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Explicit Solutions of a Three-Dimensional System of Nonlinear Difference Equations

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ABSTRACT

In this paper, we show that the system of difference equations

$$x_{n+1}=\frac{x_n+y_n}{1+x_ny_n},\ y_{n+1}=\frac{y_n+z_n}{1+y_nz_n},\ z_{n+1}=\frac{z_n+x_n}{1+z_nx_n},\quad n\in\mathbb{N}_0,$$

where the initial values x , y , z are real numbers, are solvable in explicit form via some changes of variables and tricks. Also, we determine the forbidden set of the initial values x , y , z for the above mentioned system and investigate asymptotic behavior of the well-defined solutions by using these explicit formulas.

Keywords:
Asymptotic behavior; explicit solution; rational difference equation; system

AMS Classification:
39A10

INTRODUCTION

Nonlinear difference equations constitute an important class of difference equations and studying of this kind of equations have recently attracted great interest. One can see this in some recent studies. See, for example [2-4,7-8,15-16,18,24-27]. Particularly, there have been a renewed interest on solvable ones of such equations and systems. For example, published papers on solvability of some types can be found in the references [5-6,10-14,28,31]. Additionally, there are some equations and systems whose solvability are newly discovered. For example, the solvability of the nonlinear difference equation

$$x_{n+1}=\frac{a+x_nx_{n-1}}{x_n+x_{n-1}},\ n\in\mathbb{N}_0, \tag{1}$$

where $a\in[0,\infty)$ and the initial values x_{-1},x_0 are real numbers, which was studied by Xianyi and Deming in [29], is newly discovered. The fact remains that the equations and the systems in the references [1,19-21,23,30], are so.

Our aim in this study is to show that the following systems of difference equations

$$x_{n+1}=\frac{x_n+y_n}{1+x_ny_n},y_{n+1}=\frac{y_n+z_n}{1+y_nz_n},\ z_{n+1}=\frac{z_n+x_n}{1+z_nx_n},\ n\in\mathbb{N}_0, \tag{2}$$

where real initial values x_0, y_0 and z_0 are real numbers, can be solved in explicit form. Also, we determine the forbidden set of the initial values x_0 , y_0 , z_0 for the system and investigate asymptotic behavior of the well-defined solutions by using these formulas.

Definition

Let

$$x_{n+1}=f(x_n,y_n,z_n),\ y_{n+1}=g(x_n,y_n,z_n),$$
$$z_{n+1}=h(x_n,y_n,z_n),\ n\in\mathbb{N}_0, \tag{3}$$

where $f:\mathbb{R}^3\to\mathbb{R}$, $g:\mathbb{R}^3\to\mathbb{R}$ and $h:\mathbb{R}^3\to\mathbb{R}$ is given functions, be a system of difference equations of first-order, and D_f , D_g and D_h be the domains of the functions f , g and h , respectively. Forbidden set of difference equation (3) is given by

$$F=\left\{\begin{array}{l}(x_0,y_0,z_0)\in\mathbb{R}^3\ :\ (x_i,y_i,z_i)\in D \text{ for}\\ i\in\overline{0,n-1},\ n\in\mathbb{N}_0,\ \text{and } (x_n,y_n,z_n)\notin D\end{array}\right\}$$

where $D := D_f \times D_g \times D_h$. This set contains all the initial values which causes the undefinable solutions of the system. That is, the initial values chosen from the complement of the forbidden set always produce the well-defined solutions.

RESULTS

In this section, we give our main results by obtaining the general solution in explicit form of the system (2). Next, we determine the forbidden set of the initial values x_0 , y_0 z_0 for the system (2). Additionally, we investigate asym-'ptotic behavior of the well-defined solutions by using the-ir explicit formulas.

The Explicit General Solution Of The System

To solve the system (2), we apply the changes of variables

$$x_n = \frac{e^{u_n} - 1}{e^{u_n} + 1}, \; y_n = \frac{e^{v_n} - 1}{e^{v_n} + 1}, \; z_n = \frac{e^{w_n} - 1}{e^{w_n} + 1} \tag{4}$$

to the system. Then, we have the linear system

$$u_{n+1} = u_n + v_n, \; v_{n+1} = v_n + w_n, \; w_{n+1} = w_n + u_n, \; n \in \mathrm{N}_0 \tag{5}$$

The system (5) can be written as

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - 2u_n = 0, \; n \in \mathrm{N}_0, \tag{6}$$

$$v_{n+3} - 3v_{n+2} + 3v_{n+1} - 2v_n = 0, \; n \in \mathrm{N}_0, \tag{7}$$

and

$$w_{n+3} - 3w_{n+2} + 3w_{n+1} - 2w_n = 0, \; n \in \mathrm{N}_0, \tag{8}$$

which are disjoint. Note that the equations (6)-(8) are in the same form. Therefore, we only solve one of them. Let choose Eq. (6) which can be written as

$$(u_{n+3} - 2u_{n+2}) - (u_{n+2} - 2u_{n+1}) + (u_{n+1} - 2u_n) = 0, \; n \in \mathrm{N}_0.$$

Set $u_{n+2} - 2u_{n+1} = \tilde{u}_n$, So, we get the linear equation of order two

$$\tilde{u}_{n+1} = \tilde{u}_n - \tilde{u}_{n-1}, \; n \in \mathrm{N}_0, \tag{9}$$

Eq. (9) is periodic with period 6 such that $\tilde{u}_{6n+i} = \tilde{u}_i$, $\left(i = \overline{0,5}\right)$ which implies

$$u_{6n+2+i} - 2u_{6n+1+i} = u_{i+2} - 2u_{i+1}, \; n \in \mathrm{N}_0. \tag{10}$$

By adding the backward iterate of (10) to own itself, we get

$$u_{6n+2+i} - 2u_{6n+1+i} + u_{6n+1+i} - 2u_{6n+i} = u_{i+2} - 2u_{i+1} + u_{i+1} - 2u_i$$

or

$$u_{6n+2+i} - u_{6n+1+i} - 2u_{6n+i} = u_{i+2} - u_{i+1} - 2u_i, \; n \in \mathrm{N}_0. \tag{11}$$

Eq. (6) can be also written in the further form

$$(u_{n+3} - u_{n+2} + u_{n+1}) - 2(u_{n+2} - u_{n+1} + u_n) = 0. \tag{12}$$

If we apply the change of variables

$$u_{n+2} - u_{n+1} + u_n = \hat{u}_n \tag{13}$$

to Eq. (12), then, from Eq. (12), it follows that

$$\hat{u}_{n+1} - 2\hat{u}_n = 0 \tag{14}$$

whose the general solution is given by

$$\hat{u}_n = 2^n \hat{u}_0 \tag{15}$$

which implies

$$u_{n+2} - u_{n+1} + u_n = 2^n (u_2 - u_1 + u_0). \tag{16}$$

Eq. (16) can be decomposed in terms of its own subscript as follows;

$$u_{6n+2+i} - u_{6n+1+i} + u_{6n+i} = 2^{6n+i}(u_2 - u_1 + u_0), \; i = \overline{0,5} \tag{17}$$

By subtracting (11) from (17), we get the formula

$$u_{6n+i} = \frac{1}{3}\left(2^{6n+i}(v_1 + u_0) - (v_{i+1} - 2u_i)\right), \; i = \overline{0,5}, \tag{18}$$

for the solution (u_n) of Eq. (6). We also state the formula (18) explicitly such that

$$u_{6n} = \frac{2^{6n} + 2}{3} u_0 + \frac{2^{6n} - 1}{3} v_0 + \frac{2^{6n} - 1}{3} w_0, \tag{19}$$

$$u_{6n+1} = \frac{2^{6n+1} + 1}{3} u_0 + \frac{2^{6n+1} + 1}{3} v_0 + \frac{2^{6n+1} - 2}{3} w_0, \tag{20}$$

$$u_{6n+2} = \frac{2^{6n+2} - 1}{3} u_0 + \frac{2^{6n+2} + 2}{3} v_0 + \frac{2^{6n+2} - 1}{3} w_0, \tag{21}$$

$$u_{6n+3} = \frac{2^{6n+3} - 2}{3} u_0 + \frac{2^{6n+3} + 1}{3} v_0 + \frac{2^{6n+3} + 1}{3} w_0, \tag{22}$$

$$u_{6n+4} = \frac{2^{6n+4} - 1}{3} u_0 + \frac{2^{6n+4} - 1}{3} v_0 + \frac{2^{6n+4} + 2}{3} w_0, \tag{23}$$

$$u_{6n+5} = \frac{2^{6n+5} + 1}{3} u_0 + \frac{2^{6n+5} - 2}{3} v_0 + \frac{2^{6n+5} + 1}{3} w_0 \tag{24}$$

for $n \in \mathrm{N}_0$, Consequently, by the change of variables $x_n = \frac{e^{u_n} - 1}{e^{u_n} + 1}$, we have the formulas

$$x_{6n} = 1 - \frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n}+2}{3}} \left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n}-1}{3}} \left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n}-1}{3}} + 1} \tag{25}$$

$$x_{6n+1} = 1 - \frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+1}+1}{3}} \left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+1}-1}{3}} \left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+1}-2}{3}} + 1} \tag{26}$$

$$x_{6n+2}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+2}-1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+2}+2}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+2}-1}{3}}+1} \quad (27)$$

$$x_{6n+3}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+3}-2}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+3}+1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+3}+1}{3}}+1} \quad (28)$$

$$x_{6n+4}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+4}-1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+4}-1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+4}+2}{3}}+1} \quad (29)$$

$$x_{6n+5}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+5}+1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+5}-2}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+5}+1}{3}}+1} \quad (30)$$

which are the formulas of the variable x_n, for $n\in N_0$, The formulas of y_n can be obtained by the first equation of (2). That is, by solving y_n in this equation, we have

$$y_n=\frac{x_{n+1}-x_n}{1-x_{n+1}x_n}. \quad (31)$$

From (25)-(31), it follows that

$$y_{6n}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n}-1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n}+2}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n}-1}{3}}+1} \quad (32)$$

$$y_{6n+1}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+1}-2}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+1}+1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+1}+1}{3}}+1} \quad (33)$$

$$y_{6n+2}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+2}-1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+2}-1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+2}+2}{3}}+1} \quad (34)$$

$$y_{6n+3}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+3}+1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+3}-2}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+3}+1}{3}}+1} \quad (35)$$

$$y_{6n+4}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+4}+2}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+4}-1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+4}-1}{3}}+1} \quad (36)$$

$$y_{6n+5}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+5}+1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+5}+1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+5}-2}{3}}+1} \quad (37)$$

for $n\in N_0$, Similarly, from the second equation of (2), we obtain

$$z_n=\frac{y_{n+1}-y_n}{1-y_{n+1}y_n} \quad (38)$$

which yields

$$z_{6n}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n}-1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n}-1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n}+2}{3}}+1}$$

$$z_{6n+1}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+1}+1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+1}-2}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+1}+1}{3}}+1}$$

$$z_{6n+2}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+2}+1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+2}-1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+2}-1}{3}}+1}$$

$$z_{6n+3}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+3}+1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+3}+1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+3}-2}{3}}+1}$$

$$z_{6n+4}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+4}-1}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+4}+2}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+4}-1}{3}}+1}$$

$$z_{6n+5}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+5}-2}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+5}+1}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+5}+1}{3}}+1}$$

for $n\in N_0$, along with (32)-(38).

The Forbidden Set Of The İnitial Values

The above obtained formulas exactly determine the solutions of the system (2). But, some initial values yield undefinable solution of the system. Now, we give the set of such initial values by using the formulas. To do this, we use the changes of variables (4) along with the formula (18) and so get the closed formula of x_n as follows:

$$x_{6n+i}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+x_i}{1-x_i}\right)^{2}\left(\frac{1-y_{i+1}}{1+y_{i+1}}\right)+1} \quad (39)$$

for $i=\overline{0,5}$ and $n\in N_0$, It is easy to see that the formula (39) is undefinable, if

$$\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+x_i}{1-x_i}\right)^{2}\left(\frac{1-y_{i+1}}{1+y_{i+1}}\right)=-1$$

for $i=\overline{0,5}$ and $n\in N_0$ Similarly, we have the closed formulas of y_n and z_0 as follows

$$y_{6n+i}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_i}{1-y_i}\right)^{2}\left(\frac{1-z_{i+1}}{1+z_{i+1}}\right)+1} \quad (40)$$

and

$$z_{6n+i}=1-\frac{2}{\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_i}{1-z_i}\right)^{2}\left(\frac{1-x_{i+1}}{1+x_{i+1}}\right)+1} \quad (41)$$

for $i=\overline{0,5}$ and $n\in N_0$ respectively. Consequently, we find the forbidden set of the initial values x_0, y_0, z_0 as follows:

$$F=\left\{\begin{array}{l}(x_0,y_0,z_0)\in \mathrm{R}^3\ :\\ \left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+x_i}{1-x_i}\right)^2\left(\frac{1-y_{i+1}}{1+y_{i+1}}\right)=-1\\ \left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_i}{1-y_i}\right)^2\left(\frac{1-z_{i+1}}{1+z_{i+1}}\right)=-1,\\ \left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_i}{1-z_i}\right)^2\left(\frac{1-x_{i+1}}{1+x_{i+1}}\right)=-1\\ i=\overline{0,5},\ n\in \mathrm{N}_0\end{array}\right\}$$

Asymptotic Behavior Of The Well-Defined Solutions

We investigate the asymptotic behavior of the well-defined solutions of the system (2). The main result of this subsection is given by the following theorem.

Theorem

Suppose that the sequence $(x_n,y_n,z_n)_{n\geq 0}$ is a well-defined solution of the system (2), that is, $(x_0,y_0,z_0)\notin F$, Then,

$$\lim_{n\to\infty}(|x_n|,|y_n|,|z_n|)=(1,1,1).$$

Proof

To prove, we use the function $f(x)=\frac{1+x}{1-x}$ along with the formulas (39)-(41). Before proving, we can say that the points $(0,0,0)$, $(1,1,1)$ and $(-1,-1,-1)$ are equilibrium points of the system (2). That is, equilibrium solutions of the system (system) are given by $(x_n,y_n,z_n)_{n\geq 0}=(0,0,0)$, $(x_n,y_n,z_n)_{n\geq 0}=(1,1,1)$ and $(x_n,y_n,z_n)_{n\geq 0}=(-1,-1,-1)$, respectively. We here deal with nonequilibrium solutions of the system (2). We observe for the function f that if $x\in(-\infty,0)$, then $f(x)\in(-1,1)$; if $x\in(0,1)\cup(1,\infty)$, then $f(x)\in(-\infty,-1)\cup(1,\infty)$, Hence, we prove the theorem in three cases:

(i) If $x_0,y_0,z_0\in(-\infty,0)\setminus F$, then $f(x_0),f(y_0),f(z_0)\in(-1,1)$, Therefore, from the formulas (39)-(41), we get the limit

$$\lim_{n\to\infty}(x_n,y_n,z_n)=(1,1,1).\ \text{as}\ n\to\infty,$$

(ii) If $x_0,y_0,z_0\in(0,1)\cup(1,\infty)\setminus F$, then $f(x_0),f(y_0),f(z_0)\in(-\infty,-1)\cup(1,\infty)$, Therefore, from the formulas (39)-(41), we get

$$\lim_{n\to\infty}(x_n,y_n,z_n)=(-1,-1,-1).$$

(iii) If $x_0,y_0,z_0\in(-\infty,\infty)\setminus F$, then we can not say about the quantities $f(x_0)$, $f(y_0)$ and $f(z_0)$ exactly. But, the sequences

$$s_n^{(1)}=\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+x_i}{1-x_i}\right)^2\left(\frac{1-y_{i+1}}{1+y_{i+1}}\right),$$

$$s_n^{(2)}=\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_i}{1-y_i}\right)^2\left(\frac{1-z_{i+1}}{1+z_{i+1}}\right)$$

and

$$s_n^{(3)}=\left(\frac{1+x_0}{1-x_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+y_0}{1-y_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_0}{1-z_0}\right)^{\frac{2^{6n+i}}{3}}\left(\frac{1+z_i}{1-z_i}\right)^2\left(\frac{1-x_{i+1}}{1+x_{i+1}}\right)$$

tend to either to 0 or to ∞ . So, by the formulas (39)-(41), we obtain the limit

$$\lim_{n\to\infty}(|x_n|,|y_n|,|z_n|)=(1,1,1).$$

which completes the proof.

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