Hittite Journal of Science and Engineering, 2016, 3 (2) 99-104
ISSN NUMBER: 2149-2123
DOI: 10.17350/HJSE19030000038

The Representation, Generalized Binet Formula and Sums of The Generalized Jacobsthal *p*-Sequence

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ABSTRACT

In this study, a new generalization of the usual Jacobsthal sequence is presented, which is called the generalized Jacobsthal p-sequence. The generating matrix, the generalized Binet formula, the generating functions and the combinatorial representations of the generalized Jacobsthal p-sequence are investigated. Moreover, certain sum formula consisting of the terms of the generalized Jacobsthal p-sequence are given.

Keywords:
Jacobsthal sequence; Generating Matrix; Binet Formula; Combinatorial representation.

Article History:
Received: 2016/08/09
Accepted: 2016/12/08
Online: 2016/12/31

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INTRODUCTION

Over the years, several articles have been appeared in many journals relating the integer sequences to growth patterns in plants. Among these integer sequences, Fibonacci sequence has achieved a kind of celebrity status. It is famous for possessing wonderful and amazing properties. For example, it is defined by a recurrence relation, and the ratios of its consecutive terms converge to the golden mean. Since this sequence has very wide applications, ones can find many interesting generalizations, i.e., one of them is given by Stakhov [1]. Under the special assumptions, the Fibonacci p-sequence reduces to the classical Fibonacci sequence. In addition, Stakhov and Rozin have presented number of properties and many applications of the Fibonacci p-sequence [2]. Kilic has studied the combinatorial representations, Binet formula and sums of Fibonacci p-sequence [3].

With the development of computer science and the onset of the digital age, the usual Jacobsthal sequence has extensively been investigated. It is defined by a recurrence relation, as the Fibonacci sequence. Horadam has given the important results for the Jacobsthal sequence [4]. Cerin has studied the sums of the terms of the Jacobsthal sequence [5]. In an investigation of the integer sequence defined by a recurrence relation, matrix theory has played an important and effective role. Quite apart from pursuing the discovery of the additional formulas by the matrix technics, the different matrices for obtaining new results can be introduced. Chen and Louck have investigated an *nxn* companion matrix and shown the combinatorial representation of the sequence generated by the nth power of the matrix [6]. Considering the matrix theory, Koken and Bozkurt have presented the Jacobsthal F-matrix and some results [7]. In the literature, there exist many other references on the subject which are not given here.

The object of this article is to give a new definition for the generalization of the usual Jacobsthal sequence. The generating matrix, the Binet formula, characteristic equations, generating functions, combinatorial representations and sums of the terms of the generalized Jacobsthal sequence are respectively studied.

Generalized Jacobsthal *p*-sequence

Generalization of Jacobsthal Sequence

First of all, the generalization of the usual Jacobsthal sequence is denoted by $J_p(n)$ and defined as follows: for $\forall p \in Z^+$ and $n > p+1$,

$$J_p(n) = J_p(n-1) + 2J_p(n-p-1) \tag{1}$$

with initial conditions

$$J_p(1) = J_p(2) = \cdots = J_p(p+1) = 1 \quad . \tag{2}$$

Obviously, when p=1, the generalized Jacobtshal p-sequence reduces to the usual Jacobtshal sequence. If the generalized Jacobthsal p-sequence is extended to backwards by using Eqs. (1)-(2), the following statements are obtained:

$$\begin{gathered} J_p(0) = J_p(-1) = \cdots = J_p(-p+1) = 0 \\ 2J_p(-p) = 1 \\ J_p(-p-1) = J_p(-p-2) = \cdots = J_p(-2p+1) = 0 \end{gathered} \tag{3}$$

Depending the choice of the value of p, both the recurrence relation and the initial conditions of considered sequence change. Hence, it is difficult and troublesome to compute the terms of the generalized Jacobsthal p-sequence for all the values of p. To facilitate this process, the generating matrix of the generalized Jacobsthal p-sequences is now presented as in the form

$$\mathbf{G_p} = \left[g_{ij} \right]_{(p+1)\times(p+1)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

Additionally, a new matrix is defined as follows:

$$\mathbf{F_p} = \left[\left\{ v_{n+1}^{p} \right\} \quad \left\{ v_{n-p+1}^{2p} \right\} \quad \left\{ v_{n-p}^{2p} \right\} \quad \cdots \quad \left\{ v_{n}^{2p} \right\} \right]_{(p+1)\times(p+1)} \tag{5}$$

where

$$v_n^{tp} = t.\left[J_p(n) \quad J_p(n-1) \quad \cdots \quad J_p(n-p) \right]^T. \tag{6}$$

The matrix $\mathbf{F_n}$ will be called the generalized Jacobsthal p-matrix later. It should be noted that, for p=1, the generalized Jacobsthal p-matrix reduces to the usual form given by Koken and Bozkurt [7].

From Eq. (1), the following matrix can immediately be written

$$\mathbf{F_{n+1}} = \mathbf{G_p F_n} \quad . \tag{7}$$

Then, the following theorem can be given.

Theorem 1 For any $n;p>0$,

$$\mathbf{F_n} = \mathbf{G_p}^{\mathbf{n}}. \tag{8}$$

Proof. To prove the theorem, the induction method on n is used. Taking n=1 and considering Eqs. (1)-(3), $\mathbf{F_1}=\mathbf{G_p}$ is obtained. It is thus to be true for n=1. Now suppose that Eq. (8) holds for any n-1, namely $\mathbf{F_{n-1}} = \mathbf{G_p}^{\mathbf{n-1}}$. From Eqs. (1) and (7) and the assumption, $\mathbf{G_p}^{\mathbf{n}} = \mathbf{G_p G_p}^{\mathbf{n-1}} = \mathbf{G_p F_{n-1}} = \mathbf{F_n}$ is found, which is the desired result.

Theorem 2 Let $\mathbf{F_n}$ be defined as in (5). Then,

$$\det \mathbf{F_n} = 2^n (-1)^{np} \quad . \tag{9}$$

Proof. Taking Theorem 1 into account, computing the determinant of the matrix $\mathbf{G_p}$ by the Laplace expansion with respect to (p+1)th column and considering the matrix identities, the proof can easily be obtained.

The following corollary can be written from the fundamental matrix identities such that $\mathbf{F_{n+m}}=\mathbf{F_n F_m}$ or $\mathbf{F_{n-m}}=\mathbf{F_n F_{-m}}$. It therefore is given without the proof.

Corollary 3 Let $Jp(n)$ be the nth generalized Jacobsthal p-number. Then

$$J_p(n+m) = J_p(n) J_p(m+1) + 2\sum_{i=1}^{p} J_p(n-p-1+i) J_p(m+1-i) \quad . \tag{10}$$

Actually, for p-1, Eq. (10) becomes the well-known following formula given by Koken and Bozkurt [7]:

$$J(n+m) = J(n)J(m+1) + 2J(n-1)J(m).$$

Binet Formula and Generating Functions

In this section, the Binet formula and the generating functions of the generalized Jacobsthal p-sequence will be studied. To do this, the limit of the ratio of the adjacent generalized Jacobsthal p-sequence for the case where $n \to \infty$ is considered. First of all, the following definition is introduced:

$$\lim_{n\to\infty} \frac{J_p(n)}{J_p(n-1)} = x \tag{11}$$

The ratio of the adjacent generalized Jacobsthal p -sequence is rearranged in the form

$$\frac{J_p(n)}{J_p(n-1)} = 1 + \frac{2}{\dfrac{J_p(n-1)J_p(n-2)\cdots J_p(n-p)}{J_p(n-2)J_p(n-3)\cdots J_p(n-p-1)}} \tag{12}$$

Substituting the last equation into Eq. (11), the following algebraic equation for the generalized Jacobsthal p -sequence is obtained:

$$x^{p+1} - x^p - 2 = 0\,. \tag{13}$$

It should be noted that Eq. (13) possesses the $(p+1)$th degree and $(p+1)$ roots such as $x_1, x_2, \ldots, x_{p+1}$ according to the famous "Fundamental Theorem of Algebra". Also, when $p=1$ Eq. (13) reduces to well-known form for the usual Jacobsthal sequence.

The Binet formula for the generalized Jacobsthal p-sequence will be investigated. But the following lemma is first recalled [3].

Lemma 4 Let $a_p = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$ Then $a_p > a_{p+1}$ for any $p>1$.

Then, the following lemma can be written.

Lemma 5 The characteristic equation of the generalized Jacobsthal p -sequence $x^{p+1}-x^p-2=0$ does not have multiple roots for $p>1$

Proof. Let $f(x) = x^{p+1}-x^p-2$. Suppose that α is a multiple root of $f(x) = 0$. Note that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, $f(\alpha) = \alpha^{p+1}-\alpha^p-2=0$ and $f'(\alpha) = \alpha^{p-1}((p+1)\alpha-p)=0$. Then, $\alpha = p/(p+1)$ Consequently,

$$0 = -f(\alpha) = -\alpha^{p+1} + \alpha^p + 2 = \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p + 2 = a_{p+1} + 2$$

Considering Lemma 4, $\alpha_2=1/4<1$, and $\alpha_p > \alpha_{p+1}$ for $p > 1$, $\alpha_{p+1} \neq -2$, which is a contradiction. The equation $f(z) = 0$ does therefore not have multiple roots.

Suppose that $f(\lambda)$ is the characteristic polynomial of the generalized Jacobsthal p-matrix $\mathbf{F_n}$. Considering the identities of the companion matrix, then $f(\lambda)= \lambda^{p+1}-\lambda^p-2$. Also $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ represent the eigenvalues of the matrix $\mathbf{G_p}$. By Lemma 5, it is known that each of $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ are distinct from the other. Let $\wedge$ be a Van*d*ermonde matrix of order $(p + 1) \times (p + 1)$ as follows:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1^{\,p} & \lambda_1^{\,p-1} & \cdots & \lambda_1 & 1 \\ \lambda_2^{\,p} & \lambda_2^{\,p-1} & \cdots & \lambda_2 & 1 \\ \lambda_3^{\,p} & \lambda_3^{\,p-1} & \cdots & \lambda_3 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{p+1}^{\,p} & \lambda_{p+1}^{\,p-1} & \cdots & \lambda_{p+1} & 1 \end{bmatrix} \tag{14}$$

In addition, the following column vector is defined:

$$\mathbf{d_i}^{\mathbf{k}} = \begin{bmatrix} \lambda_1^{\,n+p+1-i} & \lambda_2^{\,n+p+1-i} & \cdots & \lambda_{p+1}^{\,n+p+1-i} \end{bmatrix}^T$$

The transpose of the matrix $\wedge$ is denoted by $\mathbf{V}$, and $\mathbf{V_j^{(i)}}$ represents a $(p+1)\mathrm{x}(p+1)$ matrix constructed by replacing the jth column of $\mathbf{V}$ by $\mathbf{d_i}^{\mathbf{k}}$. Then, the generalized Binet formula for the generalized Jacobtshal p -sequence can be given by the following theorem.

Theorem 6 Let $J_p(n)$ be the nth generalized Jacobsthal p-sequence. Then

$$f_{ij} = \frac{\det\left(\mathbf{V_j^{(i)}}\right)}{\det(\mathbf{V})}\,,$$

where $\mathbf{F_n} = [f_{ij}]$.

Proof. To prove the theorem, a well-known method is applied. Since the eigenvalue of the matrix $\mathbf{G_p}$ are distinct, this matrix is diagonalizable. It is easy to show that

$$\mathbf{G_p V = VD},$$

where $\mathbf{D} = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_{p+1})$. Considering the fact that Vandermonde matrix $\mathbf{V}$ is invertible, $\mathbf{V^{-1}G_pV = D}$. Hence, the matrix $\mathbf{G_p}$ is similar to the diagonal matrix $\mathbf{D}$. So, $\mathbf{F_nV=VD^n}$. Since $\mathbf{F_p} = [f_{ij}]$, the following linear system of equations:

$$\begin{aligned} f_{i1}\lambda_1^{\,p} + f_{i2}\lambda_1^{\,p-1} + \cdots + f_{i,p+1} &= \lambda_1^{\,p+n+1-i} \\ f_{i1}\lambda_2^{\,p} + f_{i2}\lambda_2^{\,p-1} + \cdots + f_{i,p+1} &= \lambda_2^{\,p+n+1-i} \\ &\vdots \\ f_{i1}\lambda_{p+1}^{\,p} + f_{i2}\lambda_{p+1}^{\,p-1} + \cdots + f_{i,p+1} &= \lambda_{p+1}^{\,p+n+1-i} \end{aligned}$$

By the Cramer's rule, the desired result is obtained.

Consequently, the following corollary can be directly obtained from Theorem 6.

Corollary 7 For the nth term of the generalized Jacobsthal p-sequence,

$$J_p(n) = \frac{\det\left(\mathbf{V_1}^{(2)}\right)}{\det(\mathbf{V})} = \frac{1}{2}\frac{\det\left(\mathbf{V_{p+1}}^{(1)}\right)}{\det(\mathbf{V})}.$$

Now the generating functions of the generalized Jacobsthal p-sequence is presented by the following theorem.

Theorem 8 Let $J_p(n)$ be the nth term of the generalized Jacobsthal p-sequence. Then for $n > 1$,

$$x^n = J_p(n-p+1)x^p + 2\sum_{j=1}^{p} J_p(n-p+1-j)x^{j-1}.$$

Proof. (Induction method on n) It is clear that the equation holds for $n = p+1$. Suppose that the equation holds for any $n > p+1$. Hence, by the assumption and the definition of the generalized Jacobsthal p-sequence,

$$\begin{aligned} x^{n+1} = x^n x &= J_p(n-p+1)x^{p+1} + 2\sum_{j=1}^{p} J_p(n-p+1-j)x^j \\ &= \left(J_p(n-p+1) + 2J_p(n-2p+1)\right)x^p + 2J_p(n-2p+2)x^{p-1} + \cdots + 2J_p(n-p)x + 2J_p(n-p+1) \\ &= J_p(n-p+2)x^p + 2\sum_{j=1}^{p} J_p(n-p+2-j)x^{j-1} \end{aligned}$$

is obtained. So, the proof is completed.

Combinatorial Representations

Now the combinatorial representations of the generalized Jacobsthal p-sequence are investigated. First of all, introduce the following companion matrix:

$$C(c_1, c_2, \ldots, c_k) = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{k\times k} \tag{15}$$

Also, recall that the following theorem which give the opportunity to derive the elements in the nth power of the matrix C [6].

Theorem 9 Let the matrix $C = (c_{ij})_{k\times k}$ be as in (15). The element $c_{ij}^{(n)}$ in the matrix C^n is given by the formula

$$c_{ij}^{(n)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1,\ldots,t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + t_2 + \cdots + t_k}{t_1, t_2, \cdots, t_k} c_1^{t_1} \ldots c_k^{t_k}, \tag{16}$$

where the summation is over non-negative integers satisfying $t_1 + 2t_2 + \ldots + kt_k = n - i + j$, and the coefficients are defined as 1 for $n = i-j$.

Thus the following lemma can immediately be obtained from the above theorem without the proof.

Lemma 10 Let the matrix $\mathbf{G_p}^{\mathbf{n}} = [g_{ij}^{(n)}]$ be as in (6). Then,

$$g_{ij}^{(n)} = \sum_{(m_1,\ldots,m_{p+1})} \frac{m_j + m_{j+1} + \cdots + m_{p+1}}{m_1 + m_2 + \cdots + m_{p+1}} \times \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \cdots, m_{p+1}} 2^{m_{p+1}},$$

where the summation is over non-negative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n - i + j$.

Finally, the following corollaries can directly be written from Lemma 10.

Corollary 11 Let $J_p(n)$ be the nth term of the generalized Jacobsthal p-sequence. Then

i. $$J_p(n) = \frac{1}{2} \sum_{(m_1,\ldots,m_{p+1})} \frac{m_{p+1}}{m_1 + m_2 + \cdots + m_{p+1}} \times \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \cdots, m_{p+1}} 2^{m_{p+1}}$$

where the summation is over non-negative integers satisfying $m_1 + 2m_2 + \ldots + (p+1)m_{p+1} = n + p$.

ii. $$J_p(n) = \sum_{(m_1,\ldots,m_{p+1})} \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \cdots, m_{p+1}} 2^{m_{p+1}}$$

where the summation is over non-negative integers satisfying $m_1 + 2m_2 + \ldots + (p+1)m_{p+1} = n - 1$.

Sum Formula

To find the sum of terms of the generalized Jacobsthal p-sequence, certain methods are now used. To do this, some generating matrices by extending the matrix $\mathbf{G_p}$ will be used. Let S_n be the sums of the generalized Jacobsthal p-sequence as follows:

$$S_n = \sum_{i=1}^{n} J_p(i) \tag{17}$$

Also, the following matrices are defined:

$$\mathbf{T} = \left[\begin{array}{c|ccc} 1 & 0 & 0 \cdots & 0 \quad 0 \\ \hline 1 & & & \\ 0 & & & \\ \vdots & & \mathbf{G_p} & \\ 0 & & & \\ 0 & & & \end{array}\right] \tag{18}$$

$$\mathbf{A_n} = \left[\begin{array}{c|ccc} 1 & 0 & 0 \cdots & 0 \quad 0 \\ \hline S_n & & & \\ S_{n-1} & & & \\ \vdots & & \mathbf{F_n} & \\ S_{n-p+1} & & & \\ S_{n-p} & & & \end{array}\right] \tag{19}$$

Thus, the following theorem can be given.

Theorem 12 For the matrices $\mathbf{T}$ and $\mathbf{A_n}$,

$$\mathbf{A_n} = \mathbf{T}^n \tag{20}$$

Proof. (Induction method on n) When $n = 1$, it is clear that the equation holds. Suppose that Eq. (20) holds for n. On the other hand, by the assumption and $S_{n+}1 = J_p(n+1) + S_n$,

$$\mathbf{T}^{n+1} = \mathbf{T}^n\mathbf{T} = \mathbf{A_n}\mathbf{T} = \mathbf{A_{n+1}},$$

which completes the proof.

Before the main result, the following useful lemma is presented.

Lemma 13 Let $J_p(n)$ be the nth term of the generalized Jacobsthal p-sequence. Then, for all the integers $n,m \geq 0$,

$$J_p(n+m+p+1) = J_p(n+m+1) + 2\sum_{i=1}^{p} J_p(n+m-p+i)$$

Proof. The proof can easily be obtained by the definition of the generalized Jacobsthal p-sequence.

A new matrix is defined in the form

$$\mathbf{W}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & \lambda_1{}^p & \lambda_2{}^p & \cdots & \lambda_p{}^p & \lambda_{p+1}{}^p \\ -\frac{1}{2} & \lambda_1{}^{p-1} & \lambda_2{}^{p-1} & \cdots & \lambda_p{}^{p-1} & \lambda_{p+1}{}^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{2} & \lambda_1 & \lambda_2 & \cdots & \lambda_p & \lambda_{p+1} \\ -\frac{1}{2} & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}, \quad (21)$$

where λ_i ($i = 1, 2, \ldots, p + 1$) have been defined before.

Then the following theorem is given to compute the sums of the generalized Jacobsthal p-sequence by using matrix method.

Theorem 14 Let S_n be as in (17). Then

$$S_n = \frac{1}{2}\left(J_p\left(n+p+1\right)-1\right)$$

Proof. Computing det**W** by the Laplace expansion of the determinant with respect to the first row, det**W**=det**V** is obtained, where **V** is defined as before. Hence the characteristic equation of the matrix **W** is $(x-1)x(x^p-x^{p-1}-1)$. It can be said from Lemma 5 that the eigenvalues of the matrix **W** are $1, \lambda_1, \ldots, \lambda_{p+1}$ and different from each other. Therefore, $\mathbf{TW}=\mathbf{W}\bar{\mathbf{D}}$ can be written, where $\bar{\mathbf{D}} = diag(1, \lambda_1, \ldots, \lambda_{p+1})$. Consequently, $\mathbf{A_n W}=\mathbf{W}\bar{\mathbf{D}}^{\mathbf{n}}$. The element (2,1)th in the matrix $\mathbf{A_n} = [a_{ij}]_{(p-2)\mathrm{x}(p+2)}$ is $a_{21}=S_n$, and by Lemma 13, the desired result is directly obtained.

CONCLUSION

In this study, the new generalization of the usual Jacobsthal sequence is presented, which is called as "the generalized Jacobsthal p-sequence". The generating matrix of this generalized sequence is given, and a few important results are obtained by employing the matrix. Also the generating matrix is extended to certain matrix representations, and it is shown that the sums of the generalized Jacobsthal p-sequence could be derived directly by using the representations. Moreover the generalized Binet formula, the generating functions and the combinatorial representations of the generalized Jacobsthal p-sequence are presented.

ACKNOWLEDGEMENTS

This study was supported by Kastamonu University Scientific Research and Development Office under the project number KÜ-BAP01/2016-4. The author also would like to thank the referees for their very helpful and detailed comments.

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