$$U = a_0 + a_1\left(\frac{1}{G'}\right), \tag{24}$$

where $G(\xi)$ satisfy the ordinary linear differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu = 0 , \tag{25}$$

with a_0, a_1, λ and μ as constants to be determined. This equation contains the solution

$$G(\xi) = c_1 e^{-\lambda\xi} - \frac{\mu}{\lambda}\xi + c_2 , \tag{26}$$

where ξ is defined in Eq. (3) and

$$\left(\frac{1}{G'}\right) = \frac{\lambda}{-\mu + \lambda c_1\left(\cosh(\lambda\xi) - \sinh(\lambda\xi)\right)}. \tag{27}$$

Using Eq. (24) in Eq. (5) and then comparing coefficients of different powers of $\frac{1}{G'}$, we obtain system of equations

$$\left(\frac{1}{G'}\right)^0 : \quad -wa_0 + \frac{a}{2}a_0^2 = 0,$$

$$\left(\frac{1}{G'}\right)^1 : \quad -wa_1 + aa_0a_1 + na_1\lambda = 0, \tag{28}$$

$$\left(\frac{1}{G'}\right)^2 : \quad \frac{1}{2}aa_1^2 + \mu na_1 = 0.$$

Solving the system (3.26), we arrive at the following solutions:

$$\textbf{Set 1:} \quad \lambda = \frac{w}{n}, \quad a_1 = -\frac{2n\mu}{a}, \quad a_0 = 0. \tag{29}$$

Putting values from Eq. (27) and Eq. (29) in Eq. (24), we have

$$U_7(\xi) = \frac{-2nw\mu}{a(wc_1 - n\mu)\left(\cosh\left(\frac{w}{n}\xi\right) - \sinh\left(\frac{w}{n}\xi\right)\right)}. \tag{30}$$

$$\textbf{Set 2:} \quad \lambda = -\frac{w}{n}, \quad a_1 = -\frac{2n\mu}{a}, \quad a_0 = 0. \tag{31}$$

substituting values from Eq. (27) and Eq. (31) in Eq. (24), it yields

$$U_8(\xi) = \frac{2w}{a} - \frac{2nw\mu}{a(wc_1 + n\mu)\left(\cosh\left(\frac{w}{n}\xi\right) + \sinh\left(\frac{w}{n}\xi\right)\right)}, \tag{32}$$

where ξ is given in Eq. (3).

CONCLUSION

In this article, we achieved some exact solutions of (n+1)-dimensional fractional Zakharov-Kuznetsov equation. We apply the complex fractional wave transformation which converts the original nonlinear PDE into another nonlinear ODE. Then, (1/G')-expansion and modified simple equation methods are used to find the exact solutions. Obtained solutions are new and of significant importance to study the waves in the magnetized and dust plasmas. For different values of the parameters, graphical representations of the solutions are provided below with the help of computer software Maple 13. We have considered the case for n=1 for graphical representation (i.e. $u(x,t)$). In Fig. 1, for $w=-1.5$, $\alpha=0.7$, $a=3$, plots for the solution U_2 plots for the solution U_2 and its contour plot are given. In Fig.2, For $w=2$, $\alpha=0.7$, $a=-3$, plots for the solution U_3 and its contour plot are provided. In Fig.3, For $w=1.75$, $\alpha=0.3$, $a=0.4$, $c_1=5$, $\mu=5$, plots for the solution U_8 and its contour plot are presented. We have presented few of the solutions in graphical format. One can easily obtain the plots of others solutions easily.

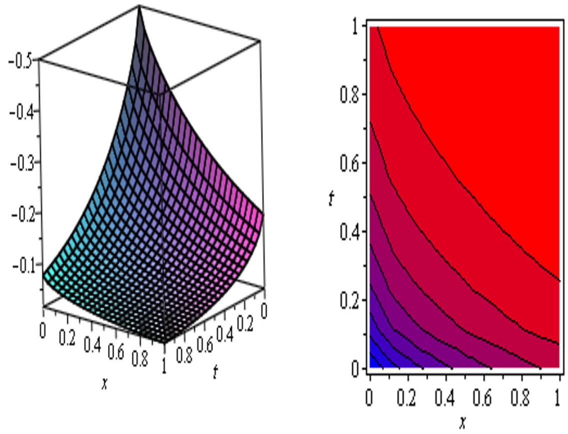

Figure 1. U_2 and its contour plot

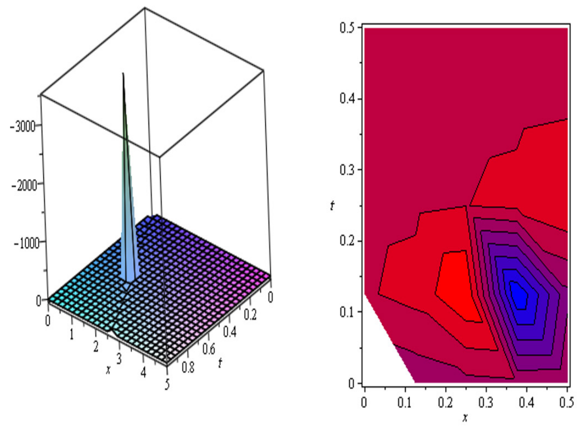

Figure 2. U_2 and its contour plot

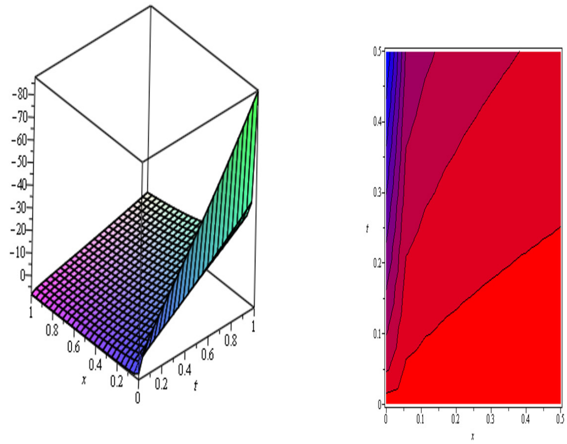

Figure 3. U_3 and its contour plot

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