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Positive Solutions for a Fractional Thermostat Model via Sum Operators Methods

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Abstract

In this paper, we consider a fractional thermostat model involving Caputo fractional derivatives. Based on recent fixed point theorems of sum operators on cones, we give the existence and uniqueness of positive solutions for the model and we can construct an iterative scheme to approximate the unique solution. In the last section, we list two concrete examples to illustrate our main results.

Keywords: *fractional thermostat model, Caputo fractional derivative, Green's function, fixed point theorems of sum operators*
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1. Introduction

It is well-known that fractional differential equations arise in many fields, such as economics, mechanics, physics and biological sciences, etc; for more details we refer the reader to [2, 3, 5, 8, 10, 11, 12, 13, 14, 15] and the references therein. Many authors have investigated the existence of positive solutions for fractional differential equation boundary value problems, see [4, 5, 12, 14, 15, 16, 17, 21, 22, 24] and the references therein. Besides, the uniqueness of positive solutions for fractional problems has been studied widely, see [14, 15, 19, 20, 21, 22, 24] for instance.

In [9], the authors studied the following fractional boundary value problem:

$$\begin{cases} -{}^cD_a^{\alpha}u(t) = f(t,u(t)), \ \ 0 < t < 1, \\ u^{'}(0) = 0, \ \ \beta {}^cD_1^{\alpha-1}u(1) + u(\eta) = 0, \end{cases}$$

where $1 < \alpha \leq 2$, ${}^cD_a^{\alpha}$ denotes the Caputo fractional derivative of order α, $\beta > 0$, $0 \leq \eta \leq 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. Some existence results were established by using Guo-Krasnoselskii fixed point theorem. But the uniqueness of solutions was not studied in [9]. For other related results to the problem, see [2, 3, 8, 10] for example.

In [2], the authors studied the following fractional thermostat model:

$$\begin{cases} -{}^cD_a^{\alpha}u(t) = y(t), \ \ a < t < b, \\ u^{'}(a) = 0, \ \ \beta {}^cD_a^{\alpha-1}u(b) + u(\eta) = 0, \end{cases}$$

where $1 < \alpha \leq 2$, $\beta > 0$, $a \leq \eta \leq b$. The authors present some Lyapunov-type inequalities for a nonlinear fractional heat equation with nonlocal boundary conditions depending on a positive parameter. As an application, a lower bound for the eigenvalues of corresponding equations was obtained. However, the authors do not provide the existence of the solution in this article.

Inspired by the above works, we mainly consider the existence and uniqueness of positive solutions for the following fractional thermostat model:

$$\begin{cases} -{}^cD_a^{\alpha}u(t) = g(t,u(t)) + f(t,u(t)), \ \ a < t < b, \\ u^{'}(a) = 0, \ \ \beta {}^cD_a^{\alpha-1}u(b) + u(\eta) = 0, \end{cases} \tag{1.1}$$

where ${}^cD_a^{\alpha}$ denotes the Caputo fractional derivative of order α, $1 < \alpha \leq 2$, $\beta > 0$ and $a \leq \eta \leq b$. $f, g : [a,b] \times [0,\infty) \to [0,\infty)$ are continuous functions. In this paper, we mainly prove the existence and uniqueness of positive solutions for the corresponding model. Our methods are two fixed point theorems of sum operators on cones. Moreover, we can construct an iterative scheme to approximate the unique positive solution.

The rest of the paper is organized as follows. In Sect.2, some preliminaries on fractional calculus and fixed point theory are presented. Next, we state and prove our main results in Sect.3 and two examples are provided.

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2. Preliminaries and previous results

In this section, we present the basic results about fractional calculus theory which will be used later. We refer the reader to [1,2,3] and the references therein.
We denote by $\mathbf{N}$ the set of positive natural numbers, that is: $\mathbf{N}=\{1,2,3,\cdots\}$.

Definition 2.1. *[1] Let $f:[a,b]\to\mathbf{R}$ be a given function. For $\alpha>0$, the Riemann-Liouville fractional integral of order α of f is defined by*

$$(I_a^{\alpha}f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,$$

where $\Gamma(\alpha)$ denotes the classical gamma function.

Definition 2.2. *[1] Let $f:[a,b]\to\mathbf{R}$ be a given function. For $\alpha>0$, the Caputo derivative of fractional order α of f is given by*

$${}^cD_a^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^n(s)ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes integer part of α.

Consider the fractional boundary value problem:

$$\begin{cases} -{}^cD_a^{\alpha}u(t)=y(t), \ \ a<t<b, \\ u^{'}(a)=0, \ \ \beta{}^cD_a^{\alpha-1}u(b)+u(\eta)=0, \end{cases} \tag{2.1}$$

where $1<\alpha<2$, $\beta>0$, $a\le\eta\le b$, $(a,b)\in\mathbf{R}^2$, and $y\in C[a,b]$.

Lemma 2.3. *[2] Suppose that $u\in C^2[a,b]$ is a solution to (2.1) if and only if*

$$u\in C[a,b],\, u(x)=\int_a^b G(t,s)y(s)ds, \ \ a<t<b,$$

where G is the Green's function given by:

$$G(t,s)=\beta+H_{\eta}(s)-H_t(s) \tag{2.2}$$

and for $r\in[a,b]$, $H_r:[a,b]\to\mathbf{R}$ is the function defined as

$$H_r(s)=\begin{cases} \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)}, a\le s\le r\le b, \\ 0, \quad a\le r\le s\le b. \end{cases}$$

Lemma 2.4. *[2] The Green's function given by (2.2) satisfies the following properties:*
(i) G is continuous in $[a,b]\times[a,b]$;
(ii) We have:

$$\max\{G(t,s):a\le t,s\le b\}=\beta+\frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.3}$$

and

$$\min\{G(t,s):a\le t,s\le b\}=\beta-\frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.4}$$

Suppose that E is a real Banach space which is partially ordered by a cone $P\subset E$. We say that $x\le y$ if and only if $y-x\in P$. $\overset{\circ}{P}$ denotes the interior of P. An operator $A:P\to P$ is increasing if $x\le y$ implies $Ax\le Ay$ for $x,y\in P$.For $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\le y\le\mu x$. Clearly, $\sim$ is an equivalence relation. Given $h>\theta(i.e., h\ge\theta$ and $h\neq\theta)$, we denote by P_h the set $P_h=\{x\in E|x\sim h\}$. It is easy to see that $P_h\subset P$.

Definition 2.5. *[15] An operator $A:E\to E$ is said to be homogeneous if it satisfies*

$$A(\lambda x)=\lambda Ax, \qquad \forall\lambda>0, x\in E.$$

An operator $A:P\to P$ is said to be sub-homogeneous if it satisfies $A(tx)\ge tAx$ for all $t>0, x\in P$.

Our main tools are the following lemmas.

Lemma 2.6. *[4] Let M be nonempty closed convex subset of $\overset{\circ}{P}$, $A:\overset{\circ}{P}\to\overset{\circ}{P}$ and $B:M\to\overset{\circ}{P}$, such that*
(i) A is increasing, and there exists $\alpha\in(0,1)$ such that $A(tx)\ge t^{\alpha}A(x)$ for any $x\in\overset{\circ}{P}$, $t\in(0,1)$;
(ii) B is continuous, and $B(M)$ resides in a compact subset of $\overset{\circ}{P}$;
(iii) $x=Ax+By$ and $y\in M$ implies $x\in M$.
Then there exists $x^*\in M$ such that $(A+B)x^*=x^*$.

Lemma 2.7. *[15] Let P be a normal cone, $A:P\to P$ be an increasing $\beta-$concave operator and $B:P\to P$ be an increasing sub-homogeneous operator. Assume that*
(i) there is $h>\theta$ such that $Ah\in P_h$ and $Bh\in P_h$;
(ii) there exists a constant $\delta_0>0$ such that $Ax\ge\delta_0 Bx$ for $x\in P$.
Then the operator equation $Ax+Bx=x$ has a unique solution x^* in P_h. Moreover, constructing the sequence $y_n=Ay_{n-1}+By_{n-1}, n=1,2,\cdots$ for any initial value $y_0\in P_h$, we have $y_n\to x^*$ as $n\to\infty$.

3. Main results

In this paper, we work in Banach space $E = C[a,b]$, equipped with the norm:

$$\|u\| = \max\{|u(t)| : t \in [a,b]\}.$$

Let P be the cone in E given by:

$$P = \{u \in C[a,b] \mid u(t) \geq 0, t \in [a,b]\}.$$

$\overset{\circ}{P}$ denotes the interior of P, then

$$\overset{\circ}{P} = \{u \in C[a,b] \mid u(t) > 0, t \in [a,b]\}.$$

Take $R > 0$, $C_2 = \beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}$, $d = \max\limits_{\substack{a \leq t \leq b \\ 0 \leq u \leq \infty}} |g(t,u)|, e = \max\limits_{\substack{a \leq t \leq b \\ 0 \leq u \leq R}} |f(t,u)|$.

Define

$$M = \left\{ u \in \overset{\circ}{P} \mid \ \|u\| \leq R \right\},$$

where R satisfies $C_2(b-a)(d-e) \leq R$, then M is closed and convex in $\overset{\circ}{P}$.

From Lemma 2.6, we know that the solution of the problem (1.1) can be expressed as

$$u(t) = \int_a^b G(t,s)[g(s,u(s)) + f(s,u(s))]ds. \tag{3.1}$$

For the convenience, we define two operators:

$$Au(t) = \int_a^b G(t,s)g(s,u(s))ds,$$

$$Bu(t) = \int_a^b G(t,s)f(s,u(s))ds.$$

Theorem 3.1. *Suppose that $\beta\Gamma(\alpha) > (b-\eta)^{\alpha-1}$ and $f,g : [a,b] \times [0,\infty) \to [0,\infty)$ are continuous. In addition,*
(H_1) $g(t,u) : [a,b] \times [0,\infty) \to [0,\infty)$ is increasing in u with $g(t,0) \not\equiv 0$, $\sup\{g(t,u) : t \in [a,b], u \in [0,\infty)\} < +\infty$, and there exists a $\gamma \in (0,1)$, such that $g(t,\lambda u) \geq \lambda^\gamma g(t,u)$ for all $\lambda \in (0,1)$, $u \geq 0$;
(H_2) when $u > 0$, $f(t,u) \not\equiv 0$ for all $t \in [a,b]$.
Then the problem (1.1) has a positive solution $u \in M$.

Proof. We apply Lemma 2.3 to discuss that the problem (1.1) has a positive solution in $C[a,b]$.
First, we will show that $A : \overset{\circ}{P} \to \overset{\circ}{P}$, $B : M \to \overset{\circ}{P}$. In fact, for $u \in \overset{\circ}{P}$, then $u(t) > 0, t \in [a,b]$ and by (H_1) and Lemma 2.4,

$$\begin{aligned} Au(t) &= \int_a^b G(t,s)g(s,u(s))ds \\ &\geq \int_a^b (\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)})g(s,0)ds \\ &= (\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)}) \int_a^b g(s,0)ds. \end{aligned}$$

Since $g(t,0) \not\equiv 0$ for $t \in [a,b]$ and $g(t,0)$ is continuous, we obtain $\int_a^b g(s,0)ds > 0$. Hence, $Au \in \overset{\circ}{P}$, that is, $A : \overset{\circ}{P} \to \overset{\circ}{P}$.
For $u \in M$, then $u(t) > 0$ for $t \in [a,b]$, then by (H_2) and Lemma 2.4,

$$\begin{aligned} Bu(t) &= \int_a^b G(t,s)f(s,u(s))ds \\ &\geq \int_a^b (\beta - \frac{(b-\eta)^{\alpha-1}}{\Gamma(\alpha)})f(s,u(s))ds > 0. \end{aligned}$$

Hence, $B : M \to \overset{\circ}{P}$.
It is clear to show that A is increasing. In fact, from Lemma 2.4 and (H_1),

$$Au_1(t) = \int_a^b G(t,s)g(s,u_1(s))ds \leq \int_a^b G(t,s)g(s,u_2(s))ds = Au_2(t)$$

for $u_1, u_2 \in \overset{\circ}{P}$ with $u_1 \leq u_2$. Also from (H_1),

$$\begin{aligned} A(\lambda u)(t) &= \int_a^b G(t,s)g(s,\lambda u(s))ds \\ &\geq \int_a^b G(t,s)\lambda^\gamma g(s,u(s))ds \\ &= \lambda^\gamma (Au)(t). \end{aligned}$$

We know $A(\lambda u) \geq \lambda^{\gamma} Au$ for $\lambda \in (0,1)$.
For the second step, we will prove that B is a completely continuous operator in M. From the continuity and nonnegativity of $G(t,s)$ and $f(t,u)$, $B : P \to P$ is continuous.
Let $\Omega \subset P$ be bounded, there is a constant $C_1 > 0$ such that $\|u\| \leq C_1$ for all $u \in \Omega$. Set $L_1 = \max\limits_{\substack{a\leq t\leq b\\ u\in\Omega}} f(t,u)+1$, then use Lemma 2.4, for $u \in \Omega$,

$$(Bu)(t) = \int_a^b G(t,s)f(s,u(s))ds \leq L_1 \int_a^b G(t,s)ds \leq N,$$

where $N = L_1(\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)})(b-a)$. Therefore, B is uniformly bounded.
Next, for $u \in M$ and $t_1, t_2 \in [a,b]$, let $L_2 = \max\limits_{\substack{a\leq t\leq b\\ u\in M}} f(t,u)$,

$$\begin{aligned}
|(Bu)(t_2)-(Bu)(t_1)| &= \left| \int_a^b G(t_2,s)f(s,u(s))ds - \int_a^b G(t_1,s)f(s,u(s))ds \right| \\
&= \left| \int_a^b [G(t_2,s)-G(t_1,s)]f(s,u(s))ds \right| \\
&\leq L_2 \int_a^b |G(t_2,s)-G(t_1,s)|\, ds \\
&\leq L_2 \left[\int_a^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds - \int_a^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds \right] \\
&= \frac{L_2}{\Gamma(\alpha+1)} [(t_1-a)^{\alpha} - (t_2-a)^{\alpha}],
\end{aligned}$$

then $(Bu)(t_1) \to (Bu)(t_2)$ as $t_1 \to t_2$. So we claim that B is equi-continuous. Hence, $B(M)$ is precompact by Ascoli-Arzel$\grave{a}$ theorem. Since $(Bu)(t) > 0$, $B(M)$ resides in a compact subset of $\overset{\circ}{P}$.
For all $v \in M$, let $u = Au + Bv$, by Lemma 2.6, we need to prove $u \in M$. Indeed,

$$\begin{aligned}
u(t) &= \int_a^b G(t,s)g(s,u(s))ds + \int_a^b G(t,s)f(s,v(s))ds \\
&\leq \int_a^b (\beta + \frac{(\eta-a)^{\alpha-1}}{\Gamma(\alpha)})|g(s,u(s)) + f(s,v(s))|ds \\
&\leq C_2(d+e)(b-a) = R,
\end{aligned}$$

and thus $\| u \| \leq R$, which shows $u \in M$.
So, assumption (iii) of Lemma 2.6 is satisfied, hence the problem (1.1) has a solution $u \in M$ by using Lemma 2.6. □ □

Let $h(t) = \int_a^b G(t,s)ds$, then

$$\begin{aligned}
h(t) &= \int_a^b (\beta + H_\eta(s) - H_t(s))ds \\
&= \int_a^b \beta ds + \int_a^b H_\eta(s)ds - \int_a^b H_t(s))ds \\
&= \beta(b-a) + \int_a^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&= \beta(b-a) + \frac{(\eta-a)^{\alpha} - (t-a)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Theorem 3.2. *Suppose that $\beta\Gamma(\alpha) > (b-\eta)^{\alpha-1}$ and $f, g : [a,b] \times [0,+\infty) \to [0,+\infty)$ are continuous. In addition,*
(H_3) *$f, g : [a,b] \times [0,+\infty) \to [0,+\infty)$ are increasing in u, $\min\limits_{a\leq t\leq b} f(t, \beta(b-a) + \frac{(\eta-a)^{\alpha}-(b-a)^{\alpha}}{\Gamma(\alpha+1)}) > 0$;*
(H_4) *$f(t,\lambda u) \geq \lambda f(t,u)$ for $\lambda \in (0,1), t \in [a,b], u \in [0,+\infty)$, and there exists a constant $\gamma \in (0,1)$ such that $g(t,\lambda u) \geq \lambda^{\gamma} g(t,u)$ for all $t \in [a,b]$, $\lambda \in (0,1)$, $u \in [0,\infty)$;*
(H_5) *there exists a constant $\delta_0 > 0$ such that $g(t,u) \geq \delta_0 f(t,u)$, $t \in [a,b]$, $u \geq 0$.*
Then the problem (1.1) *has a unique positive solution u^* in P_h, where*

$$h(t) = \beta(b-a) + \frac{(\eta-a)^{\alpha} - (t-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad t \in [a,b].$$

Moreover, for any initial value $u_0 \in P_h$, the sequence

$$u_{n+1}(t) = \int_a^b G(t,s)g(s,u_n(s))ds + \int_a^b G(t,s)f(s,u_n(s))ds, n = 0,1,2,\cdots$$

satisfies $u_n(t) \to u^*(t)$ as $n \to \infty$.

Proof. From Theorem 3.1, we easily prove that $A : P \to P$ and $B : P \to P$. Next we verify that operators A and B satisfy assumptions of Lemma 2.7.
Obviously, A is an increasing operator. Let $u_1(t) \le u_2(t)$, $t \in [a,b]$, by (H_3) and Lemma 2.4,

$$Bu_1(t) = \int_a^b G(t,s)f(s,u_1(s))ds \le \int_a^b G(t,s)f(s,u_2(s))ds = Bu_2(t),$$

so we know that B is increasing.
It has been proved that the operator A is a γ-concave operator in Theorem 3.1. For any $\lambda \in (0,1)$ and $u \in P$, from (H_4),

$$B(\lambda u)(t) = \int_a^b G(t,s)f(s,\lambda u(s))ds \ge \lambda \int_a^b G(t,s)f(s,u(s))ds = \lambda B(u)(t),$$

this is, $B(\lambda u) \ge \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. So the operator B is sub-homogeneous.
Now we show that $Ah \in P_h$ and $Bh \in P_h$. Set

$$h_{\max} = \max\{h(t) : t \in [a,b]\} = \beta(b-a) + \frac{(\eta-a)^\alpha}{\Gamma(\alpha+1)},$$

$$h_{\min} = \min\{h(t) : t \in [a,b]\} = \beta(b-a) + \frac{(\eta-a)^\alpha - (b-a)^\alpha}{\Gamma(\alpha+1)},$$

then $h_{\max} \ge h_{\min} > 0$. Denote $d_1 = \max\limits_{a\le t\le b} g(t,h_{\max})$, $d_2 = \min\limits_{a\le t\le b} g(t,h_{\min})$, $e_1 = \max\limits_{a\le t\le b} f(t,h_{\max})$, $e_2 = \min\limits_{a\le t\le b} f(t,h_{\min})$.
From (H_3) and Lemma 2.4,

$$Ah(t) = \int_a^b G(t,s)g(s,h(s))ds \le \max_{a\le s\le b} |g(s,h_{\max})| \int_a^b G(t,s)ds = d_1 h(t),$$

$$Ah(t) = \int_a^b G(t,s)g(s,h(s))ds \ge \min_{a\le s\le b} |g(s,h_{\min})| \int_a^b G(t,s)ds = d_2 h(t).$$

From (H_3) and (H_5), we can obtain

$$d_1 \ge d_2 \ge \delta_0 e_2 > 0,$$

$$d_2 h(t) \le Ah(t) \le d_1 h(t),\ t \in [a,b],$$

so $Ah \in P_h$. Similarly,

$$e_2 h(t) \le Bh(t) \le e_1 h(t),$$

from (H_3) we easily prove $Bh \in P_h$. Hence the condition (i) of Lemma 2.7 is satisfied. In the following we show that the condition (ii) of Lemma 2.7 is satisfied. For $u \in P$, from (H_5),

$$Au(t) = \int_a^b G(t,s)g(s,u(s))ds \ge \delta_0 \int_a^b G(t,s)f(s,u(s))ds = \delta_0 Bu(t).$$

Then we get $Au \ge \delta_0 Bu$, $u \in P$. Finally, an application of Lemma 2.7 implies: the operator equation $Au + Bu = u$ has a unique solution u^* in P_h. That is, the problem (1.1) has a unique positive solution u^* in P_h. Moreover, for any initial value $u_0 \in P_h$, we construct a sequence

$$u_{n+1}(t) = \int_a^b G(t,s)g(s,u_n(s))ds + \int_a^b G(t,s)f(s,u_n(s))ds, n = 0,1,2,\cdots,$$

then $u_n(t) \to u^*(t)$ as $n \to \infty$. □ □

Example 3.3. *Consider the fractional boundary value problem:*

$$\begin{cases} -{}^cD_{\frac{1}{3}}^{\frac{3}{2}} u(t) = g(t,u(t)) + f(t,u(t)), & \frac{1}{3} < t < 1, \\ u'(\frac{1}{3}) = 0, \ \frac{4}{5}{}^cD_{\frac{1}{3}}^{\frac{1}{2}} u(1) + u(\frac{2}{3}) = 0, \end{cases} \tag{3.2}$$

where $a = \frac{1}{3}$, $b = 1$, $\alpha = \frac{3}{2}$, $\beta = \frac{4}{5}$, $\eta = \frac{2}{3}$, $f(t,u) = \frac{1}{2}t^2 \sin u + \frac{3}{4}$,

$$g(t,u) = \begin{cases} \sqrt{u} + t^2, & 0 \le u \le 1, \\ 1 + t^3, & u > 1, \end{cases}$$

and $g(t,u) \ge 0$ *is increasing in u with* $g(t,0) = t^2 \not\equiv 0$, *where* $(t,u) \in [\frac{1}{3},1] \times [0,\infty)$, *and there exist* $\gamma = \frac{1}{2}$, *such that* $g(t,(\lambda u)) = \sqrt{\lambda u} + t^2 \ge \lambda^{\frac{1}{2}}(\sqrt{u} + t^2) = \lambda^{\frac{1}{2}} g(t,u)$, *when* $0 \le u \le 1$; $g(t,\lambda u) = 1 + t^3 \ge \lambda^{\frac{1}{2}} g(t,u)$, *when* $u > 1$ *for all* $\lambda \in (0,1)$. *It is obvious that* $f(t,u) = \frac{1}{2}t^2 \sin u + \frac{3}{4}$ *is continuous on* $[\frac{1}{3},1] \times [0,\infty)$, *when* $u > 0$, $f(t,u) \not\equiv 0$ *for all* $t \in [\frac{1}{3},1]$. *And* $C_2 = 0.1316$, $b - a = \frac{2}{3}$, $d = 2$, $e = \frac{1}{2}\sin 1 + \frac{3}{4}$ *and* $R = C_2 \cdot (d+e)(b-a) = 0.1316 \times [2 + (\frac{1}{2}\sin 1 + \frac{3}{4})] \times \frac{2}{3} = 0.2782$ *is bounded. So all the hypotheses of Theorem* 3.1 *are fulfilled. Therefore, it follows from Theorem* 3.1 *that the boundary value problem (3.2) has a positive solution.*

Example 3.4. *Consider the fractional boundary value problem:*

$$\begin{cases} -{}^{c}D_{\frac{1}{3}}^{\frac{3}{2}}u(t)=u^{\frac{1}{2}}(t)+\frac{u(t)}{1+u(t)}q(t)+t^{3}+n, & \frac{1}{3}<t<\frac{2}{3}, \\ u^{'}(\frac{1}{3})=0, \ 2{}^{c}D_{\frac{1}{3}}^{\frac{1}{2}}u(\frac{2}{3})+u(\frac{2}{3})=0, \end{cases} \tag{3.3}$$

where $a=\frac{1}{3}$, $b=\frac{2}{3}$, $\alpha=\frac{3}{2}$, $\beta=2$, $\eta=\frac{2}{3}$, *where* $n>0$ *is a constant,* $q:[0,1]\to[0,+\infty)$ *is continuous with* $q\not\equiv 0$. *In this example, take* $0<m<n$ *and let*

$$g(t,u)=u^{\frac{1}{2}}+t^{3}+m, \qquad f(t,u)=\frac{u}{1+u}q(t)+n-m.$$

$$\gamma=\frac{1}{2}, \quad q_{\max}=\max\{q(t):t\in[a,b]\}.$$

Obviously, $q_{\max}>0$, $f,g:[a,b]\times[0,\infty)\to[0,\infty)$ *are continuous and increasing in u.* $f(t,\frac{2}{3}+\frac{4\sqrt{\pi}[(\frac{1}{27})^{\frac{1}{2}}-(\frac{1}{3})^{\frac{1}{2}}]}{3\pi})\geq n-m>0$. *Besides, for* $\lambda\in(0,1),\ t\in[a,b],\ u\in[0,+\infty)$, *we have*

$$g(t,\lambda u)=(\lambda u)^{\frac{1}{2}}+t^{3}+m\geq\lambda^{\frac{1}{2}}(u^{\frac{1}{2}}+t^{3}+m)=\lambda^{\frac{1}{2}}g(t,u),$$

$$f(t,\lambda u)=\frac{\lambda u}{1+\lambda u}q(t)+n-m\geq\frac{\lambda u}{1+u}q(t)+\lambda(n-m)=\lambda f(t,u).$$

In addition, if we take $\delta_0\in(0,\frac{m}{q_{\max}+n-m}]$, *then we have*

$$g(t,u)=u^{\frac{1}{2}}+t^{3}+m\geq m=\frac{m}{q_{\max}+n-m}(q_{\max}+n-m)\geq\delta_0[\frac{u}{1+u}q(t)+n-m]=\delta_0 f(t,u).$$

Therefore, all the conditions of Theorem 3.2 are satisfied. This implies that (3.3) has a unique positive solution in P_h, *where*

$$h(t)=\frac{2}{3}+\frac{4\sqrt{\pi}[(\frac{1}{27})^{\frac{1}{2}}-(t-\frac{1}{3})^{\frac{1}{2}}]}{3\pi}.$$

4. Conclusion

In this paper, we proved the existence and the uniqueness of solution for the fractional thermostat model involving Caputo fractional derivatives (1.1) under different conditions. Our methods are two recent fixed point theorems of sum operators. Moreover, we can give a sequence to approximate the unique solution. As applications, we list two concrete examples to illustrate our main results.

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