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ON THE RESOLVENT OF SINGULAR q-STURM-LIOUVILLE OPERATORS

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Abstract. In this paper, we investigate the resolvent operator of the singular q-Sturm-Liouville problem defined as

$$-\frac{1}{q}D_{q^{-1}}\left[D_q y\left(x\right)\right]+\left[r\left(x\right)-\lambda\right]y\left(x\right)=0,$$

with the boundary condition

$$y\left(0,\lambda\right)\cos\beta+D_{q^{-1}}y\left(0,\lambda\right)\sin\beta=0,$$

where $\lambda\in\mathbb{C}$, r is a real-valued function defined on $[0,\infty)$, continuous at zero and $r\in L^1_{q,loc}[0,\infty)$. We give a representation for the resolvent operator and investigate some properties of this operator. Furthermore, we obtain a formula for the Titchmarsh-Weyl function of the singular q-Sturm-Liouville problem.

1. Introduction

Quantum (or q) calculus is a very interesting field in mathematics. It has numerous in statistic physics, quantum theory, the calculus of variations and number theory; see, e.g., [12, 1, 11, 14, 15, 18, 21, 24]). The first results in q-calculus belong to the Euler. In 2005, Annaby and Mansour investigated q-Sturm-Liouville problems [10]. Later in [9], the authors studied the Titchmarsh-Weyl theory for q-Sturm-Liouville equations. In [3,4], the authors proved the existence of a spectral function for q-Sturm-Liouville operator.

In this article, we investigate the following q-Sturm-Liouville problem defined as

$$-\frac{1}{q}D_{q^{-1}}D_q y\left(x\right)+u\left(x\right)y\left(x\right)=\lambda y\left(x\right),\tag{1}$$

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where $0 < x < \infty$. The resolvent operator for this problem is constructed. Using the spectral function, an integral representation is obtained. Furthermore, some properties of this operator are investigated. A formula for the Titchmarsh-Weyl function of Eq. (1) is given. Historically, in 1910, H. Weyl was first obtained a representation theorem for the resolvent of Sturm-Liouville problem defined by

$$-\left(py'\right)' + qy = \lambda y,\ x \in (0, \infty),$$

where p, q are real-valued and $p^{-1}, q \in L^1_{loc}[0, \infty)$. Similar representation theorems were proved in [25, 20, 2, 5, 6, 7].

2. Preliminaries

In this section, we give a brief introduction to quantum calculus and refer the interested reader to [17, 8, 12].

Let $0 < q < 1$ and let $A \subset \mathbb{R}$ is a q-geometric set, i.e., $qx \in A$ for all $x \in A$. The *Jackson q-derivative is* defined by

$$D_q y(x) = \mu^{-1}(x)\left[y(qx) - y(x)\right],$$

where $\mu(x) = qx - x$ and $x \in A$. We note that there is a connection the Jackson q-derivative between and q-deformed Heisenberg uncertainty relation (see [23]). The q-derivative at zero is defined as

$$D_q y(0) = \lim_{n\to\infty} \left[q^n x\right]^{-1} \left[y(q^n x) - y(0)\right] \quad (x \in A), \tag{2}$$

if the limit in (2) exists and does not depend on x. The *Jackson q-integration* is given by

$$\int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t,$$

where $a, b \in A$. The q-integration for a function over $[0, \infty)$ defined by the formula ([13])

$$\int_0^{\infty} f(t)\, d_q t = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

Let f be a function on A and let $0 \in A$. For every $x \in A$, if

$$\lim_{n\to\infty} f(xq^n) = f(0),$$

then f is called q-regular at zero. Throughout the paper, we deal only with functions q-regular at zero.

The following relation holds

$$\int_0^a g(t)\, D_q f(t)\, d_q t + \int_0^a f(qt)\, D_q g(t)\, d_q t = f(a)\, g(a) - f(0)\, g(0),$$

where f and g are q-regular at zero.

Let $L_q^2[0,\infty)$ be the Hilbert space consisting of all functions f satisfying ([9])

$$\|f\| := \sqrt{\int_0^\infty |f(x)|^2\, d_q x} < +\infty$$

with the inner product

$$(f,g) := \int_0^\infty f(x)\, \overline{g(x)} d_q x.$$

The q-Wronskian of the functions $y(.)$ and $z(.)$ is defined by the formula

$$W_q(y,z)(x) := y(x)\, D_q z(x) - z(x)\, D_q y(x),$$

where $x \in [0,\infty)$.

3. Main Results

Consider the q-Sturm-Liouville equation

$$L(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + r(x)\, y(x) = \lambda y(x), \tag{3}$$

satisfying the conditions

$$y(0,\lambda)\cos\beta + D_{q^{-1}} y(0,\lambda)\sin\beta = 0, \tag{4}$$

$$y\left(q^{-n},\lambda\right)\cos\alpha + D_{q^{-1}} y\left(q^{-n},\lambda\right)\sin\alpha = 0,\ \alpha,\beta \in \mathbb{R},\ n \in \mathbb{N} := \{1,2,...\}, \tag{5}$$

where $\lambda \in \mathbb{C}$, r is a real-valued function defined on $[0,\infty)$, continuous at zero and $r \in L_{q,loc}^1[0,\infty)$.

Let $\varphi(x,\lambda)$ and $\theta(x,\lambda)$ be the solutions of the Eq. (3) satisfying the following conditions

$$\begin{array}{l} \varphi(0,\lambda) = \sin\beta,\ D_{q^{-1}}\varphi(0,\lambda) = -\cos\beta, \\ \theta(0,\lambda) = \cos\beta,\ D_{q^{-1}}\theta(0,\lambda) = \sin\beta. \end{array} \tag{6}$$

Lemma 1 ([9])**.** *Let $\lambda \notin \mathbb{R}$ and let*

$$\chi_{q^{-n}}(x,\lambda) = \theta(x,\lambda) + l\left(\lambda, q^{-n}\right)\varphi(x,\lambda) \in L_q^2(0,\infty),$$

where $n \in \mathbb{N}$. Then we have

$$\chi_{q^{-n}}(x,\lambda) \quad \to \quad \chi(x,\lambda),$$

$$\int_0^{q^{-n}} \left|\chi_{q^{-n}}(qt,\lambda)\right|^2 d_q x \quad \to \quad \int_0^\infty |\chi(x,\lambda)|^2\, d_q x,\ n \to \infty.$$

Putting

$$G_{q^{-n}}(x,t,\lambda)=\begin{cases}\chi_{q^{-n}}(x,\lambda)\,\varphi(t,\lambda), & t\leq x\\ \varphi(x,\lambda)\,\chi_{q^{-n}}(t,\lambda), & t>x,\end{cases}$$

$$y(x,\lambda):=\left(R_{q^{-n}}f\right)(x,\lambda)=$$

$$\int_0^{q^{-n}} G_{q^{-n}}(x,t,\lambda)\,f(t)\,d_qt,\ \ (\lambda\in\mathbb{C},\ \ \operatorname{Im}\lambda\neq 0), \tag{7}$$

where $f\in L_q^2[0,q^{-n}]$. Now, we shall show that the equality (7) satisfies the equation $L(y)-\lambda y(x)=f(x)$, $x\in(0,q^{-n})$ ($\lambda\in\mathbb{C}$, $\operatorname{Im}\lambda\neq 0$) and the boundary conditions (4)-(5). From (7), we get

$$\begin{aligned} y(x,\lambda) &= q\chi_{q^{-n}}(x,\lambda)\int_0^x \varphi(qt,\lambda)\,f(qt)\,d_qt \\ &\quad + q\varphi(x,\lambda)\int_x^{q^{-n}} \chi_{q^{-n}}(qt,\lambda)\,f(qt)\,d_qt. \end{aligned} \tag{8}$$

From (8), it follows that

$$\begin{aligned} D_qy(x,\lambda) &= qD_q\chi_{q^{-n}}(x,\lambda)\int_0^x \varphi(qt,\lambda)\,f(qt)\,d_qt \\ &\quad + qD_q\varphi(x,\lambda)\int_x^{q^{-n}} \chi_{q^{-n}}(qt,\lambda)\,f(qt)\,d_qt, \end{aligned}$$

and

$$\begin{aligned} D_{q^{-1}}D_qy(x,\lambda) &= qD_{q^{-1}}D_q\chi_{q^{-n}}(x,\lambda)\int_0^x \varphi(qt,\lambda)\,f(qt)\,d_qt \\ &\quad + qD_{q^{-1}}D_q\varphi(x,\lambda)\int_x^{q^{-n}} \chi_{q^{-n}}(qt,\lambda)\,f(qt)\,d_qt \\ &\quad - qW_q\left(\chi_{q^{-n}},\varphi\right)f(x). \end{aligned}$$

Hence, by $W_q\left(\varphi,\chi_{q^{-n}}\right)=1$ ($n\in\mathbb{N}$), we deduce that

$$\begin{aligned} & -\frac{1}{q}D_{q^{-1}}D_qy(x,\lambda) \\ =\ & (\lambda-r(x))\,q\chi_{q^{-n}}(x,\lambda)\int_0^x \varphi(qt,\lambda)\,f(qt)\,d_qt \\ & +(\lambda-r(x))\,q\varphi(x,\lambda)\int_x^{q^{-n}} \chi_{q^{-n}}(qt,\lambda)\,f(qt)\,d_qt+f(x) \\ =\ & (\lambda-r(x))\,y(x,\lambda)+f(x), \end{aligned}$$

i.e., the function $y(x,\lambda)$ satisfies the equation $L(y)-\lambda y(x)=f(x)$, $x\in(0,q^{-n})$.

Moreover,

$$\begin{aligned} y(0,\lambda) &= q\varphi(0,\lambda)\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt \\ &= q\cos\beta\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt, \\ D_{q^{-1}}y(0,\lambda) &= qD_{q^{-1}}\varphi(0,\lambda)\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt \\ &= -q\sin\beta\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt, \end{aligned}$$

i.e., $y(x,\lambda)$ satisfies (4). Similarly, we may infer that $y(x,\lambda)$ satisfies (5).

Note that the problem (3)-(5) has a purely discrete spectrum [10].

Let $\lambda_{m,q^{-n}}$ be the eigenvalues of the problem (3)-(5). Let $\varphi_{m,q^{-n}}$ be the corresponding eigenfunctions and

$$\alpha_{m,q^{-n}}:=\left\|\varphi_{m,q^{-n}}\right\|=\left(\int_0^{q^{-n}}\varphi_{m,q^{-n}}^2(x)\,d_qx\right)^{\frac{1}{2}},$$

where $\varphi_{m,q^{-n}}(x):=\varphi_{m,q^{-n}}\left(x,\lambda_{m,q^{-n}}\right)$ and $m\in\mathbb{N}$.

Then we have the following *Parseval equality* (see [8])

$$\int_0^{q^{-n}}|f(x)|^2\,d_qx=\sum_{m=1}^{\infty}\frac{1}{\alpha_{m,q^{-n}}^2}\left\{\int_0^{q^{-n}}f(x)\,\varphi_{m,q^{-n}}(x)\,d_qx\right\}^2, \tag{9}$$

where $f(.)\in L_q^2[0,q^{-n}]$.

Now, let us define the nondecreasing step function $\varrho_{q^{-n}}$ on $[0,\infty)$ by

$$\varrho_{q^{-n}}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_{m,q^{-n}}<0}\frac{1}{\alpha_{m,q^{-n}}^2}, & \text{for } \lambda\le 0\\ \sum_{0\le\lambda_{m,q^{-n}}<\lambda}\frac{1}{\alpha_{m,q^{-n}}^2} & \text{for } \lambda>0. \end{cases}$$

It follows from (9) that

$$\int_0^{q^{-n}}|f(x)|^2\,d_qx=\int_{-\infty}^{\infty}F^2(\lambda)\,d\varrho_{q^{-n}}(\lambda), \tag{10}$$

where

$$F(\lambda)=\int_0^{q^{-n}}f(x)\,\varphi(x,\lambda)\,d_qx.$$

Lemma 2. *Let $\kappa > 0$. Then the following relation holds*

$$\overset{\kappa}{\underset{-\kappa}{V}}\left\{\varrho_{q^{-n}}(\lambda)\right\} = \sum_{-\kappa\leq\lambda_{m,q^{-n}}<\kappa}\frac{1}{\alpha^2_{m,q^{-n}}} = \varrho_{q^{-n}}(\kappa) - \varrho_{q^{-n}}(-\kappa) < \Upsilon, \tag{11}$$

where $\Upsilon = \Upsilon(\kappa)$ is a positive constant not depending on q^{-n}.

Proof. Let $\sin\beta \neq 0$. Since $\varphi(x,\lambda)$ is continuous at zero, by condition $\varphi(0,\lambda) = \sin\beta$, there exists a positive number h and nearby 0 such that

$$|\varphi(x,\lambda)| > \frac{1}{\sqrt{2}}|\sin\beta|, \; 0\leq x\leq h$$

and

$$\left(\frac{1}{h}\int_0^h\varphi(x,\lambda)\,d_qx\right)^2 > \left(\frac{1}{\sqrt{2}h}\sin\beta\int_0^h d_qx\right)^2 = \frac{1}{2}\sin^2\beta. \tag{12}$$

Let us define $f_h(x)$ by

$$f_h(x) = \begin{cases} 0, & x > h \\ \frac{1}{h}, & 0\leq x\leq h. \end{cases}$$

It follows from (10) and (12) that

$$\begin{aligned}\int_0^h f_h^2(x)\,d_qx &= \frac{1}{h} = \int_{-\infty}^{\infty}\left(\frac{1}{h}\int_0^h\varphi(x,\lambda)\,d_qx\right)^2 d\varrho_{q^{-n}}(\lambda)\\ &\geq \int_{-\kappa}^{\kappa}\left(\frac{1}{h}\int_0^h\varphi(x,\lambda)\,d_qx\right)^2 d\varrho_{q^{-n}}(\lambda)\\ &> \frac{1}{2}\sin^2\beta\left\{\varrho_{q^{-n}}(\kappa)-\varrho_{q^{-n}}(-\kappa)\right\},\end{aligned}$$

which proves the inequality (11).

Let $\sin\beta = 0$ and

$$f_h(x) = \begin{cases} 0, & x > h \\ \frac{1}{h^2}, & 0\leq x\leq h. \end{cases}$$

By (10), we can get the desired result. □

We now return to the formula (7), whose right-hand side has been called the resolvent. The resolvent is known to exist for all λ which are not eigenvalues of the problem (3)-(5). Now, we will get the expansion of the resolvent.

Since the function $y(x,\lambda)$ satisfies the equation $L(y)-\lambda y(x) = f(x)$, $x\in(0,q^{-n})$ ($\lambda\in\mathbb{C}$, $\lambda\neq\lambda_{m,q^{-n}}$, $m\in\mathbb{N}$) and the boundary conditions (4), (5), via the q-integration by parts, we find (the operator A generated by the expression L and the boundary conditions (4), (5) is a self-adjoint (see [10]))

$$(Ay, \varphi_{m,q^{-n}})$$

$$
\begin{aligned}
&= \int_0^{q^{-n}} \left[-\frac{1}{q}D_{q^{-1}}D_q y(x,\lambda) + r(x)\, y(x,\lambda)\right] \varphi_{m,q^{-n}}(x)\, d_q x \\
&= (y, A\varphi_{m,q^{-n}}) \\
&= \int_0^{q^{-n}} y(x,\lambda) \left[-\frac{1}{q}D_{q^{-1}}D_q \varphi_{m,q^{-n}}(x) + r(x)\, \varphi_{m,q^{-n}}(x)\right] d_q x \\
&= \lambda_{m,q^{-n}} \int_0^{q^{-n}} y(x,\lambda)\, \varphi_{m,q^{-n}}(x)\, d_q x.
\end{aligned}
$$

The set of all eigenfunctions $\frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}}$ $(m \in \mathbb{N})$ of the self-adjoint operator A form an orthonormal basis for $L_q^2(0, q^{-n})$ (see [10]). Then, the function $y(.,\lambda) \in L_q^2(0, q^{-n})$ $(\lambda \in \mathbb{C},\ \lambda \neq \lambda_{m,q^{-n}},\ m \in \mathbb{N})$ can be expanded into Fourier series of eigenfunctions $\frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}}$ $(m \in \mathbb{N})$ of the problem (3)-(5) (or of the operator A). Then we have

$$
y(x,\lambda) = \sum_{m=1}^{\infty} t_m(\lambda) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}},
$$

where $t_m(\lambda)$ is the Fourier coefficient, i.e.,

$$
t_m(\lambda) = \int_0^{q^{-n}} y(x,\lambda) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} d_q x,\ m \in \mathbb{N}.
$$

Since $y(x,\lambda)$ $(\lambda \in \mathbb{C},\ \lambda \neq \lambda_{m,q^{-n}},\ m \in \mathbb{N})$ satisfies the equation

$$
-\frac{1}{q}D_{q^{-1}}D_q y(x,\lambda) + (r(x) - \lambda)\, y(x,\lambda) = f(x),\ x \in (0, q^{-n}),
$$

we get

$$
\begin{aligned}
a_m &:= \int_0^{q^{-n}} f(x) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} d_q x \\
&= \int_0^{q^{-n}} \left[-\frac{1}{q}D_{q^{-1}}D_q y(x,\lambda) + (r(x) - \lambda)\, y(x,\lambda)\right] \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} d_q x \\
&= \int_0^{q^{-n}} \left[-\frac{1}{q}D_{q^{-1}}D_q \varphi_{m,q^{-n}}(x) + (r(x) - \lambda)\varphi_{m,q^{-n}}(x)\right] \frac{y(x,\lambda)}{\alpha_{m,q^{-n}}} d_q x \\
&= \int_0^{q^{-n}} \left[\lambda_{m,q^{-n}}\varphi_{m,q^{-n}}(x) - \lambda\varphi_{m,q^{-n}}(x)\right] \frac{y(x,\lambda)}{\alpha_{m,q^{-n}}} d_q x \\
&= \lambda_{m,q^{-n}} t_m(\lambda) - \lambda t_m(\lambda),\ m \in \mathbb{N}.
\end{aligned}
$$

Thus, we have

$$t_m(\lambda) = \frac{a_m}{\lambda_{m,q^{-n}} - \lambda},$$

and

$$\begin{aligned} y(x,\lambda) &= \int_0^{q^{-n}} G_{q^{-n}}(x,t,\lambda) f(t)\, d_q t \\ &= \sum_{m=1}^{\infty} \frac{a_m}{\lambda_{m,q^{-n}} - \lambda} \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} \ (\lambda \in \mathbb{C},\ \lambda \neq \lambda_{m,q^{-n}},\ m \in \mathbb{N}). \end{aligned}$$

Then

$$y(x,z) = \left(R_{q^{-n}} f\right)(x,z)$$

$$\begin{aligned} &= \sum_{m=1}^{\infty} \frac{\varphi_{m,q^{-n}}(x)}{\alpha^2_{m,q^{-n}}\left(\lambda_{m,q^{-n}} - z\right)} \int_0^{q^{-n}} f(t)\, \varphi_{m,q^{-n}}(t)\, d_q t \\ &= \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} \left\{ \int_0^{q^{-n}} f(t)\, \varphi_{m,q^{-n}}(t,\lambda)\, d_q t \right\} d\varrho_{q^{-n}}(\lambda). \end{aligned} \tag{13}$$

Lemma 3. *The following formula holds*

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda) < K, \tag{14}$$

where x is a fixed number and z is a non-real number.

Proof. Let $f(t) = \frac{\varphi_{m,q^{-n}}(t)}{\alpha_{m,q^{-n}}}$. By (13), we conclude that

$$\frac{1}{\alpha_{m,q^{-n}}} \int_0^{q^{-n}} G_{q^{-n}}(x,t,z)\, \varphi_{m,q^{-n}}(t)\, d_q t = \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}\left(\lambda_{m,q^{-n}} - z\right)}. \tag{15}$$

Under (15) and (9), we see that

$$\begin{aligned} \int_0^{q^{-n}} \left| G_{q^{-n}}(x,t,z) \right|^2 d_q t &= \sum_{m=1}^{\infty} \frac{\left|\varphi_{m,q^{-n}}(x)\right|^2}{\alpha^2_{m,q^{-n}} \left|\lambda_{m,q^{-n}} - z\right|^2} \\ &= \int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda). \end{aligned}$$

It follows from Lemma 1 that the last integral is convergent. The proof is complete □

Now, we present below for the convenience of the reader.

Theorem 4 ([19]). *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real non-decreasing function on a finite interval $[a,b]$. Then*
(i) there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a non-decreasing function w such that

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda),$$

where $a \leq \lambda \leq b$.

(ii) suppose

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),$$

where $a \leq \lambda \leq b$. Then, we have

$$\lim_{n\to\infty} \int_a^b f(\lambda)\, dw_n(\lambda) = \int_a^b f(\lambda)\, dw(\lambda),$$

where $f \in C[a,b]$.

By Lemma 2 and Theorem 4, one can find a sequence $\{q^{-n_k}\}$ such that

$$\lim_{k\to\infty} \varrho_{q^{-n_k}}(\lambda) \to \varrho(\lambda),$$

where $\varrho(\lambda)$ is a monotone function.

Lemma 5. *Let $z \notin \mathbb{R}$. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho(\lambda) \leq K, \tag{16}$$

where x is a fixed number.

Proof. Let $\eta > 0$. It follows from (14) that

$$\int_{-\eta}^{\eta} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda) < K.$$

Then

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho(\lambda) = \lim_{\substack{\eta\to\infty \\ n\to\infty}} \int_{-\eta}^{\eta} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda) < K.$$

□

Lemma 6. *Let $\eta > 0$. Then we have*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty, \ \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty. \tag{17}$$

Proof. Let $\sin\beta \neq 0$. From (16), we deduce that

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty.$$

Let $\sin\beta = 0$. Hence we see that

$$\frac{1}{\alpha_{m,q^{-n}}}\int_0^{q^{-n}} \varphi_{m,q^{-n}}(t)\, D_{q,x}\left[G_{q^{-n}}(x,t,z)\right] d_q t = \frac{D_{q,x}\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}\left(\lambda_{m,q^{-n}} - z\right)}.$$

It follows from (9) that

$$\int_0^{q^{-n}} \left|D_{q,x}\left[G_{q^{-n}}(x,t,z)\right]\right|^2 d_q t = \int_{-\infty}^{\infty} \left|\frac{D_{q,x}\varphi(x,\lambda)}{\lambda - z}\right|^2 d\varrho_{q^{-n}}(\lambda).$$

Proceeding similarly, we can get the desired result. □

Lemma 7. *Let*

$$G(x,t,z) = \begin{cases} \chi(x,z)\,\varphi(t,z), & x \geq t \\ \varphi(x,z)\,\chi(t,z), & x < t, \end{cases}$$

and let $f(.) \in L_q^2[0,\infty)$. Then we have

$$\int_0^{\infty} \left|(Rf)(x,z)\right|^2 d_q x \leq \frac{1}{v^2}\int_0^{\infty} \left|f(x)\right|^2 d_q x,$$

where

$$(Rf)(x,z) = \int_0^{\infty} G(x,t,z)\, f(t)\, d_q t,$$

and $z = u + iv$.

Proof. See [9]. □

Now we shall state the main result of this paper.

Theorem 8. *The following relation holds*

$$(Rf)(x,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F(\lambda)\, d\varrho(\lambda), \tag{18}$$

where $f(.) \in L_q^2[0,\infty)$,

$$F(\lambda) = \lim_{\xi\to\infty}\int_0^{q^{-\xi}} f(x)\,\varphi(x,\lambda)\, d_q x,$$

and $z \notin \mathbb{R}$.

Proof. Define the function $f_\xi(x)$ as

$$f_\xi(x) = \begin{cases} f_\xi(x), & x \in [0, q^{-\xi}], \\ 0, & x \notin [0, q^{-\xi}] \end{cases} \quad (q^{-\xi} < q^{-n})$$

such that $f_\xi(x)$ satisfies (4). By (13), we conclude that

$$\begin{aligned} &\left(R_{q^{-n}} f_\xi\right)(x,z) \\ = &\int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_{q^{-n}}(\lambda) = \int_{-\infty}^{-a} \frac{\varphi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_{q^{-n}}(\lambda) \end{aligned}$$

$$
\begin{aligned}
& +\int_{-a}^{a} \frac{\varphi(x,\lambda)}{\lambda - z} F_{\xi}(\lambda)\, d\varrho_{q^{-n}}(\lambda) + \int_{a}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F_{\xi}(\lambda)\, d\varrho_{q^{-n}}(\lambda) \\
= \; & I_1 + I_2 + I_3, \qquad (19)
\end{aligned}
$$

where

$$
F_{\xi}(\lambda) = \int_{0}^{q^{-\xi}} f(x)\,\varphi(x,\lambda)\, d_q x,
$$

and $a > 0$.

It follows from (13) that

$$
\begin{aligned}
|I_1| &= \left| \int_{-\infty}^{-a} \frac{\varphi(x,\lambda)}{\lambda - z} F_{\xi}(\lambda)\, d\varrho_{q^{-n}}(\lambda) \right| \\
&\leq \sum_{\lambda_{k,q^{-n}} < -a} \frac{\left|\varphi_{k,q^{-n}}(x)\right| \left| \int_0^{q^{-\xi}} f_{\xi}(t)\,\varphi_{k,q^{-n}}(t)\, d_q t \right|}{\alpha_{k,q^{-n}}^2 \left|\lambda_{k,q^{-n}} - z\right|} \\
&\leq \left(\sum_{\lambda_{k,q^{-n}} < -a} \frac{\varphi_{k,q^{-n}}^2(x)}{\alpha_{k,q^{-n}}^2 \left|\lambda_{k,q^{-n}} - z\right|^2} \right)^{1/2} \\
&\quad \times \left(\sum_{\lambda_{k,q^{-n}} < -a} \frac{1}{\alpha_{k,q^{-n}}^2} \left[\int_0^{q^{-\xi}} f_{\xi}(x)\,\varphi_{k,q^{-n}}(x)\, d_q x \right]^2 \right)^{1/2}. \qquad (20)
\end{aligned}
$$

Using the q-integration-by-parts formula in the integral below, we have

$$
\begin{aligned}
& \int_0^{q^{-\xi}} f_{\xi}(x)\,\varphi_{k,q^{-n}}(x)\, d_q x \\
= \; & \frac{1}{\lambda_{k,q^{-n}}} \int_0^{q^{-\xi}} f_{\xi}(x) \left\{ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{k,q^{-n}}(x) + r(x)\,\varphi_{k,q^{-n}}(x) \right\} d_q x \\
= \; & \frac{1}{\lambda_{k,q^{-n}}} \int_0^{q^{-\xi}} \left\{ -\frac{1}{q} D_{q^{-1}} D_q f_{\xi}(x) + r(x)\, f_{\xi}(x) \right\} \varphi_{k,q^{-n}}(x)\, d_q x. \qquad (21)
\end{aligned}
$$

From Lemma 3, we get

$$
|I_1| \leq \frac{K^{1/2}}{a} \left(\begin{array}{c} \sum_{\lambda_{k,q^{-n}} < -a} \frac{1}{\alpha_{k,q^{-n}}^2} \\ \times \left[\int_0^{q^{-\xi}} \left\{ -\frac{1}{q} D_{q^{-1}} D_q f_{\xi}(x) + r(x)\, f_{\xi}(x) \right\} \varphi_{k,q^{-n}}(x)\, d_q x \right]^2 \end{array} \right)^{1/2}.
$$

Application of Bessel inequality yields

$$|I_1| \leq \frac{K^{1/2}}{a}\left[\int_0^{q^{-\xi}}\left\{-\frac{1}{q}D_{q^{-1}}D_q f_\xi(x) + r(x) f_\xi(x)\right\}^2 d_q x\right]^{1/2} = \frac{C}{a}.$$

Likewise, we show that $|I_3| \leq \frac{C}{a}$. Then $I_1, I_3 \to 0$, as $a \to \infty$, uniformly in q^{-n}. By virtue of (19) and Theorem 4, we see that

$$(Rf_\xi)(x,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho(\lambda). \tag{22}$$

We can find a sequence $\{f_\xi(x)\}_{\xi=1}^{\infty}$ which satisfies the previous conditions and tend to $f(x)$ as $\xi \to \infty$, since $f(.) \in L_q^2[0,\infty)$. It follows from (9) that the sequence of Fourier transform converges to the transform of $f(x)$. Using Lemmas 5 and 7, one can pass to the limit $\xi \to \infty$ in (22). □

Remark 9. *The following formula holds.*

$$\int_0^{\infty} (Rf)(x,z)\, g(x)\, d_q x = \int_{-\infty}^{\infty} \frac{F(\lambda)\, G(\lambda)}{\lambda - z} d\varrho(\lambda), \tag{23}$$

where

$$G(\lambda) = \lim_{\xi\to\infty} \int_0^{q^{-\xi}} g(x)\, \varphi(x,\lambda)\, d_q x,$$

and

$$F(\lambda) = \lim_{\xi\to\infty} \int_0^{q^{-\xi}} f(x)\, \varphi(x,\lambda)\, d_q x.$$

Now, we will study some properties of the resolvent operator. We give the following definition and theorems.

Definition 10. *Let $M(x,t)$ be a complex-valued function, where $x,t \in (a,b)$. If*

$$\int_a^b \int_a^b |M(x,t)|^2\, d_q x d_q t < +\infty,$$

then $M(x,t)$ is called the q-Hilbert-Schmidt kernel.

Theorem 11 ([22])**.** *Let us define the operator A as*

$$A\{x_i\} = \{y_i\},$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \ \ i \in \mathbb{N}. \tag{24}$$

If

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < +\infty \tag{25}$$

then A is a compact operator in the sequence space l^2.

Theorem 12. *Let the limit circle case holds for Eq. (3) and*

$$G(x,t) = G(x,t,0) = \begin{cases} \varphi(x)\chi(t), & x < t \\ \chi(x)\varphi(t), & x \geq t. \end{cases} \tag{26}$$

Then the function $G(x,t)$ defined by (26) is a q-Hilbert-Schmidt kernel.

Proof. It follows from (26) that

$$\int_0^\infty d_q x \int_0^x |G(x,t)|^2 d_q t < +\infty,$$

and

$$\int_0^\infty d_q x \int_x^\infty |G(x,t)|^2 d_q t < +\infty,$$

since the integrals

$$\int_0^\infty |G(x,t)|^2 d_q x$$

and

$$\int_0^\infty |G(x,t)|^2 d_q t$$

exist and are a linear combination of the products $\varphi(x)\chi(t)$, and these products belong to $L_q^2[0,\infty) \times L_q^2[0,\infty)$. Then

$$\int_0^\infty \int_0^\infty |G(x,t)|^2 d_q x d_q t < +\infty. \tag{27}$$

□

Theorem 13. *Let us define the operator R as*

$$(Rf)(x) = \int_0^\infty G(x,t) f(t) d_q t$$

Under the assumptions of Theorem 12, R is a compact operator.

Proof. Let $\phi_i = \phi_i(t)$ ($i \in \mathbb{N}$) be a complete, orthonormal basis of $L_q^2[0,\infty)$. By Theorem 12, we can define

$$\begin{aligned} x_i &= (f,\phi_i) = \int_0^\infty \overline{\phi_i(t)} f(t) d_q t, \\ y_i &= (g,\phi_i) = \int_0^\infty \overline{\phi_i(t)} g(t) d_q t, \\ a_{ik} &= \int_0^\infty \int_0^\infty \overline{\phi_k(t)\phi_i(x)} G(x,t) d_q x d_q t, \end{aligned}$$

where $i,k \in \mathbb{N}$. Then, $L_q^2[0,\infty)$ is mapped isometrically l^2. Therefore, the operator R transforms into A defined by (24) in l^2 by this mapping, and (27) is translated into (25). It follows from Theorem 11 that A is compact operator. Consequently, R is a compact operator. □

Now, we will give some auxiliary lemmas.

Lemma 14. *The following equalities hold.*

$$\lim_{x\to\infty} W_q\left(\chi\left(x,\lambda\right),\chi\left(x,\lambda'\right)\right) = 0, \tag{28}$$

$$\int_0^\infty \chi\left(x,\lambda\right),\chi\left(x,\lambda'\right)d_qx = \frac{m\left(\lambda\right)-m\left(\lambda'\right)}{\lambda-\lambda'}, \tag{29}$$

where λ and λ' are any fixed nonreal numbers.

Proof. See [9]. □

Using (29) and setting $\lambda = u + iv$ and $\lambda' = \overline{\lambda}$, we obtain

$$\int_0^\infty \left|\chi\left(x,\lambda\right)\right|^2 d_qx = -\frac{\operatorname{Im}\left\{m\left(\lambda\right)\right\}}{v}. \tag{30}$$

Lemma 15. *For fixed u_1 and u_2, we have*

$$\int_{u_1}^{u_2} -\operatorname{Im}\left\{m\left(u+i\delta\right)\right\}du = O\left(1\right), \text{ as } \delta\to 0. \tag{31}$$

Proof. Let $\sin\beta \neq 0$. It follows from (9) and (18) that

$$\int_0^\infty \left|\chi\left(t,z\right)\right|^2 d_qt = \int_{-\infty}^\infty \frac{d\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+v^2}, \tag{32}$$

where $z = u + iv$.

Let $\sin\beta = 0$. If the equality (15) is q-differentiated throughout with respect to x, and the limit is taken as $n\to\infty$, then we can get the desired result.

By virtue of (30) and (32), we conclude that

$$-\operatorname{Im}\left\{m\left(u+i\delta\right)\right\} = \delta\int_{-\infty}^\infty \frac{d\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+\delta^2}.$$

Then we have

$$-\int_{u_1}^{u_2}\operatorname{Im}\left\{m\left(u+i\delta\right)\right\}du = \delta\int_{u_1}^{u_2}du\int_{-\infty}^\infty \frac{d\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+\delta^2}.$$

Let (a,b) be a finite interval where $a < u_1$ and $b > u_2$. From (17), we see that

$$\delta\int_{u_1}^{u_2}du\int_{-\infty}^{a} \frac{d\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+\delta^2} = O\left(1\right),$$

$$\delta\int_{u_1}^{u_2}du\int_{b}^{\infty} \frac{d\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+\delta^2} = O\left(1\right).$$

Hence, we get

$$\delta\int_{u_1}^{u_2}du\int_a^b \frac{d\varrho\left(\lambda\right)}{\left(u-\lambda\right)^2+\delta^2} = \int_a^b d\varrho\left(\lambda\right)\int_{\frac{u_1-\lambda}{\delta}}^{\frac{u_2-\lambda}{\delta}}\frac{dv}{1+v^2} = O\left(1\right).$$

□

Assume that $\sigma(\lambda) = \sigma_1(\lambda) + i\sigma_2(\lambda)$ is a complex bounded variation on the entire line. Set

$$\begin{aligned}\varphi(z) &= \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z},\ \psi(\sigma,\tau) = \frac{sgn\tau}{\pi}\frac{\varphi(z) - \varphi(\overline{z})}{2i} \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|\, d\sigma(\lambda)}{(\lambda-\sigma)^2 + \tau^2},\ z = \sigma + i\tau.\end{aligned}$$

Theorem 16 ([20])**.** *Let the points a, b are points of continuity of $\sigma(\lambda)$. Then we obtain*

$$\sigma(b) - \sigma(a) = \lim_{\tau\to 0}\int_a^b -\psi(\sigma,\tau)\, d\sigma.$$

Theorem 17. *Let the endpoints of $\Delta = (\lambda, \lambda+\Delta)$ be the points of continuity of $\varrho(\lambda)$. Then, we deduce that*

$$\varrho(\lambda+\Delta) - \varrho(\lambda) = \frac{1}{\pi}\lim_{\delta\to 0}\int_\Delta -\operatorname{Im}\{m(u+i\delta)\}\, du. \tag{33}$$

Proof. Let $f(.), g(.) \in L_q^2[0,\infty)$ vanish outside a finite interval. By (23), we deduce that

$$\begin{aligned}y(\lambda) &= \int_0^{\infty} (Rf)(x,z)\, g(x)\, d_q x \\ &= \int_{-\infty}^{\infty} \frac{F(\lambda)\, G(\lambda)}{\lambda - z} d\varrho(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{\lambda - z},\end{aligned}$$

where

$$\rho(\Delta) = \int_\Delta F(\lambda)\, G(\lambda)\, d\varrho(\lambda).$$

It follows from Theorem 16 that

$$\rho(\Delta) = -\frac{1}{\pi}\lim_{\delta\to 0}\int_\Delta \operatorname{Im}\{\psi(u+i\delta)\}\, du. \tag{34}$$

Furthermore, we have

$$\begin{aligned}&\operatorname{Im}\{\psi(u+i\delta)\} = \int_0^{\infty} g(x)\, d_q x \\ &\times\{\int_0^x [\theta(x,u+i\delta) + m(u+i\delta)\varphi(x,u+i\delta)]\,\varphi(t,u+i\delta)\, f(t)\, d_q t \\ &+ \int_x^{\infty} [\theta(t,u+i\delta) + m(u+i\delta)\varphi(t,u+i\delta)]\,\varphi(x,u+i\delta)\, f(t)\, d_q t\},\end{aligned}$$

where $\theta(x,u)$, $\varphi(x,u)$, $g(x)$ and $f(x)$ are real-valued functions. It follows from (34) and Lemma 15 that

$$\rho(\Delta) = \frac{1}{\pi}\lim_{\delta\to 0}\int_\Delta -\operatorname{Im}\{m(u+i\delta)\}\, G(u)\, F(u)\, du. \tag{35}$$

If we choose $g(x)$ and $f(x)$ conveniently, we can make $G(u)$ and $F(u)$ differ as little from unity in the fixed interval Δ. From Lemma 15 and (33), we get the desired result. □

Theorem 18. *Let $z \notin \mathbb{R}$. Then we have*

$$m(z) = -\cot\beta + \int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{\lambda - z}. \tag{36}$$

Proof. It follows from (18) that

$$G(x,t,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)\,\varphi(t,\lambda)\,d\varrho(\lambda)}{\lambda - z}, \tag{37}$$

since $f(x)$ is an arbitrary function. By definition, we get

$$G(x,t,z) = \begin{cases} [\theta(t,z) + m(z)\,\varphi(t,z)]\,\varphi(x,z)\,, & t > x \\ [\theta(x,z) + m(z)\,\varphi(x,z)]\,\varphi(t,z)\,, & t \leq x. \end{cases}$$

By virtue of (6) and (37), we conclude that

$$\begin{aligned} G(0,0,z) &= \sin\beta\,\{\cos\beta + m(z)\sin\beta\} \\ &= \int_{-\infty}^{\infty} \frac{\sin^2\beta}{\lambda - z}\,d\varrho(\lambda)\,, \end{aligned}$$

i.e.,

$$m(z) = -\cot\beta + \int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{\lambda - z}.$$

□

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