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Numerical Solution of a Quadratic Integral Equation through Classical Schauder Fixed Point Theorem

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Abstract
In this paper, we investigate the existence of at least one solution on the closed interval for quadratic integral equations with non-linear modification of the argument in Hölder spaces using the technique in the classical Schauder fixed point theorem.

Keywords: Fredholm integral equation, Hölder condition, Schauder fixed point theorem.
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1. Introduction

Integral equations arise naturally in various applications in describing numerous real universe problems. As well, quadratic integral equations have numerous useful applications in describing uncountable events and problems of the real world. For instance, quadratic integral equations are often applicable in the traffic theory, in the theory of radiative transfer, in the theory of neutron transport and kinetic theory of gases. Several authors have comprehensively studied the integral equations and the solution of them in this references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Moreover, M. Benchohra and M. A. Darwish et al. [1] study the existence of the unique solution, defined on a semi-infinite interval $J:[0,\infty)$ for the following quadratic integral equations with a linear modification of the argument

$$x(t)=f(t)+(Ax)(t)\int_0^T u(t,s,x(s),x(\alpha s))ds,\ \ t\in J.$$

where $f:J\to\mathbb{R}$, $u:J\times J_T\times\mathbb{R}^2\to\mathbb{R}$ are given functions, $0<\alpha<1$, $J_T=[0,T]$ and $A:C(J;\mathbb{R})\to C(J;\mathbb{R})$ is an appropriate operator. Here $C(J;\mathbb{R})$ denotes the space of continuous functions $x:J\to\mathbb{R}$.

This article concerns the entity of solutions of the following a quadratic integral equation of Fredholm type,

$$x(t)=(T_1x)(t)+(T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau,\ \ t\in I=[0,1]. \tag{1.1}$$

where k is given function, T_1,T_2,T_3 are given operators satisfying conditions specified later and x is unknown function.

2. Preliminaries

Let $[a,b]$ be a closed interval in $\mathbb{R}$, by $C[a,b]$ we indicate the space of continuous functions defined on $[a,b]$ equipped with the supremum norm, i.e.,

$$\|x\|_\infty=\sup\left\{|x(t)|:t\in[a,b]\right\}$$

for $x \in C[a,b]$. For a fixed α with $0 < \alpha \leq 1$, by $H_\alpha[a,b]$ we will indicate the spaces of the real functions x defined on $[a,b]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that

$$|x(t) - x(s)| \leq H_x^\alpha |t-s|^\alpha \tag{2.1}$$

for all $t,s \in [a,b]$. It is well proved that $H_\alpha[a,b]$ is a linear subspaces of $C[a,b]$. Also, for $x \in H^\alpha[a,b]$, by H_x^α we will indicate the least possible stable for which inequality (2.1) is satisfied. Rather, we put

$$H_x^\alpha = \sup\left\{ \frac{|x(t)-x(s)|}{|t-s|^\alpha} : t,s \in [a,b] \text{ and } t \neq s \right\}. \tag{2.2}$$

The space $H_\alpha[a,b]$ with $0 < \alpha \leq 1$ may be equipped with the norm

$$\|x\|_\alpha = |x(a)| + H_x^\alpha$$

for $x \in H_\alpha[a,b]$. Here, H_x^α is defined by (2.2). In [2], the authors demonstrated that $(H_\alpha[a,b], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space.

Lemma 2.1. *For $0 < \alpha \leq 1$ and $x \in H_\alpha[a,b]$, we have:*

$$\|x\|_\infty \leq \max\left(1, (b-a)^\alpha\right) \|x\|_\alpha.$$

In particular, the inequality $\|x\|_\infty \leq \|x\|_\alpha$ is satisfied for $a = 0$ and $b = 1$, [2].

Lemma 2.2. *For $0 < \alpha < \beta \leq 1$, we have*

$$H_\beta[a,b] \subset H_\alpha[a,b] \subset C[a,b].$$

Furthermore, for $x \in H_\beta[a,b]$, we have:

$$\|x\|_\alpha \leq \max\left(1, (b-a)^{\beta-\alpha}\right) \|x\|_\beta.$$

Particularly, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\beta$ is satisfied for $a = 0$ and $b = 1$, [2].

Lemma 2.3. *Let's assume that $0 < \alpha < \beta \leq 1$ and E is a bounded subset in $H_\beta[a,b]$, then E is a relatively compact subset in $H_\alpha[a,b]$, [3].*

Lemma 2.4. *Assume that $0 < \alpha < \beta \leq 1$ and by B_r^β we indicate the ball centered at θ and radius r in the space $H_\beta[a,b]$, i.e., $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$. Then B_r^β is a closed subset of $H_\alpha[a,b]$, [3].*

Corollary 2.5. *Assume that $0 < \alpha < \beta \leq 1$ and $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$, then B_r^β is a compact subset in the space $H_\alpha[a,b]$, [3].*

Theorem 2.6 (Schauder's fixed point theorem)**.** *Let E be a nonempty, compact and convex subset of a Banach space $(X, \|\cdot\|)$, convex and let $T : E \to E$ be a continuity mapping. Then T has at least one fixed point in E, [4].*

3. Main Result

Theorem 3.1. *Assume that the following conditions $(i)-(iv)$ are satisfied:*

(i) *The operators $T_1, T_2 : H_\beta[0,1] \to H_\beta[0,1]$ are continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$. Also, T_1 and T_2 hold the inequalities*

$$\|T_1 x\|_\beta \leq f_1(\|x\|_\beta) \text{ and } \|T_2 x\|_\beta \leq f_2(\|x\|_\beta)$$

for any $x \in H_\beta[0,1]$, where α and β are the fixed constants satisfying $0 < \alpha < \beta \leq 1$ and the functions $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing on $\mathbb{R}_+$.

(ii) *$k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function such that there exists a constant $k_\beta > 0$ satisfying*

$$|k(t,\tau) - k(s,\tau)| \leq k_\beta |t-s|^\beta,$$

for any $t,s,\tau \in [0,1]$.

(iii) *The operators $T_3 : H_\beta[0,1] \to C[0,1]$ is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$. Also, T_3 holds the inequality*

$$\|T_3x\|_\infty \leq f_3(\|x\|_\beta)$$

for any $x \in H_\beta[0,1]$, where α and β are the fixed constants satisfying $0 < \alpha < \beta \leq 1$ and the functions $f_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing on $\mathbb{R}_+$.

(iv) *There exists a positive solution r_0 of the inequality*

$$f_1(r) + (2K + k_\beta) f_2(r) f_3(r) \leq r,$$

where the constant K is defined by

$$\sup\left\{\int_0^1 |k(t,\tau)| d\tau : t \in [0,1]\right\} \leq K.$$

Then the equation (1.1) has at least one solution $x = x(t)$ belonging to space $H_\alpha[0,1]$.

Proof. We take for arbitrarily fixed $t, s \in [0,1], (t \neq s)$ and let us consider $x \in H_\beta[0,1]$ and the operator F defined on the space $H_\beta[0,1]$ by the formula:

$$(Fx)(t) = (T_1x)(t) + (T_2x)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau,$$

for $t \in [0,1]$. Then, in view of our assumptions we get

$$\begin{aligned}
(Fx)(t) - (Fx)(s) &= (T_1x)(t) + (T_2x)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau - (T_1x)(s) - (T_2x)(s) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau \\
&= (T_1x)(t) - (T_1x)(s) + (T_2x)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau - (T_2x)(s) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau \\
&\quad + (T_2x)(s) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau - (T_2x)(s) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau \\
&= (T_1x)(t) - (T_1x)(s) + ((T_2x)(t) - (T_2x)(s)) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau \\
&\quad + (T_2x)(s) \int_0^1 (k(t,\tau) - k(s,\tau))(T_3x)(\tau) d\tau
\end{aligned}$$

and

$$\begin{aligned}
\frac{|(Fx)(t) - (Fx)(s)|}{|t-s|^\beta} &\leq \frac{|(T_1x)(t) - (T_1x)(s)|}{|t-s|^\beta} + \frac{|(T_2x)(t) - (T_2x)(s)|}{|t-s|^\beta} \int_0^1 |k(t,\tau)| \, |(T_3x)(\tau)| d\tau \\
&\quad + \frac{|(T_2x)(s)|}{|t-s|^\beta} \int_0^1 |k(t,\tau) - k(s,\tau)| \, |(T_3x)(\tau)| d\tau \leq H^\beta_{T_1x} + \|T_2x\|_\beta \|T_3x\|_\infty \int_0^1 |k(t,\tau)| d\tau \\
&\quad + \|T_2x\|_\infty \|T_3x\|_\infty \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\beta} d\tau \\
&\leq H^\beta_{T_1x} + \|T_2x\|_\beta \|T_3x\|_\infty K + \|T_2x\|_\beta \|T_3x\|_\infty \int_0^1 k_\beta \frac{|t-s|^\beta}{|t-s|^\beta} d\tau \\
&\leq H^\beta_{T_1x} + f_2(\|x\|_\beta) f_3(\|x\|_\beta) K + f_2(\|x\|_\beta) f_3(\|x\|_\beta) k_\beta \\
&= H^\beta_{T_1x} + (K + k_\beta) f_2(\|x\|_\beta) f_3(\|x\|_\beta).
\end{aligned} \tag{3.1}$$

This demonstrates that the operator F maps $H_\beta[0,1]$ into itself. Besides, for any $x \in H_\beta[0,1]$, we get

$$\begin{aligned}
|(Fx)(0)| &\leq |(T_1x)(0)| + |(T_2x)(0)| \int_0^1 |k(0,\tau)| (T_3x)(\tau)| d\tau \\
&\leq |(T_1x)(0)| + \|T_2x\|_\infty \|T_3x\|_\infty K \\
&\leq |(T_1x)(0)| + \|T_2x\|_\beta \|T_3x\|_\infty K \\
&\leq |(T_1x)(0)| + f_2(\|x\|_\beta) f_3(\|x\|_\beta) K.
\end{aligned} \tag{3.2}$$

By the inequalities by (3.1) and (3.2), we derive that

$$\begin{aligned}\|Fx\|_\beta &\le \|T_1x\|_\beta + (2K+k_\beta)f_2(\|x\|_\beta)f_3(\|x\|_\beta)\\ &\le f_1(\|x\|_\beta) + (2K+k_\beta)f_2(\|x\|_\beta)f_3(\|x\|_\beta).\end{aligned} \tag{3.3}$$

Since positive number r_0 is the solution of the inequality given in hypothesis (iv), from (3.3), we conclude that the inequality

$$\|Fx\|_\beta \le f_1(r_0) + (2K+k_\beta)f_2(r_0)f_3(r_0) \le r_0 \tag{3.4}$$

holds. As a results, it follows that F transforms the ball

$$B^\beta_{r_0} = \{x \in H_\beta[0,1] : \|x\|_\beta \le r_0\}$$

into itself. That is, $F : B^\beta_{r_0} \to B^\beta_{r_0}$. Thus, we have that the set $B^\beta_{r_0}$ is relatively compact in $H_\alpha[0,1]$ for any $0<\alpha<\beta\le 1$. Furthermore, $B^\beta_{r_0}$ is a compact subset in $H_\alpha[0,1]$.

We will show that the operator F is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$, where $0<\alpha<\beta\le 1$. Let $y \in B^\beta_{r_0}$ be an arbitrary point in $B^\beta_{r_0}$. Then, we get

$$\begin{aligned}(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s)) &= (T_1x)(t) + (T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau\\ &\quad - (T_1y)(t) - (T_2y)(t)\int_0^1 k(t,\tau)(T_3y)(\tau)d\tau\\ &\quad - (T_1x)(s) - (T_2x)(s)\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau\\ &\quad + (T_1y)(s) + (T_2y)(s)\int_0^1 k(s,\tau)(T_3y)(\tau)d\tau\end{aligned} \tag{3.5}$$

for any $x \in B^\beta_{r_0}$ and $t,s\in[0,1]$. The equality (3.5) can be rewritten as:

$$\begin{aligned}(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s)) &= (T_1x)(t)-(T_1y)(t)-((T_1x)(s)-(T_1y)(s))\\ &\quad + (T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau - (T_2y)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau\\ &\quad + (T_2y)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau - (T_2y)(t)\int_0^1 k(t,\tau)(T_3y)(\tau)d\tau\\ &\quad - (T_2x)(s)\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau + (T_2y)(s)\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau\\ &\quad - (T_2y)(s)\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau + (T_2y)(s)\int_0^1 k(s,\tau)(T_3y)(\tau)d\tau.\end{aligned} \tag{3.6}$$

By (3.6), we have

$$\begin{aligned}(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s)) &= (T_1x)(t)-(T_1y)(t)-((T_1x)(s)-(T_1y)(s))\\ &\quad +((T_2x)(t)-(T_2y)(t))\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau\\ &\quad + (T_2y)(t)\int_0^1 k(t,\tau)((T_3x)(\tau)-(T_3y)(\tau))d\tau\\ &\quad -((T_2x)(s)-(T_2y)(s))\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau\\ &\quad - (T_2y)(s)\int_0^1 k(s,\tau)((T_3x)(\tau)-(T_3y)(\tau))d\tau.\end{aligned} \tag{3.7}$$

(3.7) yields the following equality:

$$\begin{aligned}((Fx)(t)-(Fy)(t))-((Fx)(s)-(Fy)(s)) &= (T_1x)(t)-(T_1y)(t)-((T_1x)(s)-(T_1y)(s))\\ &\quad+[((T_2x)(t)-(T_2y)(t))-((T_2x)(s)-(T_2y)(s))]\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau\\ &\quad+((T_2x)(s)-(T_2y)(s))\int_0^1 (k(t,\tau)-k(s,\tau))(T_3x)(\tau)d\tau\\ &\quad+((T_2y)(t)-(T_2y)(s))\int_0^1 k(t,\tau)((T_3x)(\tau)-(T_3y)(\tau))d\tau\\ &\quad+(T_2y)(s)\int_0^1 (k(t,\tau)-k(s,\tau))((T_3x)(\tau)-(T_3y)(\tau))d\tau. \end{aligned} \tag{3.8}$$

Since $|(T_3x)(\tau)|\le\|T_3x\|_\infty\le f_3(\|x\|_\beta)$ and $|(T_3x)(\tau)-(T_3y)(\tau)|\le\|T_3x-T_3y\|_\infty$ for all $x,y\in B_{r_0}^\beta$ and $\tau\in[0,1]$, taking into account (3.8) and hypotheses, we can write:

$$\begin{aligned}\frac{|(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s))|}{|t-s|^\alpha} &\le \frac{|(T_1x)(t)-(T_1y)(t)-((T_1x)(s)-(T_1y)(s))|}{|t-s|^\alpha}\\ &\quad+\frac{|(T_2x)(t)-(T_2y)(t)-((T_2x)(s)-(T_2y)(s))|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)||(T_3x)(\tau)|d\tau\\ &\quad+\frac{|(T_2x)(s)-(T_2y)(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)-k(s,\tau)||(T_3x)(\tau)|d\tau\\ &\quad+\frac{|(T_2y)(t)-(T_2y)(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)||(T_3x)(\tau)-(T_3y)(\tau)|d\tau\\ &\quad+\frac{|(T_2y)(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)-k(s,\tau)||(T_3x)(\tau)-(T_3y)(\tau)|d\tau\\ &\le \|T_1x-T_1y\|_\alpha+\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty K+\|T_2x-T_2y\|_\infty\|T_3x\|_\infty\int_0^1 k_\beta|t-s|^{\beta-\alpha}d\tau\\ &\quad+\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty K+\|T_2y\|_\infty\|T_3x-T_3y\|_\infty\int_0^1 k_\beta|t-s|^{\beta-\alpha}d\tau\\ &\le \|T_1x-T_1y\|_\alpha+K\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+k_\beta\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty\\ &\quad+K\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty+k_\beta\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty\\ &= \|T_1x-T_1y\|_\alpha+(K+k_\beta)\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+(K+k_\beta)\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty \end{aligned} \tag{3.9}$$

for all $t,s\in[0,1]$ with $t\neq s$. Besides, for $x,y\in B_{r_0}^\beta$, we obtain following equality:

$$\begin{aligned}(Fx)(0)-(Fy)(0) &= (T_1x)(0)+(T_2x)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau-(T_1y)(0)-(T_2y)(0)\int_0^1 k(0,\tau)(T_3y)(\tau)d\tau\\ &= (T_1x)(0)-(T_1y)(0)+(T_2x)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau\\ &\quad-(T_2y)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau+(T_2y)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau\\ &\quad-(T_2y)(0)\int_0^1 k(0,\tau)(T_3y)(\tau)d\tau\\ &= (T_1x)(0)-(T_1y)(0)+((T_2x)(0)-(T_2y)(0))\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau\\ &\quad+(T_2y)(0)\int_0^1 k(0,\tau)((T_3x)(\tau)-(T_3y)(\tau))d\tau. \end{aligned} \tag{3.10}$$

By (3.10), we get that

$$\begin{aligned}|(Fx)(0)-(Fy)(0)| &= |(T_1x)(0)-(T_1y)(0)|+|(T_2x)(0)-(T_2y)(0)|K\int_0^1|(T_3x)(\tau)|d\tau \\ &\quad +|(T_2y)(0)|K\int_0^1|(T_3x)(\tau)-(T_3y)(\tau)|d\tau \\ &\le \|T_1x-T_1y\|_\infty+\|T_2x-T_2y\|_\infty K\|T_3x\|_\infty+\|T_2y\|_\infty K\|T_3x-T_3y\|_\infty \\ &\le \|T_1x-T_1y\|_\alpha+\|T_2x-T_2y\|_\alpha K\|T_3x\|_\infty+\|T_2y\|_\alpha K\|T_3x-T_3y\|_\infty. \end{aligned} \tag{3.11}$$

From (3.9) and (3.11), we have that

$$\begin{aligned}\|Fx-Fy\|_\alpha &= |(Fx-Fy)(0)|+H^\alpha_{Fx-Fy} \\ &= |(Fx)(0)-(Fy)(0)|+\sup\left\{\frac{|(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s))|}{|t-s|^\alpha}:t,s\in[0,1]\text{ and }t\neq s\right\} \\ &\le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+(2K+k_\beta)\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty \\ &\le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+(2K+k_\beta)\|T_2y\|_\beta\|T_3x-T_3y\|_\infty \\ &\le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)\|T_2x-T_2y\|_\alpha f_3(\|x\|_\beta)+(2K+k_\beta)f_2(\|y\|_\beta)\|T_3x-T_3y\|_\infty. \end{aligned} \tag{3.12}$$

Moreover, since $\|x\|_\beta\le r_0$ and $\|y\|_\beta\le r_0$, we derive from (3.12) that the following inequality holds:

$$\|Fx-Fy\|_\alpha \le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)f_3(r_0)\|T_2x-T_2y\|_\alpha+(2K+k_\beta)f_2(r_0)\|T_3x-T_3y\|_\infty. \tag{3.13}$$

Since the operators $T_1,T_2:H_\beta[0,1]\to H_\beta[0,1]$ and $T_3:H_\beta[0,1]\to C[0,1]$ are continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, they are also continuous at the point $y\in B^\beta_{r_0}$. Let us take an arbitrary $\varepsilon>0$, then there exists the number $\delta=\delta(\varepsilon)>0$. The inequalities

$$\|T_1x-T_1y\|_\alpha<\frac{\varepsilon}{6},\|T_2x-T_2y\|_\alpha<\tfrac{\varepsilon}{3(2K+k_\beta)f_3(r_0)}$$

and

$$\|T_3x-T_3y\|_\infty<\frac{\varepsilon}{3(2K+k_\beta)f_2(r_0)}$$

hold for all $x\in B^\beta_{r_0}$. Then, taking into account (3.13), we derive the following inequality:

$$\|Fx-Fy\|_\alpha<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

for all $x\in B^\beta_{r_0}$ with $\|x-y\|_\alpha<\delta$. Eventually, we infer that the operator F is continuous at the point $y\in B^\beta_{r_0}$. Since y was chosen arbitrarily, we conclude that F is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$. Because $B^\beta_{r_0}$ is compact in $H_\alpha[0,1]$, by the classical Schauder fixed point theorem, we get the desired consequence. □

4. Conclusion

This article concerns the entity of solutions of the following a quadratic integral equation of Fredholm type,

$$x(t)=(T_1x)(t)+(T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau,\ \ t\in I=[0,1].$$

where k is given function, T_1,T_2,T_3 are given operators satisfying conditions specified later and x is unknown function.

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