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# On Critical Buckling Loads of Euler Columns With Elastic End Restraints

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## ABSTRACT

In recent years, a great number of analytical approximate solution techniques have been introduced to find a solution to the nonlinear problems that arised in applied sciences. One of these methods is the homotopy analysis method (HAM). HAM has been successfully applied to various kinds of nonlinear differential equations. In this paper, HAM is applied to find buckling loads of Euler columns with elastic end restraints. The critical buckling loads obtained by using HAM are compared with the exact analytic solutions in the literature. Perfect match of the results veries that HAM can be used as an efficient, powerfull and accurate tool for buckling analysis of Euler columns with elastic end restraints.

**Key Words:**
Homotopy analysis method (HAM); Euler column; Buckling load; Elastic restraint.

## INTRODUCTION

Many phenomena in science and engineering involve nonlinear problems. However, the majority of these nonlinear problems have no exact analytical solutions since it is generally dicult to solve nonlinear equations analytically. In recent years, these nonlinear equations have been solved by analytical approximate solution techniques, such as perturbation and nonperturbation techniques. Perturbation techniques usually depend on small/large physical parameters. Although non-perturbation techniques do not depend on small/large physical parameters, these methods cannot ensure the convergence of the solution series. In fact, neither perturbation methods nor nonperturbation techniques can adjust or control the convergence region and the rate of the approximation series. On the other hand, the homotopy analysis method (HAM) which is proposed by Liao [1, 2] is an analytic approach to obtain series solution of various types of linear and nonlinear differential equations, such as ordinary differential equations, partial differential equations, integro-differential equations, difference equations, differential-difference equations, integro-differential difference equations [1, 2, 3, 4, 5, 6] and it provides a convenient way to adjust and control the convergence region and the rate of the approximation series by an auxiliary parameter $\hbar$ and auxiliary function H(x) [2]. Since HAM is independent of small/large physical parameters, it can be applied to many nonlinear problems whether there exist small/large physical parameters or not. One of the most important advantages of HAM is the freedom to choose the so-called auxiliary operator L, the auxiliary parameter $\hbar$ and auxiliary function H(x), the initial approximations and the set of base functions [1, 2]. One of the important topics of the elds of structural, mechanical and aeronautical engineering has been the stability analysis of the commonly used basic structural elements; namely columns. There are various studies on elastic stability of columns. However, it is not easy to determine exact analytical solutions for many kinds of buckling problems. Many researchers successfully applied analytical approximate solution techniques to stability analysis of uniform and nonuniform Euler columns and beams with various end conditions. Atay and Coskun investigated the elastic stability of homogenous and non-homogenous Euler columns by using variational iteration method and homotopy perturbation method [7, 8, 9, 10, 11, 12, 13]. Pinarbasi

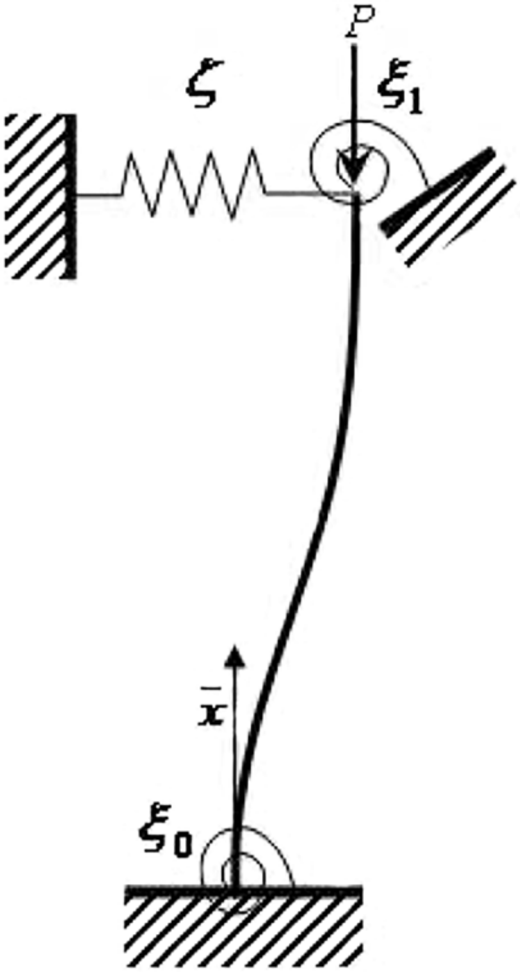


**Figure 1.** Column with elastic end restraints.

et al. studied on elastic buckling behavior of nonuniform rectangular beams [14, 15], nonuniform columns with elastic end restraints [16] and stepped columns [17, 18] using variational iteration method and homotopy perturbation method. By transforming the governing equation with varying coefficients to linear algebric equations and by using various end boundary conditions, critical buckling loads of beams with arbitrarily axial inhomogenity is solved by Huang and Luo [19]. Recently, Yuan and Wang [20] applied a new differential quadrature based iterative numerical integration method to solve post-buckling differential equations of extensible beam-colums with six dierent cases. By using HAM, Eryilmaz et al. investigated the buckling loads of Euler columns with continuous elastic restraint [21] and Basbuk investigated buckling loads of elastic columns with constant cross-section [22]. These studies showed that HAM has been eciently used in such stability problems. In this study HAM is used to nd the buckling loads of Euler columns with elastic end restraints.

## BUCKLING OF ELASTIC COLUMNS

The general case of a uniform column with elastic restraints is shown in Fig. 1. The moment-displacement relation according to the EulerBernoulli beam theory is given by

$$M = \frac{d^2\overline{w}}{d\overline{x}^2} \qquad (2.1)$$

where x is the longitudinal coordinate measured from the column base, w is the transverse displacement and M is the bending moment [23]. Bazant and Cedolin stated the equilibrium equations as follows [24]:

$$\frac{dM}{d\overline{x}} = Q \qquad (2.2)$$

$$\frac{dQ}{d\overline{x}} = P\frac{d^2\overline{w}}{d\overline{x}^2} \qquad (2.3)$$

where Q is the shear force normal to the deflected column axis. By substituting equations (1) and (2) into equation (3) yields the following governing Euler column buckling equation:

$$\frac{d^4\overline{w}}{d\overline{x}^4} + \alpha\frac{d^2\overline{w}}{d\overline{x}^2} = 0, \qquad \alpha = \frac{PL^2}{El} \qquad (2.4)$$

where w = w=L and x = x=L. The boundary conditions of a column with elastic end restraints has the form,

$$w(0) = 0, \qquad (2.5)$$

$$\left[\xi_0\frac{dw}{dx} - \frac{d^2w}{dx^2}\right]\Big|_{x=0} = 0, \qquad (2.6)$$

$$\left[\xi_1\frac{dw}{dx} + \frac{d^2w}{dx^2}\right]\Big|_{x=1} = 0, \qquad (2.6)$$

$$\zeta w(1) + \left[\alpha\frac{dw}{dx} + \frac{d^3w}{dx^3}\right]\Big|_{x=1} = 0, \qquad (2.7)$$

where, $\xi_0$ and $\xi_1$ represent the rotational spring constants, denotes the translational spring constant against sidesway [25].

## BASIC IDEA OF HOMOTOPY ANALYSIS METHOD (HAM)

Liao introduced the Homotopy Analysis Method (HAM) in [1, 2]. To demonstrate the method, let us consider the following differential equation

$$\mathcal{N}[w(x)] = 0 \tag{3.1}$$

where N is a nonlinear operator, x denotes the independent variable and w(x) is an unknown function. Liao [2] constructs the so-called zero-order deformation equation as follows:

$$(1-q)\mathcal{L}\left[\phi(x;q) - w_0(x)\right] = q\hbar H(x)\mathcal{N}[\phi(x;q)] \tag{3.2}$$

where q ∈ [0; 1] is the embedding parameter, $\hbar$ is an embedding parameter, H(x) is nonzero auxiliary function, $w_0(x)$ is the initial guess to w(x), L is an auxiliary linear operator and (x; q) is an unknown function. As q increases from 0 to 1, (x; q) varies from the initial guess $w_0(x)$ to the exact solution w(x). By expanding (x; q) in a Taylors series with respect to q, one has

$$\phi(x;q) = w_0(x) + \sum_{m=1}^{\infty} w_m(x)q^m, \tag{3.3}$$

$$w_m(x) = \frac{1}{m!}\frac{\partial^m \mathcal{N}[\phi(x;q)]}{\partial q^m}\bigg|_{q=0}. \tag{3.4}$$

If the initial guess $w_0(x)$, auxiliary linear operator L, embedding parameter $\hbar$ and auxiliary function H(x) are properly chosen, the series in equation (11) converges at q = 1, then we have

$$w(x) = w_0(x) + \sum_{m=1}^{\infty} w_m(x) \tag{3.5}$$

Now, lets define the vector

$$\overrightarrow{w}(x) = w_1(x), w_2(x), \ldots, w_m(x) \tag{3.6}$$

Differentiating equation (10) m-times with respect to q and then setting q = 0 and finally dividing by m!, Liao has the so-called mth order deformation equation

$$\mathcal{L}\left[\phi(x;q) - \chi_m w_{m-1}(x)\right] = q\hbar\mathcal{H}(x)\mathcal{R}[\overrightarrow{w}_{m-1}(x)] \tag{3.7}$$

where

$$\mathcal{R}[\overrightarrow{w}_m(x)] = \frac{1}{m!}\frac{\partial^m \mathcal{N}[\phi(x;q)]}{\partial q^m}\bigg|_{q=0}, \tag{3.8}$$

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & else. \end{cases} \tag{3.9}$$

## NUMERICAL RESULTS AND HAM FORMULATION OF THE PROBLEM

Due to the boundary conditions (5)-(8), the rule of solution

$w_0(x) = ax^3 + bx^2 + cx + d,$ t is straightforward to

choose

(4.1)

$\mathcal{L}[\phi(x;q)] = \dfrac{\partial^4 \mathcal{N}[\phi(x;q)]}{\partial q^4}$ of w(x) and the auxiliary

(4.2)

$$\mathcal{L}[c_0 + c_1 x + c_2 x^2 + c_3 x^3] = 0.$$

with the property

(4.3)

To solve equation (4) by means of homotopy analysis

$\mathcal{N}[\phi(x;q)] = \dfrac{\partial^4 \mathcal{N}[\phi(x;q)]}{\partial x^4} + \alpha \dfrac{\partial^2 \mathcal{N}[\phi(x;q)]}{\partial x^2}$ N[(x; q)] as

follows

$$\mathcal{H}(x) = 1 \tag{4.4}$$

Let $\hbar$ denote a nonzero embedding parameter and

(4.5)

$$(1-q)\mathcal{L}\left[\phi(x;q) - w_0(x)\right] = q\hbar\mathcal{H}(x)\mathcal{N}[\phi(x;q)]$$

an auxiliary function. Then, we construct the zero-order deformation equation

$$\mathcal{L}\left[\phi(x;q) - \chi_m w_{m-1}(x)\right] = q\hbar\mathcal{H}(x)\mathcal{N}[\phi(x;q)]$$

(4.6)

The high order deformation equation is as follows

(4.7)

$$w_m(x) = \chi_m w_{m-1}(x) + \hbar \int_0^x \int_0^\tau \int_0^\zeta \int_0^\psi \left[w_{m-1}^{iv}(\xi) + \alpha w_{m-1}''(\xi)\right] d\xi d\psi d\zeta d\tau \tag{4.8}$$

By substituting equations (19), (21) and (22) into equation (24) the high order deformation equation takes the form where $^{iv}$ and '' denote the fourth and second derivatives with respect to ξ respectively. Starting with initial approximation $w_0(x)$, we successively obtained wi(x),

$$w(x) = w_0(x) + \sum_{m=1}^{\infty} w_m(x)$$ high-order deformation is of the form

(4.9)

Since the governing equation (4) is a fourth order differential equation we choose the initial approximation as $w_0(x) = ax^3 + bx^2 + cx + d$; i.e., a polynomial of third degree with four unknown coefficients a; b; c; d. Then we

$$W_{10}(x, \hbar) = \sum_{m=0}^{10} w_m(x) = w_0(x) + w_1(x) + \cdots + w_{10}(x)$$

obtained $w_i(x)$, i = 1; 2; 3; ... as follows

After ten iterations, $W_{10}$ is obtained as follows

(4.10)

By substituting equation (27) into the boundary conditions, we obtained four homogeneous equations. By representing these equations in the matrix form by

$$[C(\alpha, \xi_0, \xi_1, \zeta, \hbar)]\,[a \quad b \quad c \quad d]^T = [0 \quad 0 \quad 0 \quad 0]^T$$

coefficient matrix [C(α, $\xi_0$, $\xi_1$, ς, ħ)], we obtained the

$$\begin{aligned}
w_1(x) &= \frac{1}{12}bx^4\alpha\hbar + \frac{1}{20}ax^5\alpha\hbar \\
w_2(x) &= \frac{1}{12}bx^4\alpha\hbar + \frac{1}{20}ax^5\alpha\hbar + \frac{1}{12}bx^4\alpha\hbar^2 + \frac{1}{20}ax^5\alpha\hbar^2 + \frac{1}{360}bx^6\alpha^2\hbar^2 \\
&\quad + \frac{1}{840}ax^7\alpha^2\hbar^2 \\
w_3(x) &= \frac{1}{12}bx^4\alpha\hbar + \frac{1}{20}ax^5\alpha\hbar + \frac{1}{6}bx^4\alpha\hbar^2 + \frac{1}{10}ax^5\alpha\hbar^2 + \frac{1}{180}bx^6\alpha^2\hbar^2 \\
&\quad + \frac{1}{420}ax^7\alpha^2\hbar^2 + \frac{1}{12}bx^4\alpha\hbar^3 + \frac{1}{20}ax^5\alpha\hbar^3 + \frac{1}{180}bx^6\alpha^2\hbar^3 \\
&\quad + \frac{1}{420}ax^7\alpha^2\hbar^3 + \frac{bx^8\alpha^3\hbar^3}{20160} + \frac{ax^9\alpha^3\hbar^3}{60480} \\
w_4(x) &= \frac{1}{12}bx^4\alpha\hbar + \frac{1}{20}ax^5\alpha\hbar + \frac{1}{4}bx^4\alpha\hbar^2 + \frac{3}{20}ax^5\alpha\hbar^2 + \frac{1}{120}bx^6\alpha^2\hbar^2 \\
&\quad + \frac{1}{280}ax^7\alpha^2\hbar^2 + \frac{1}{4}bx^4\alpha\hbar^3 + \frac{3}{20}ax^5\alpha\hbar^3 + \frac{1}{60}bx^6\alpha^2\hbar^3 + \frac{1}{140}ax^7\alpha^2\hbar^3 \\
&\quad + \frac{bx^8\alpha^3\hbar^3}{6720} + \frac{ax^9\alpha^3\hbar^3}{20160} + \frac{1}{12}bx^4\alpha\hbar^4 + \frac{1}{20}ax^5\alpha\hbar^4 + \frac{1}{120}bx^6\alpha^2\hbar^4 \\
&\quad + \frac{1}{280}ax^7\alpha^2\hbar^4 + \frac{bx^8\alpha^3\hbar^4}{6720} + \frac{ax^9\alpha^3\hbar^4}{20160} + \frac{bx^{10}\alpha^4\hbar^4}{1814400} + \frac{ax^{11}\alpha^4\hbar^4}{6652800} \\
&\;\vdots
\end{aligned}$$

(4.13)

following equation:

(4.11)

where a; b; c and d are the unknown constants of initial approximation $w_0(x)$. For nontrivial solution, the

$$Det[C(\alpha, \xi_0, \xi_1, \zeta, \hbar)] = 0$$ matrix [C(α, $\xi_0$, $\xi_1$, ς, ħ)]

must vanish. Then, the problem takes the following form

(4.12)

The smallest positive real root of the equation (29) is the critical buckling load. The equation (29) depends on the stability parameter the rotational spring constants 120 and 121, the translational spring constant and the

$$W(\alpha, \xi_0, \xi_1, \zeta, \hbar) = Det[C(\alpha, \xi_0, \xi_1, \zeta, \hbar)]$$ define the function W(α, $\xi_0$, $\xi_1$, ς, ħ) as follows:

Then, we pilot the ħ-curves of the W(α, $\xi_0$, $\xi_1$, ς, ħ) and

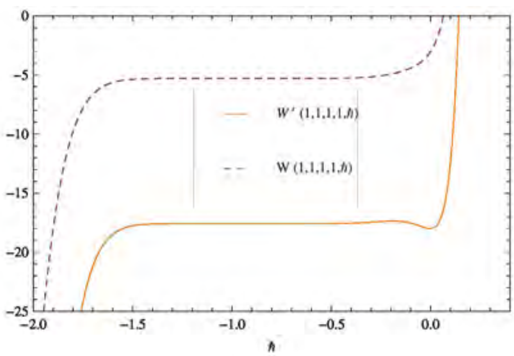


**Figure 2.** The ħ curves of ħ-curves of W(α, $\xi_0$, $\xi_1$, ς, ħ) and W' (α, $\xi_0$, $\xi_1$, ς, ħ).

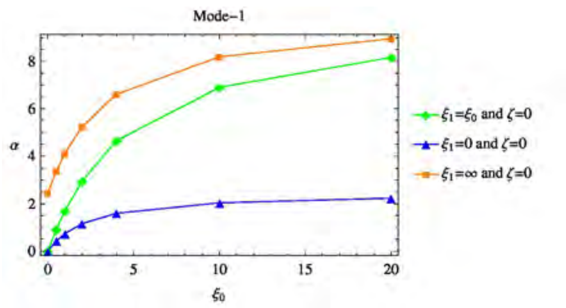


**Figure 3.** Critical buckling loads for columns with the top end free to slide laterally ($\varsigma$ = 0) for various values of $\xi_0$ and $\xi_1$.

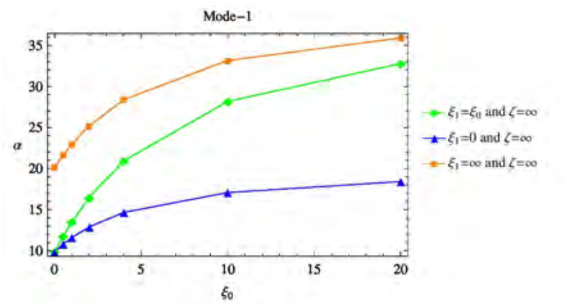


**Figure 4.** Critical buckling loads for columns with the top end free to slide laterally ($\varsigma$ = ∞) for various values of $\xi_0$ and $\xi_1$.

$W_0(\alpha, \xi_0, \xi_1, \varsigma, \hbar)$ in order to nd the convergence region of ~. The ~-curves of $W(\alpha, \xi_0, \xi_1, \varsigma, \hbar)$ and $W0(\alpha, \xi_0, \xi_1, \varsigma, \hbar)$ are obtained in Figure 2. The valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis is about -1.4 < $\hbar$ < -0.6.

Finally the critical buckling loads are obtained from the equation (29) for $\hbar$ = -0:99. Figure 3 and Figure 4 show the nondimension-alized values of the rst mode buckling loads for columns with the top end free to slide laterally ($\varsigma$= 0) and for the columns with only end rotational restraints ($\varsigma$ = 1) for various values of $\xi_0$ and $\xi_1$, respectively. Table 1. compares HAM solutions with the exact solutions given by Wang et al. [36].

## CONCLUSION

In this work, a reliable algorithm based on the homotopy analysis method (HAM) is used to determine the buckling load of Euler columns with elastic end restraints. Several cases are studied to illustrate the validity and accuracy of this procedure. The series solution of equation (4) by HAM contains the auxiliary parameter $\hbar$. By means of the so-called $\hbar$-curve, it is straightforward to choose a proper value of $\hbar$ which ensures that the series solution is convergent. Figure 2 shows the $\hbar$-curves obtained from the 10th-order HAM approximation solutions. Figure 3 and Figure 4 show the critical buckling loads for various values of $\xi_0$; $\xi_1$ and $\varsigma$ obtained by HAM. The approximate solutions obtained by HAM and the exact solutions given in [25] are compared. The exact match of the results verify that HAM is an ecient, powerful and accurate tool for buckling loads of columns with elastic end restraints.

**Table 1.** Comparison of critical buckling loads for various values of $\xi_0$, $\xi_1$, $\varsigma$ computed from exact [25] and HAM solutions

| | | | | | $\xi_0$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\xi_1$ | 0 | 0.5 | 1 | 2 | 4 | 10 | 20 | ∞ | |
| | | | | Columns with only end rotational restraints ($\varsigma$ = ∞) | | | | | |
| $\xi_0$ | Exact | $\pi^2$ | 11.772 | 13.492 | 16.463 | 20.957 | 28.168 | 30.355 | $4\pi^2$ |
| | HAM | 9.8686 | 11.772 | 13.492 | 16.463 | 20.957 | 28.168 | 32.782 | 30.478 |
| 0 | Exact | $\pi^2$ | 10.798 | 11.598 | 12.894 | 14.660 | 17.076 | 18.417 | 20.191 |
| | HAM | 9.8688 | 10.798 | 11.598 | 12.894 | 14.660 | 17.076 | 18.417 | 20.191 |
| ∞ | Exact | 20.191 | 21.659 | 22.969 | 25.182 | 28.397 | 33.153 | 35.902 | $4\pi^2$ |
| | HAM | 20.191 | 21.659 | 22.969 | 25.182 | 28.397 | 33.153 | 35.902 | 39.478 |
| | | | | Columns with the top end free to slide laterally ($\varsigma$ = 0) | | | | | |
| $\xi_0$ | Exact | 0 | 0.9220 | 1.7071 | 2.9607 | 4.6386 | 6.9047 | 8.1667 | $\pi^2$ |
| | HAM | 0 | 0.9220 | 1.7071 | 2.9607 | 4.6386 | 6.9047 | 8.1667 | 9.8686 |
| 0 | Exact | 0 | 0.4268 | 0.7402 | 1.1597 | 1.5992 | 2.0517 | 2.2384 | $\pi^2/4$ |
| | HAM | 0 | 0.4268 | 0.7402 | 1.1597 | 1.5992 | 2.0517 | 2.2384 | 2.4674 |
| ∞ | Exact | $\pi^2/4$ | 3.3731 | 4.1159 | 5.2392 | 6.6071 | 8.1955 | 8.9583 | $\pi^2$ |
| | HAM | 2.4674 | 3.3731 | 4.1159 | 5.2392 | 6.6071 | 8.1955 | 8.9583 | 9.8686 |

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