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New hybrid of Conjugate Gradient Technique for Solving Fuzzy Nonlinear Equations

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ABSTRACT In this paper, we suggest a new technique for numerical solution of fuzzy nonlinear equations in parametric form using a new Conjugate Gradient Technique. Table of the numerical solution is given to show the efficiency of the proposed technique and which is compared with classical algorithms such as (Fletcher and Reeves (FR), Polak and Ribiere (PRP), Fletcher (CD), and (KH)) techniques.

KEYWORDS: Conjugate Gradient Technique; Fuzzy Parametric Form; Fuzzy Nonlinear Equations; Numerical Techniques; Optimization Technique; Fuzzy Interval.

1. INTRODUCTION

Simultaneous non-linear equation systems have recently played a major role in various fields of science, such as mathematics, statistics, engineering, social sciences, robot technology and medicine. The idea concept of fuzzy numbers and mathematical operation was first proposed and researched by [1-5]. The use of fuzzy numbers is important among nonlinear systems, the parameters of which are defined entirely or partially by fuzzy numbers [6-8]. Buckley and Qu are Standard analytic techniques [9][10], cannot be adapted to the solution of equations such as

i- $ax^5 + bx^4 + cx^3 + dx - e = f,$

ii- $x - \sin(x) = g,$

Where x, a, b, c, d, e, f and g are fuzzy numbers. In this way, we need to build up the numerical techniques to find the roots of such equations. Here, we think about these conditions, as a rule, as

$$F(x) = c.$$

whose parameters are all or partially represented by fuzzy numbers, [11] research the performance of Newton's scheme for getting the solve of the fuzzy nonlinear equations and reached out to systems of fuzzy nonlinear equations by [12]. Newton's technique converges quickly if the initial point is picked near the solution point. The primary downside of Newton's strategy is figuring the Hessian matrix in each epoch. Probably the easiest variation of Newton's technique was considered by[13] for solution the double fuzzy nonlinear equations. Another type of Newton's technique known as Levenberg-Marquardt alteration was used to fathom fuzzy nonlinear equations by[14]. Authors in [15] applied also the Broyden's technique to solve the fuzzy nonlinear equations. Every one of these techniques is Newton-like which requires the calculation and storage of either Hessian matrix or approximate Hessian matrix at every iterative of iterations. Newly, a diagonal update technique for solving fuzzy nonlinear equations was proposed by[14]. A gradient based technique by [16] was applied to get the optimal value of variables of fuzzy nonlinear equations. This technique is simple and requires no Hessian matrix assessment during calculations. Be that as it may, technique convergence is linear and very slow toward the optimal solution [17-19]. The steepest descent technique is additionally affected by ill-conditioning [20,21].

In this work we developed a new conjugate gradient coefficient and applied it to solve fuzzy nonlinear equations. The conjugate gradient technique is known to be easy and high proficient in taking care of optimization problem. The plan in this work is to convert the parametric form of a fuzzy nonlinear equation into an unconstrained optimization problem before applying the new conjugate gradient technique to get the optimal solution.

This paper is organized as follows: In Section 2. Preliminaries in fuzzy. In Section 3 we present new conjugate gradient (CG) technique. Section 4 show that our technique satisfies descent and global convergence conditions. Section 5 presents numerical experiments and comparisons.

2. PRELIMINARIES

Definition 1. A fuzzy number is a fuzzy set like $u: \mathbb{R} \longrightarrow I = [0,1]$ which satisfies[22, 23],

1. u is upper semi continuous,
2. $u(x) = 0$ outside some interval $[c, d]$,
3. There are real numbers a, b such that $c \leq a \leq b \leq d$ and
   i. $u(x)$ is monotonic increasing on $[c, a]$,
   ii. $u(x)$ is monotonic decreasing on $[b, d]$,
   iii. $u(x) = 1, a \leq x \leq b$.

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is also given in[24] as follows.

Definition 2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of function $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfies the following requirements:

1- $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
2- $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
3- $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A popular fuzzy number is the trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β where the membership function is

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma), & x_0 - \sigma \leq x \leq x_0, \\ 1, & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta, \\ 0, & otherwise. \end{cases}$$

Its parametric form is

$$\underline{u}(r) = x_0 - \sigma + r\sigma, \overline{u}(r) = y_0 + \beta + r\beta.$$

Let $TF(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$ and $k > 0$ we define addition $(u + v)$ and multiplication by scalar k as

$$(\underline{u}+\underline{v})(r)=\underline{u}+\underline{v}, \quad (\overline{u}+\overline{v})=\overline{u}+\overline{v}, \tag{1}$$

$$(\underline{ku})(r)=ku(r), \quad (\overline{ku})(r)=k\overline{u}(r). \tag{2}$$

3. NEW CONJUGATE GRADIENT TECHNIQUE FOR SOLVING FUZZY NONLINEAR EQUATIONS

In this section we will show some of the conjugate gradient techniques and then suggest a new algorithm for conjugate gradient algorithm for solving fuzzy nonlinear equations

$$x_{k+1}=x_k+\alpha_k d_k, \ \ k\geq 1 \tag{3}$$

Where α_k step-size that satisfy the standard wolfe conditions

$$f(x_k+\alpha_k d_k)\leq f(x_k)+\delta\alpha_k g_k^T d_k \tag{4}$$

$$d_k^T g(x_k+\alpha_k d_k)\geq \sigma d_k^T g_k \tag{5}$$

or strong Wolfe conditions

$$f(x_k+\alpha_k d_k)\leq f(x)+\delta\alpha_k g_k^T d_k \tag{6}$$

$$|d_k^T g(x_k+\alpha_k d_k)|\leq -\sigma d_k^T g_k \tag{7}$$

$$d_{k+1}=\begin{cases}-g_1, & k=1\\ -g_{k+1}+\beta_k d_k, & k\geq 1\end{cases} \tag{8}$$

The Fletcher and Reeves (FR) [25], Fletcher (CD) [26], Polak and Ribiere (PRP) [27] and β is scalar.

$\beta^{FR}=\frac{\|g_{k+1}\|^2}{\|g_k\|^2}$, see [25]

$\beta^{CD}=\frac{-\|g_{k+1}\|^2}{g_k^T d_k}$, see [26]

$\beta^{PRP}=\frac{g_{k+1}^T y_k}{\|g_k\|^2}$, see [27]

Where $g_k=\nabla f(x_k)$, and let $y_k=g_{k+1}-g_k$.

Now we suggest a new of conjugate gradient algorithm for solving fuzzy nonlinear equations depend basically on norm-1 instead of norm-2 in Fletcher-Reeves (FR) algorithm so we get new formula

$$d_{k+1}=-g_{k+1}+\beta_k^{NEW} d_k \tag{9}$$

$\beta_k^{KH}=\frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2}$ see [28]

$$\beta_k^{NEW}=\theta_k\beta_k^{FR}+(1-\theta_k)\beta_k^{KH} \tag{10}$$

Multiply the equation (9) by y_k

$y_k^T d_{k+1}=-y_k^T g_{k+1}+\beta_k^{NEW} y_k^T d_k$

By Pure Conjugacy Condition we get

$-y_k^T g_{k+1}+\beta_k^{NEW} y_k^T d_k=0$

$-y_k^T g_{k+1}+(\theta_k\beta_k^{FR}+(1-\theta_k)\beta_k^{KH})y_k^T d_k=0$

$(\theta_k(\beta_k^{FR} - \beta_k^{KH}) + \beta_k^{KH})y_k^T d_k = y_k^T g_{k+1}$

$\theta_k(\beta_k^{FR} - \beta_k^{KH}) = \frac{y_k^T g_{k+1}}{y_k^T d_k} - \beta_k^{KH}$

$$\theta_k = \frac{(y_k^T g_{k+1} \parallel g_k \parallel_1^2 - y_k^T d_k \parallel g_{k+1} \parallel_1^2) \parallel g_k \parallel^2}{(\parallel g_{k+1} \parallel^2 \parallel g_k \parallel_1^2 - \parallel g_{k+1} \parallel_1^2 \parallel g_k \parallel^2) y_k^T d_k} \quad (11)$$

substituting (11) in (10).

Algorithm (NEW)

Step (1): Initialization:

choose $x_1 \in R^n$ and calculate $f(x_1), g(x_1)$ and consider

$d_1 = -g_1$ and $k = 1$

Step (2): Check for convergent:

If $\parallel g_k \parallel \leq \varepsilon$, stop, x_k is the optimum solution

else go to step(3).

Step (3): line search:

Calculate α_k satisfying the strong Wolfe conditions and update variable $x_{k+1} = x_k + \alpha_k d_k$. Compute, $f_{k+1}, g_{k+1}, g_k, d_k$.

Step (4): Calculate direction search:

Calculate β_k from (10) and then calculate

$d_{k+1} = -g_{k+1} + \beta_k^{NEW} d_k$ then $k = k + 1$ go to step (2).

4. THE DESCENT PROPERTY AND GLOBAL CONVERGENCE STUDY OF NEW TECHNIQUE

This technique is developed satisfy The Descent Property and Global convergence because the $\beta_k^{FR} \& \frac{\parallel g_{k+1} \parallel_1^2}{\parallel g_k \parallel_1^2}$ satisfies the conditions Descent Property and Global convergence Because β_k^{NEW} it is derived from them see more [25-33].

5. NUMERICAL EXAMPLES

In this part, we present the numerical solution of some examples using the new proposed technique for CG technique for fuzzy nonlinear equations. All computations are carried out on MATLAB 9.0 using a double precision computer. We use the stop criterion $\parallel g_{k+1} \parallel < 10^{-6}$, We employ the new technique to solve fuzzy nonlinear equations .and compare with the other techniques (Fletcher and Reeves (FR), Polak and Ribiere (PRP), Fletcher (CD) and (KH)).

Example 1: Consider the fuzzy nonlinear equation [9]

$(3,4,5)x^2 + (1,2,3)x = (1,2,3)$

Without any loss of generality, assume that x is positive, and then the parametric form of this equation is as follows:

$$\begin{cases}(3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) - (1+r) = 0,\\(5-r)\overline{x}^2(r) + (3-r)\overline{x}(r) - (3-r) = 0.\end{cases}$$

The above system needs initial values as follows. For $r = 1$

$$\begin{cases}4\underline{x}^2(1) + 2\underline{x}(1) - 2 = 0,\\4\overline{x}^2(1) + 2\overline{x}(1) - 2 = 0,\end{cases}$$

For $r = 0$

$$\begin{cases}3\underline{x}^2(0) + \underline{x}(0) - 1 = 0,\\5\overline{x}^2(0) + \overline{x}(0) - 3 = 0\end{cases}$$

With initial values

$x_0 = (\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0)) = (0.434, 0.5, 0.5, 0.681).$

Example 2: Consider the fuzzy nonlinear equation [9]

$(4,6,8)x^2 + (2,3,4)\,x - (8,12,16) = (5,6,7)$

Without any loss of generality, assume that x is positive, and then the parametric form of this equation is as follows:

$$\begin{cases}(4+2r)\underline{x}^2(r) + (2+r)\underline{x}(r) - (3+3r) = 0,\\(8-2r)\overline{x}^2(r) + (4-r)\overline{x}(r) - (9-3r) = 0.\end{cases}$$

The above system needs initial values as follows. For $r = 1$

$$\begin{cases}6\underline{x}^2(1) + 3\underline{x}(1) - 6 = 0,\\6\overline{x}^2(1) + 3\overline{x}(1) - 6 = 0,\end{cases}$$

For $r = 0$

$$\begin{cases}4\underline{x}^2(0) + 2\underline{x}(0) - 3 = 0,\\8\overline{x}^2(0) + 4\overline{x}(0) - 9 = 0,\end{cases}$$

With initial values

$x_0 = (\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0)) = (0.651, 0.7808, 0.7808, 0.8397).$

Example 3: Consider the fuzzy nonlinear equation [9]

$(1,2,3)x^3 + (2,3,4)x^2 + (3,4,5) = (5,8,13)$

Without any loss of generality, assume that x is positive, and then the parametric form of this equation is as follows:

$$\begin{cases}(1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) - (2+2r) = 0,\\(3-r)\overline{x}^3(r) + (4-r)\overline{x}^2(r) - (8-4r) = 0.\end{cases}$$

The above system needs initial values as follows. For $r = 1$

$$\begin{cases}2\underline{x}^3(1) + 3\underline{x}^2(1) - 4 = 0,\\2\overline{x}^3(1) + 3\overline{x}^2(1) - 4 = 0,\end{cases}$$

For $r = 0$

$$\begin{cases}\underline{x}^3(0) + 2\underline{x}^2(0) - 2 = 0,\\3\overline{x}^3(0) + 4\overline{x}^2(0) - 8 = 0.\end{cases}$$

With initial values

$x_0 = (\underline{x}(0), \underline{x}(1), \overline{x}(1), \overline{x}(0)) = (0.76, 0.91, 0.91, 1.06)$**.**

Table of numerical results for examples above

Example	***FR***			***PRP***			***CD***			β_k^{KH}			***NEW2***		
	iter	***x_optimal***	***f_optimal***	***iter***	***x_optimal***	***f_optimal***	***iter***	***x_optimal***	***f_optimal***	***iter***	***x_optimal***	***f_optimal***	***iter***	***x_optimal***	***f_optimal***
1	8	0.4343 0.5000 0.5000 0.5307	5.2505e-014	6	0.4343 0.5000 0.5000 0.5307	1.2314e-016	8	0.4343 0.5000 0.5000 0.5307	8.1709e-014	6	0.4343 0.5000 0.5000 0.5307	1.7016e-011	**5**	**0.4343** **0.5000** **0.5000** **0.5307**	**8.6657e-016**
2	12	0.6514 0.7808 0.7808 0.8397	2.3998e-010	12	0.6514 0.7808 0.7808 0.8397	7.9887e-012	14	0.6514 0.7808 0.7808 0.8397	5.9506e-010	**9**	**0.6514** **0.7808** **0.7808** **0.8397**	**1.9513e-011**	**9**	**0.6514** **0.7808** **0.7808** **0.8397**	**7.7858e-011**
3	19	0.8393 0.9108 0.9108 1.0564	2.6011e-011	16	0.8393 0.9108 0.9108 1.0564	4.6595e-012	135	0.8393 0.9108 0.9108 1.0564	1.4978e-008	13	0.8393 0.9108 0.9108 1.0564	1.6756e-013	**9**	**0.8393** **0.9108** **0.9108** **1.0564**	**1.2232e-012**

Iter : *Numbers Iterations.*
x_optimal : *Optimal of x value.*
f_optimal : *Optimal of f value.*

6. CONCLUSIONS

We presented in this research a new hybrid conjugate gradient technique for solving fuzzy nonlinear equations, and the proposed algorithm has shown a high efficiency in solving these problems with the least number of iterations and higher accuracy in reaching the approximate solution of the function.

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