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A New Perspective On Soft Topology

Mustafa Burç KANDEMİR
Muğla Sıtkı Koçman University, Department of Mathematics, Muğla, Turkey.

ABSTRACT

In this paper, we approach the concept of soft topology with a different perspective. We define a soft topological space over given initial topological universe as a parameterization of subspaces of a topological space. In addition to this, we define some basic topological concepts such as Hausdorffness, compactness, connectedness for soft topological spaces and study their some properties. We give some results for relations between a topological space and a soft topological space that is defined on it.

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***Correspondence to**: Mustafa Burç Kandemir, Muğla Sıtkı Koçman University, Department of Mathematics, 48000, Muğla, Turkey,*
E-Mail:hbkandemir@mu.edu.tr

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INTRODUCTION AND PRELIMINARIES

Almost every branch of science has its own uncertainties and ambiguities. These uncertainties depend on the existence of many parameters. So it is not always easy to model a daily life problem mathematically using classical mathematical methods. In this sense, mankind has gone to find new mathematical models. In 1999, Molodtsov [1] established the soft set theory to model uncertainties in any phenomenon. He defined the concept of soft set as follows;

Definition 1.1.

[1] Let U be an initial universe, E be a parameters set. The pair (F,A) is called a soft set over U such that $F:A\rightarrow P(U)$ is a set-valued function where $A\subseteq E$ and $P(U)$ is the power set of U.

In fact a soft set on an initial universe is nothing more than the parameterization of some subsets of the universe.

The family of all soft sets over the initial universe U with respect to the parameters set E is denoted by $S(U,E)$.

Set-theoretic operations for soft sets given by Maji et al. and Ali et al. in [2, 3]. The operations between two soft sets such as soft union, soft intersection, soft complement etc. defined in [2, 3] as follows.

Definition 1.2.

[2, 3] Let (F,A) and (G,B) be soft sets over U where $A,B\subseteq E$. We say that (F,A) is a soft subset of (G,B) if $A\subseteq B$ and $F(p)\subseteq G(p)$ for each $p\in A$, and denoted by $(F,A)\tilde{\subseteq}(G,B)$.

Definition 1.3.

[2, 3] Let (F,A) and (G,B) be soft sets over U where $A,B\subseteq E$. Soft union of (F,A) and (G,B) is a soft set (H,C) over U such that

$$H(p)=\begin{cases} F(p) & ,if\ p\in A-B \\ G(p) & ,if\ p\in B-A \\ F(p)\cup G(p) & ,if\ p\in A\cap B \end{cases} \quad (1)$$

for each $p\in C$ where $C=A\cup B$. It is detonated by $(F,A)\tilde{\cup}(G,B)=(H,C)$.

Soft intersection of (F,A) and (G,B) is a soft set (H,C) over U such that $H(p)=F(p)\cap G(p)$ for each $p\in C$ where $C=A\cap B$. It is denoted by $(F,A)\tilde{\cap}(G,B)=(H,C)$.

Definition 1.4.

[2, 3] Let (F,A) be a soft set over U. If $F(p)=\varnothing$ for each $p\in A$, we call that (F,A) is a relative null soft set and denoted by $\widetilde{\Phi}$. Besides, if $F(p)=U$ for each $p\in A$, then we call that (F,A) is a relative whole soft set and denoted by $\widetilde{U}_A$. We note that (F,E) is called absolute soft set if $F(p)=U$ for all $p\in E$.

"And" and "Or" operations between soft sets are defined since an element in the universal set may have more than one property or an attribute. Hence, their definitions are given below.

Definition 1.5.

[2, 3] $(F,A)\,\mathbf{Or}\,(G,B)=(H,A\times B)$ is defined by $H(p_1,p_2)=F(p_1)\cup G(p_2)$ for all $(p_1,p_2)\in A\times B$. In a similar manner, $(F,A)\,\mathbf{And}\,(G,B)=(H,A\times B)$ is defined by $H(p_1,p_2)=F(p_1)\cap G(p_2)$ for all $(p_1,p_2)\in A\times B$.

Definition 1.6.

[2, 3] The complement of a soft set (F,A) is defined as $(F,A)^c=(F^c,A)$ where $F^c:A\to P(U)$ is a mapping given by $F^c(p)=U-F(p)$ for each $p\in A$.

The concept of Cartesian product introduced by Babitha et al. in [4].

Definition 1.7.

[4] The Cartesian product of the soft sets (F,A) and (G,B) is defined as $(F,A)\tilde{\times}(G,B)=(H,A\times B)$ where $H:A\times B\to P(U\times U)$ and $H(p_1,p_2)=F(p_1)\times G(p_2)$ for each $(p_1,p_2)\in A\times B$.

Kim and Min defined the concept of full soft set as follows;

Definition 1.8.

[5] Let (F,A) be a soft set over U. (F,A) is called full soft set over U if the condition $\bigcup_{p\in A}F(p)=U$ is provided for each parameters in the parameter set A.

The concept of similarity between soft sets given by Min in [6].

Definition 1.9.

[6] Let (F,A) and (G,B) be soft sets over U. We call that (F,A) is similar to (G,B) if there exists a bijective function $\phi:A\to B$ such that $F(p)=(G\circ\phi)(p)$ for each $p\in A$. If (F,A) is simİlar to (G,B), we denote by $(F,A)\cong(G,B)$.

Topology is one of the most important areas of mathematics. Main object of topology is the study of the general abstract nature of continuity or closeness on spaces. We refer readers to [7] and [8] for the basic definitions and properties for topological spaces. We can give the formal definition of topology defined in terms of set operations. Let U be a set and $\mathfrak{T}$ be a family of some subsets of U. It is called that $\mathfrak{T}$ is a topology on U if it satisfies following properties;

- U and $\varnothing$ are in $\mathfrak{T}$.
- Whenever sets X and Y are in $\mathfrak{T}$, then so is $X\cap Y$.
- Whenever two or more sets are in $\mathfrak{T}$, then so is their union.

The pair $(U,\mathfrak{T})$ is called topological space. All members of $\mathfrak{T}$ are called open sets. The set whose complement is open is called closed set [7, 8].

We can obtain new topological spaces using a given topological space via subset of a underlying set, and such topological space are called subspace. We can give this achievement method as in the following theorem.

Theorem 1.10.

[7, 8] Let $(U,\mathfrak{T})$ be a topological space and $X\subseteq U$. The family

$$\mathfrak{T}_X=\{X\cap O \mid O\in\mathfrak{T}\}$$

is a topology on X.

The topology $\mathfrak{T}$ obtained in the above theorem is called subspace topology for $X\subseteq U$, and the pair $(X,\mathfrak{T}_X)$ is called subspace of $(U,\mathfrak{T})$.

Thus we gain the following result according to the definition of subspace.

Corollary 1.11.

[8] Let $(U,\mathfrak{T})$ be a topological space and $(X,\mathfrak{T}_X)$ be a subspace of it. Then, $O_X\in\mathfrak{T}_X$ if and only if there exists

$O \in \mathfrak{T}$ such that $O_X = X \cap O$.

We will now define a new intersection and union concepts for two families of sets consisting of subsets of a set that we will use later.

Definition 1.12.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two families of some subsets of the set U. The family

$$\mathfrak{X} \widehat{\cap} \mathfrak{Y} = \{X \cap Y \mid X \in \mathfrak{X}, Y \in \mathfrak{Y}\}$$

is called Cartesian intersection of the families $\mathfrak{X}$ and $\mathfrak{Y}$. The family

$$\mathfrak{X} \widehat{\cup} \mathfrak{Y} = \{X \cup Y \mid X \in \mathfrak{X}, Y \in \mathfrak{Y}\}$$

is called Cartesian union of the families $\mathfrak{X}$ and $\mathfrak{Y}$.

Example 1.13.

Let $U = \{a,b,c,d,e\}$ be a set, $\mathfrak{X} = \{\{a,b,d\},\{c,e\},\{d\}\}$ and $\mathfrak{Y} = \{\{a,e\},\{d,e\}\}$ be families of some subsets of U. Then we obtain that

$$\mathfrak{X} \widehat{\cap} \mathfrak{Y} = \{\{a\},\{d\},\{e\},\varnothing\}$$

from Definition 1.12. Moreover,

$$\mathfrak{X} \widehat{\cup} \mathfrak{Y} = \{\{a,b,d,e\},\{a,c,e\},\{c,d,e\},\{a,d,e\},\{d,e\}\}$$

We can give the following theorem for subspaces of a topological space by using Definition 1.12.

Theorem 1.14.

Let $(U,\mathfrak{T})$ be a topological space, $(X,\mathfrak{T}_X)$ and $(Y,\mathfrak{T}_Y)$ be subspaces of U. Then $\mathfrak{T}_{X\cap Y} = \mathfrak{T}_X \widehat{\cap} \mathfrak{T}_Y$.

Proof. Suppose that $O \in \mathfrak{T}_{X\cap Y}$. Then there exists $O' \in \mathfrak{T}$ such that $O = O' \cap (X \cap Y)$. So, we have

$$O = (O' \cap X) \cap (O' \cap Y)$$

and $O' \cap X \in \mathfrak{T}_X$ and $O' \cap Y \in \mathfrak{T}_Y$ where $O' \in \mathfrak{T}$. Therefore, $O \in \mathfrak{T}_X \widehat{\cap} \mathfrak{T}_Y$.

Conversely, let $O \in \mathfrak{T}_X \widehat{\cap} \mathfrak{T}_Y$. Then there exist $M \in \mathfrak{T}_X$ and $N \in \mathfrak{T}_Y$ such that $O = M \cap N$. Therefore, there exist $O_1, O_2 \in \mathfrak{T}$ such that $M = O_1 \cap X$ and $N = O_2 \cap Y$. At that case, we have

$$\begin{aligned} O = M \cap N &= (O_1 \cap X) \cap (O_2 \cap Y) \\ &= (O_1 \cap O_2) \cap (X \cap Y) \end{aligned}$$

Since $O_1 \cap O_2 \in \mathfrak{T}$, we obtain that $O \in \mathfrak{T}_{X\cap Y}$.

Hence, $\mathfrak{T}_{X\cap Y} = \mathfrak{T}_X \widehat{\cap} \mathfrak{T}_Y$.

We can give some basic topological properties of topological spaces and subspaces.

Let $(U,\mathfrak{T})$ be a topological space and $\mathfrak{B}$ is a subfamily of $\mathfrak{T}$. We call that $\mathfrak{B}$ is a base for the topology $\mathfrak{T}$ if every open set in U is a union of sets from $\mathfrak{B}$. Let $(U,\mathfrak{T}_U)$ and $(V,\mathfrak{T}_V)$ be a topological spaces and $\mathfrak{B}_U \subseteq \mathfrak{T}_U$ and $\mathfrak{B}_V \subseteq \mathfrak{T}_V$ are bases, respectively. It is called that the topological space which generated by the family

$$\mathfrak{B} = \{B_1 \times B_2 \mid B_1 \in \mathfrak{B}_U, B_2 \in \mathfrak{B}_V\}$$

is product space of the topological spaces $(U,\mathfrak{T})$ and $(V,\mathfrak{T}_V)$, and denoted by $(U \times V, \mathfrak{T}_U \times \mathfrak{T}_V)$ [7, 8]. We can give the following lemma for subspaces of product spaces.

Lemma 1.15.

[7, 8] If $X \subseteq U$ and $Y \subseteq V$ are subsets of topological spaces U and V, respectively, then product topology of the subspace topology on X and Y coincides subspace topology of the product topology on $U \times V$.

Let $(U,\mathfrak{T})$ be a topological space. If any two distinct points of $(U,\mathfrak{T})$ are given and there are distinct open neighborhoods that separate these points, this space is called Hausdroff space. We can give the following lemma for subspaces of Hausdorff space.

Lemma 1.16.

[7, 8] Each subspace of a Hausdorff space is Hausdorff.

Lemma 1.17.

[7, 8] The product space of two topological spaces is Hausdorff if and only if each one of topological spaces is Hausdorff.

Compactness is an important concept of a topological space. We can regard the concept of compactness in topological spaces as a generalization of the concept of boundedness in the space of real numbers $\mathbb{R}$. The notion of compactness defined by the concept of open cover of given topological space. For given topological space $(U,\mathfrak{T})$, consider the family $\{X_i \mid i\in I, X_i \subseteq U\}$. If $\bigcup_{i\in I} X_i = U$, then the family $\{X_i \mid i\in I, X_i \subseteq U\}$ is called cover for U. If each element of the family $\{X_i \mid i\in I, X_i \subseteq U\}$ is in $\mathfrak{T}$ and the family is a cover, then the family $\{X_i \mid i\in I, X_i \subseteq U\}$ is called open cover for U. If the index set $J \subset I$ is finite and the family $\{X_i \mid i\in I, X_i \subseteq U\}$ is a cover for U, then the family $\{X_i \mid i\in I, X_i \subseteq U\}$ is called finite cover for U. If any subfamily of $\{X_i \mid i\in I, X_i \subseteq U\}$ is a cover for U then the subfamily is called subcover for U. So, the topological space $(U,\mathfrak{T})$ is called compact if all open cover of U has a finite subcover. Moreover, given any topological space $(U,\mathfrak{T})$ and $X \subset U$, if the subspace $(X,\mathfrak{T}_X)$ is compact, then X is called compact subset of the topological space $(U,\mathfrak{T})$ [7, 8].

Lemma 1.18.

[7, 8] All closed subsets of a compact space is compact.

Lemma 1.19.

[7, 8] Let $(U,\mathfrak{T})$ be a compact and Hausdorff topological space and $X \subset U$. X is compact if and only if X is closed.

Lemma 1.20.

[7, 8] Each compact subset of a Hausdorff space is closed.

Lemma 1.21 (Tychonoff).

[7, 8] If $(U_i,\mathfrak{T}_i)$ are compact topological spaces for each $i\in I$, then so is $U=\prod_{i\in I} U_i$ which is endowed with the product topology.

Another important concept for topological spaces is the concept of connectedness. The definition of connectedness for a topological space is related to the fact that the space is a whole. In fact, the definition of connectedness of a topological space comes from the notion of separation of space into two or more parts. Let $(U,\mathfrak{T})$ be a topological space. A separation of U is a pair X and Y of disjoint nonempty open subsets of U whose union is U. The space U is said to be connected if there does not exist a separation of U. In other words, a topological space $(U,\mathfrak{T})$ is connected if and only if the only subsets of U that are both open and closed in U are empty set and U itself. For given any subset X of a topological space $(U,\mathfrak{T})$, if the subspace $(X,\mathfrak{T}_X)$ is connected, then it is called that the subset X is connected in $(U,\mathfrak{T})$ [7, 8].

Lemma 1.22.

[7, 8] The family of non-disjoint connected subspaces of a topological space $(U,\mathfrak{T})$ is connected.

Of course, the principle of the notion of topology based on set theory. We define the specific properties of sets such as opennes, closedness, compactness, connectedness etc. We can regard the soft set theory as an unusual set theory. So, there is a necessity to define the concept of soft topology on given initial universe or on a soft set. In 2011, Shabir and Naz [9] introduced soft topological space over an initial universe with a fixed set of parameters. This article is first published about soft topology and many basic properties were examined in this article by Shabir and Naz.

Definition 1.23.

[9] A soft topology τ on U is a family of soft sets over U if

1) $\widetilde{\Phi}$ and $\widetilde{U}_E$ belong to τ,

2) the union of any number of soft sets in τ belongs to τ,

3) the intersection of any two soft sets in τ belongs to τ.

The triplet (U,τ,E) is called a soft topological space over U.

They gave following theorem in [9];

Theorem 1.24.

[9] Let (U,τ,E) be a soft topological space over U. Then the family

$$\tau_p = \{F(p) \mid (F,E)\in\tau\}, \forall p\in E$$

defines a topology on U.

Together with this theorem, they showed that soft topology is a parameterization of topologies defined on U. With this approach, they tried to capture the Molodtsov's sense. But Molodtsov's sense is the process of parameterizing the substructure of any given structure.

In this paper, we deal with this situation. In the next section, we redefine the concept soft topological space with respect to Molodtsov's sense.

RESULTS AND DISCUSSION

Soft Topology: A New Perspective

In this section, we look at the concept of soft topology from a new perspective. Molodtsov [1] defined a soft set over an initial universe as parameterization of some subsets of given initial universe as we mentioned in this previous section. Moreover, Aktaş et al. [10] defined the concept of a soft group over a group as parametrization of some subgroups of given group which is similar to Molodtsov' s sense. Acar et al. [11] defined the concept of a soft ring over a ring as a parametrization of some subrings of given any ring which is similar to Molodtsov' s sense, too. We use this notion, i.e. Molodtsov's sense, to redefine soft topology. We can give the definition as below.

Definition 2.1.

Let $(U,\mathfrak{T})$ be a topological space, E be a parameters set and (F,A) be a soft set over U where $A\subseteq E$. It is called that (F,A) is a soft topology over U if $\left(F(p),\mathfrak{T}_{F(p)}\right)$ is a subspace of $(U,\mathfrak{T})$ for each $p\in A$. $(F,A,\mathfrak{T})$ is called a soft topological space over U.

Example 2.2.

. Let $U=\{a,b,c,d\}$ and $\mathfrak{T}=\{\varnothing,U,\{a,b\}\}$, $E=\{1,2,3,4,5,6\}$ and $(F,A)=\{1=\{a,b\},4=\{a\}\}$ where $\mathfrak{T}_{\{a,b\}}=\{\varnothing,\{a,b\}\}$, and $\mathfrak{T}_{\{a\}}=\{\varnothing,\{a\}\}$. We obtain that $\left(F(p),\mathfrak{T}_{F(p)}\right)$ is a subspace of $(U,\mathfrak{T})$ for each $p\in\{1,4\}$. Thus $(F,A,\mathfrak{T})$ is a soft topological space over U.

As it can be understood from Definition 2.1, a soft topological space is a parametrization of some subspaces of a topological space.

Theorem 2.3.

If $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are soft topological space over $(U,\mathfrak{T})$, then their intersection $(F,A,\mathfrak{T})\tilde{\cap}(G,B,\mathfrak{T})$ is a soft topological space over U where the intersection of all subspaces according to each parameter is the Cartesian intersection subspace topology on the intersection of subsets according to each parameters.

Proof. Suppose that $(F,A,\mathfrak{T})\tilde{\cap}(G,B,\mathfrak{T})=(H,C,\mathfrak{T})$. We need to show that $\left(H(p),\mathfrak{T}_{H(p)}\right)$ is a subspace of U for each $p\in C=A\cap B$. From the definition of soft intersection, we have that $H(p)=F(p)\cap G(p)$ for each $p\in C$. Obviously, we obtain that $\mathfrak{T}_{H(p)}=\mathfrak{T}_{F(p)}\hat{\cap}\mathfrak{T}_{G(p)}$ for each $p\in C$ from Theorem 1.14. Thus, $\left(H(p),\mathfrak{T}_{H(p)}\right)$ is a subspace of U for all $p\in C$. Hence, $(H,C,\mathfrak{T})$ is a soft topology over U.

Theorem 2.4.

If $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are soft topological space over $(U,\mathfrak{T})$, and $A\cap B=\varnothing$, then their union $(F,A,\mathfrak{T})\tilde{\cup}(G,B,\mathfrak{T})$ is a soft topological space over U.

Proof. Say that $(F,A,\mathfrak{T})\tilde{\cup}(G,B,\mathfrak{T})=(H,C,\mathfrak{T})$. From definition of soft union and Equation 1.1, we have

$$H(p)=\begin{cases}F(p) & ,if\ p\in A-B\\ G(p) & ,if\ p\in B-A\\ F(p)\cup G(p) & ,if\ p\in A\cap B\end{cases}$$

and $C=A\cup B$. Since $A\cap B=\varnothing$, either $H(p)=F(p)$ where $p\in A$ or $H(p)=G(p)$ where $p\in B$ for each $p\in C$. Since $\left(F(p),\mathfrak{T}_{F(p)}\right)$ and $\left(G(p),\mathfrak{T}_{G(p)}\right)$ are subspace of U where $p\in A$ and $p\in B$, respectively, then we obtain that $\left(H(p),\mathfrak{T}_{H(p)}\right)$ is a subspace of U for each $p\in C$. Thus $(H,C,\mathfrak{T})$ is a soft topological space over U.

We obtain the following theorem which proof is similar to the proof of Theorem 2.3.

Theorem 2.5.

If $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are soft topological space over $(U,\mathfrak{T})$ then $(F,A,\mathfrak{T})\textbf{And}(G,B,\mathfrak{T})$ is a soft topological space over U where the intersection of all subspaces according to each parameter is the Cartesian intersection subspace topology on the intersection of subsets according to each parameters.

$(F,A,\mathfrak{T})\textbf{Or}(G,B,\mathfrak{T})$ may not be soft topological space where $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are soft topological space over $(U,\mathfrak{T})$. We can give the following example to see this.

Example 2.6.

Let $U=\{a,b,c,d\}$ be a set, $\mathfrak{T}=\{\varnothing,U,\{a,b,c\},\{a,b\},\{c\}\}$ be topology on U.

$(F,A)=\{1=\{a,c\},\ 2=\{d\}\}$ soft topological space over U such that $\mathfrak{T}_{F(1)}=\{\varnothing,\{a,c\},\{a\},\{c\}\}$ and $\mathfrak{T}_{F(2)}=\{\varnothing,\{d\}\}$ and $(G,B)=\{2=\{b,d\}\}$ soft topological space over U such that $\mathfrak{T}_{G(2)}=\{\varnothing,\{b,d\},\{b\}\}$. Now, we have

$$\begin{aligned}(F,A)\mathbf{Or}(G,B)&=(H,A\times B)\\&=\{(1,2)=\{a,b,c,d\},(2,2)=\{b,d\}\}\end{aligned}$$

and we have that $\mathfrak{T}_{F(1)}\cup\mathfrak{T}_{G(2)}\neq\mathfrak{T}_{H(1,2)}=\mathfrak{T}_{F(1)\cup G(2)}$. Thus $(H(1,2),\mathfrak{T}_{H(1,2)})$ is not subspace of $(U,\mathfrak{T})$. Hence, $(H,A\times B,\mathfrak{T})$ is not a soft topological space over U.

Note that, since every topological space is a subspace of itself, each relative whole soft set is a soft topological space over any given topological space. Such soft topological spaces are called overt soft topological space, i.e. if $(U,\mathfrak{T})$ is a topological space and $\widetilde{U}_A$ is a relative whole soft set over U then it is called that $(\widetilde{U}_A,\mathfrak{T})$ is an overt soft topological space over U.

From Theorem 1.10, we obtain a soft topological space over given any topological space and soft set defined on it, obviously.

Remark 2.7.

Let $(U,\mathfrak{T})$ be a topological space and (F,A) be a soft set over U where $A\subseteq E$. For each $p\in A$, $\mathfrak{T}_{F(p)}=\{F(p)\cap O\mid O\in\mathfrak{T}\}$ is a topology on $F(p)$, and so $(F,A,\mathfrak{T})$ is a soft topological space.

Example 2.8.

Let $U=\{a,b,c,d\}$ and $\mathfrak{T}=\{\varnothing,U,\{a,b\},\{a,b,c\},\{c,d\},\{c\}\}$ be a topology on U. Let $A=\{1,3,4\}$ with $E=\{1,2,3,4,5,6\}$, and let $(F,A)=\{1=\{a,c\},3=\{b,d\},4=U\}$. Then we have three topologies as

$$\mathfrak{T}_{F(1)}=\{\varnothing,\{a,c\},\{a\},\{c\}\}$$

$$\mathfrak{T}_{F(3)}=\{\varnothing,\{b,d\},\{b\},\{d\}\}$$

$$\mathfrak{T}_{F(4)}=\mathfrak{T}$$

Hence, we obtain the soft topological space $(F,A,\mathfrak{T})$ over U.

Theorem 2.9.

If $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are soft topological space over $(U,\mathfrak{T})$, then $(F,A,\mathfrak{T})\tilde{\times}(G,B,\mathfrak{T})$ is a soft topological space over $U\times U$.

Proof. From the definition of Cartesian product of soft sets and Lemma 1.15, it is obvious.

Definition 2.10.

The soft topological space obtained in Theorem 2.9 is called product soft topological space over $U\times U$.

Theorem 2.11.

Let $(U,\mathfrak{T})$ be a topological space, $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ be soft topological spaces over U. If (F,A) is similar to (G,B) $((F,A)\cong(G,B))$, then they are same soft topological spaces over U.

Proof. Since $(F,A)\cong(G,B)$, we have the bijection $\phi:A\to B$ such that $F(p)=(G\circ\phi)(p)$ for each $p\in A$ from definition of similarity of soft sets. So, there is one and only one parameter $p'\in B$ corresponding to each parameter $p\in A$. Therefore, we have $\mathfrak{T}_{F(p)}=\mathfrak{T}_{G(\phi(p))}=\mathfrak{T}_{G(p')}$ for each $p\in A$ and $p'\in B$. Hence, we obtain same subspaces for each parameters.

Some Topological Properties of Soft Topological Spaces

In this section, we mention about some basic topological properties such as Hausdorffness, compactness, connectedness of soft topological spaces.

Definition 3.1.

Let $(F,A,\mathfrak{T})$ be a soft topological spaces over $(U,\mathfrak{T})$. It is called that $(F,A,\mathfrak{T})$ is a Hausdorff soft topological space if $(F(p),\mathfrak{T}_{F(p)})$ is Hausdorff for each $p\in A$.

Example 3.2.

Let $U=\{a,b,c,d\}$ be a universal set, $\mathfrak{T}=\{\varnothing,U,\{a,b\},\{c,d\}\}$

be a topology on U and $E=\{1,2,3,4,5\}$ be a parameters set. Let $(F,A,\mathfrak{T})$ be a soft topological space over U such that

$$(F,A)=\{1=\{a,c\},2=\{b,d\}\} \text{ and}$$

$$\mathfrak{T}_{F(1)}=\{\varnothing,\{a,c\},\{a\},\{c\}\}$$

$$\mathfrak{T}_{F(2)}=\{\varnothing,\{b,d\},\{b\},\{d\}\}$$

Since $(F(1),\mathfrak{T}_{F(1)})$ and $(F(2),\mathfrak{T}_{F(2)})$ are Hausdorff, then $(F,A,\mathfrak{T})$ is a Hausdorff soft topological space over U from Definition 3.1.

Note that, the topological space $(U,\mathfrak{T})$ does not need to be a Hausdorff space.

We obtain following theorem for Hausdorff soft topological spaces.

Theorem 3.3.

If $(U,\mathfrak{T})$ is a Hausdorff space and $(F,A,\mathfrak{T})$ is a soft topological space over it. Then $(F,A,\mathfrak{T})$ is Hausdorff.

Proof. It is obvious from Lemma 1.16 and Definition 3.1.

Theorem 3.4.

Let $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ be soft topological spaces over $(U,\mathfrak{T})$. Their product space is Hausdorff if and only if they are Hausdorff.

Proof. Suppose that $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are Hausdorff, then their product $(H,A\times B,\mathfrak{T}\times\mathfrak{T})$ is a soft topological space over $U\times U$ from Theorem 2.9. From Lemma 1.17, we gain $(H(p_1,p_2),\mathfrak{T}_{H(p_1,p_2)})$ is Hausdorff for each $(p_1,p_2)\in A\times B$. Thus, $(H,A\times B,\mathfrak{T}\times\mathfrak{T})$ is Hausdorff.

Conversely, suppose that

$$(F,A,\mathfrak{T})\tilde{\times}(G,B,\mathfrak{T})=(H,A\times B,\mathfrak{T}\times\mathfrak{T})$$

and $(H,A\times B,\mathfrak{T}\times\mathfrak{T})$ is Hausdorff. So, $(H(p_1,p_2),\mathfrak{T}_{H(p_1,p_2)})$ is Hausdorff for each $(p_1,p_2)\in A\times B$ from Definition 3.1 and the topological space $(H(p_1,p_2),\mathfrak{T}_{H(p_1,p_2)})$ is product of $(F(p_1),\mathfrak{T}_{F(p_1)})$ and $(G(p_2),\mathfrak{T}_{G(p_2)})$ for each $p_1\in A$ and $p_2\in B$ from Definition 2.10. We have that $(F(p_1),\mathfrak{T}_{F(p_1)})$ and $(G(p_2),\mathfrak{T}_{G(p_2)})$ are Hausdorff from Lemma 1.17 for each $p_1\in A$ and $p_2\in B$. Hence, $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are Hausdorff.

Theorem 3.5.

If $(U,\mathfrak{T})$ is a Hausdorff space and $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ are soft topological space over U, then $(F,A,\mathfrak{T})\tilde{\times}(G,B,\mathfrak{T})$ is Hausdorff.

Proof. The direct result of Theorem 3.3 and Theorem 3.4.

Definition 3.6.

Let $(F,A,\mathfrak{T})$ be a soft topological space over $(U,\mathfrak{T})$. It is called that $(F,A,\mathfrak{T})$ is a compact soft topological space if $(F(p),\mathfrak{T}_{F(p)})$ is compact subspace of $(U,\mathfrak{T})$ for each $p\in A$.

Example 3.7.

Consider the usual topological space of real numbers $(\mathbb{R},\mathfrak{U})$. Define the soft set (F,A) over $\mathbb{R}$ such that $F:A\to P(\mathbb{R}), F(p)=[p-1,p+1]$ where $p\in A=\{1,2,3,4\}\subseteq\mathbb{N}$ and

$$\mathfrak{T}_{F(p)}=\{[p-1,p+1]\cap O \mid p\in A, O\in\mathfrak{U}\}$$

By definition, $(F,A,\mathfrak{U})$ is a soft topological space. Since each subspace is compact, then $(F,A,\mathfrak{U})$ is a compact soft topological space from Definition 3.6.

Note that, if U is a finite set, it is compact according to each topology defined on it. For this reason;

Example 3.8.

Soft topological space given in Example 3.2 is a compact soft topological space.

Proof. It is obvious.

Similar to the above theorems, we can also give following;

Theorem 3.12.

Let $(U,\mathfrak{T})$ be a Hausdorff space, $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ be compact soft topological spaces over U. Then $(F,A,\mathfrak{T})\textbf{And}(G,B,\mathfrak{T})$ is a compact soft topological space.

Theorem 3.13.

Let $(U,\mathfrak{T})$ be a topological space, $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ be compact soft topological spaces over U. Then $(F,A,\mathfrak{T})\tilde{\times}(G,B,\mathfrak{T})$ is a compact soft topological space over $U\times U$.

Proof. From the definition of cartesian product of soft sets and Lemma 1.21, it is straightforward.

Definition 3.14.

Let $(U,\mathfrak{T})$ be a topological space, and $(F,A,\mathfrak{T})$ be a soft topological space over U. It is called that $(F,A,\mathfrak{T})$ is a connected soft topological space if $\left(F(p),\mathfrak{T}_{F(p)}\right)$ is connected subspace of U for each $p\in A$.

Example 3.15.

Let $U=\{a,b,c,d,e\}$ be a set and

$\mathfrak{T}=\{\varnothing,U,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$

be a topology on U. Let $E=\{1,2,3,4,5,6\}$ and

$(F,A)=\{2=\{b,d,e\},\ 4=\{b,c,e\}\}$

be a soft topological space over U such that

$\mathfrak{T}_{F(2)}=\{\varnothing,\{b,d,e\},\{d\}\}$ and

$\mathfrak{T}_{F(4)}=\{\varnothing,\{b,c,e\},\{c\}\}$

Since $\left(F(p),\mathfrak{T}_{F(p)}\right)$ is a connected subspace for each $p\in A$, $(F,A,\mathfrak{T})$ is a connected soft topological space over U.

Example 3.16.

Consider the usual topological space of the real numbers $(\mathbb{R},\mathfrak{U})$. Define the soft topological space $(F,A,\mathfrak{U})$ such that $F(p)=(p-1,p+1)$ for each $p\in\mathbb{N}$, and $\mathfrak{U}_{F(p)}$ is relative topology of $F(p)$ with respect to usual topology $\mathfrak{U}$. Since each open interval $(p-1,p+1)$ is connected for each $p\in\mathbb{N}$ in $(\mathbb{R},\mathfrak{U})$, then $(F,A,\mathfrak{U})$ is a connected soft topological space.

Theorem 3.17.

Let $(U,\mathfrak{T})$ be a topological space and $(F,A,\mathfrak{T})$ and $(G,B,\mathfrak{T})$ be connected soft topological spaces over U. If $F(p_1)\cap G(p_2)\neq\varnothing$ for each $p_1\in A$ and $p_2\in B$, then $(F,A,\mathfrak{T})\tilde{\cup}(G,B,\mathfrak{T})$ is a connected soft topological space over U.

Proof. Suppose that $(F,A,\mathfrak{T})\tilde{\cup}(G,B,\mathfrak{T})=(H,C,\mathfrak{T})$. So, from definition of union of soft sets we have $H(p)=F(p)$ if $p\in A-B$, $H(p)=G(p)$ if $p\in B-A$ and $H(p)=F(p)\cup G(p)$ if $p\in A\cap B$. The desired result is obvious for the first two cases. Let's examine the third case, i.e. $H(p)=F(p)\cup G(p)$ for $p\in A\cap B$. We know from the hypothesis that subspaces are not disjoint for each parameter and all of them connected, so we obtain that $H(p)=F(p)\cup G(p)$ is connected for each $p\in A\cap B$ from Lemma 1.22. Hence, $(H,C,\mathfrak{T})$ is a connected soft topological space over U.

Theorem 3.18.

Let $(U,\mathfrak{T})$ be a topological space, $(F,A,\mathfrak{T})$ be a connected soft topological space over U. If (F,A) is a full soft set over U and $\bigcap_{p\in A}F(p)\neq\varnothing$, then the topological space $(U,\mathfrak{T})$ is connected.

Proof. It is straightforward from the definition of full soft set and Lemma 1.22.

CONCLUSION

In this article, we have approached the concept of soft topological space with a new perspective. We define soft topological space that is defined on a topological space as a parametrization of some subspaces of the space in Molodtsov's sense [1]. Besides, some basic topological concepts such as Hausdorffness, compactness and connectedness are given for soft topological spaces over given any topological space, and we have given some results for these concepts. In addition to these, we have given the notion of Cartesian intersection and Cartesian union for set-families in general. Using these definitions we have given a result for the intersection of two subspaces of any topological space.

One of the most important topics of topology is undoubtedly the concept of continuity. In future work, someone can study the concept of continuity among soft topological spaces over given any topological space. Of course, it can studied in detail in all other topological concepts.

The author hope that this article sheds light on a way of working scientists in this field.

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