ON THE RESOLVENT OF SINGULAR $q$-STURM-LIOUVILLE OPERATORS

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Abstract. In this paper, we investigate the resolvent operator of the singular $q$-Sturm-Liouville problem defined as

$$-\frac{1}{q} D_{q^{-1}} [D_q y(x)] + \frac{u(x)}{q} y(x) = 0,$$

with the boundary condition

$$y(0, \lambda) \cos \beta + D_{q^{-1}} y(0, \lambda) \sin \beta = 0,$$

where $\lambda \in \mathbb{C}$, $r$ is a real-valued function defined on $[0, \infty)$, continuous at zero and $r \in L^1_q(0, \infty)$. We give a representation for the resolvent operator and investigate some properties of this operator. Furthermore, we obtain a formula for the Titchmarsh-Weyl function of the singular $q$-Sturm-Liouville problem.

1. Introduction

Quantum (or $q$) calculus is a very interesting field in mathematics. It has numerous in statistic physics, quantum theory, the calculus of variations and number theory; see, e.g., [12, 11, 14, 15, 18, 21, 24]). The first results in $q$-calculus belong to the Euler. In 2005, Annaby and Mansour investigated $q$-Sturm-Liouville problems [10]. Later in [9], the authors studied the Titchmarsh-Weyl theory for $q$-Sturm-Liouville equations. In [3, 4], the authors proved the existence of a spectral function for $q$-Sturm-Liouville operator.

In this article, we investigate the following $q$-Sturm-Liouville problem defined as

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + u(x) y(x) = \lambda y(x),$$

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where \(0 < x < \infty\). The resolvent operator for this problem is constructed. Using the spectral function, an integral representation is obtained. Furthermore, some properties of this operator are investigated. A formula for the Titchmarsh-Weyl function of Eq. (1) is given. Historically, in 1910, H. Weyl was first obtained a representation theorem for the resolvent of Sturm-Liouville problem defined by

\[-(py')' + qy = \lambda y, \quad x \in (0, \infty),\]

where \(p, q\) are real-valued and \(p^{-1}, q \in L^1_{\text{loc}}[0, \infty)\). Similar representation theorems were proved in [25, 20, 2, 5, 6, 7].

2. Preliminaries

In this section, we give a brief introduction to quantum calculus and refer the interested reader to [17, 8, 12].

Let \(0 < q < 1\) and let \(A \subset \mathbb{R}\) is a \(q\)-geometric set, i.e., \(qx \in A\) for all \(x \in A\). The Jackson \(q\)-derivative is defined by

\[D_q y(x) = \mu^{-1}(x) \left[ y(qx) - y(x) \right],\]

where \(\mu(x) = qx - x\) and \(x \in A\). We note that there is a connection of the Jackson \(q\)-derivative between and \(q\)-deformed Heisenberg uncertainty relation (see [23]). The \(q\)-derivative at zero is defined as

\[D_q y(0) = \lim_{n \to \infty} [q^n x]^{-1} [y(q^n x) - y(0)] \quad (x \in A), \quad (2)\]

if the limit in (2) exists and does not depend on \(x\). The Jackson \(q\)-integration is given by

\[\int_0^x f(t) d_q t = x (1 - q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),\]

provided that the series converges, and

\[\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t,\]

where \(a, b \in A\). The \(q\)-integration for a function over \([0, \infty)\) defined by the formula (13)

\[\int_0^\infty f(t) d_q t = \sum_{n=-\infty}^{\infty} q^n f(q^n).\]

Let \(f\) be a function on \(A\) and let \(0 \in A\). For every \(x \in A\), if

\[\lim_{n \to -\infty} f(xq^n) = f(0),\]

then \(f\) is called \(q\)-regular at zero. Throughout the paper, we deal only with functions \(q\)-regular at zero.
The following relation holds
\[ \int_0^a g(t) D_q f(t) d_q t + \int_0^a f(qt) D_q g(t) d_q t = f(a) g(a) - f(0) g(0), \]
where \( f \) and \( g \) are \( q \)-regular at zero.

Let \( L_q(0, \infty) \) be the Hilbert space consisting of all functions \( f \) satisfying (9)
\[ k f := \sqrt{\int_0^\infty |f(x)|^2 d_q x} < \infty \]
with the inner product
\[ (f, g) := \int_0^\infty f(x) \overline{g(x)} d_q x. \]

The \( q \)-Wronskian of the functions \( y(.) \) and \( z(.) \) is defined by the formula
\[ W_q(y, z)(x) := y(x) D_q z(x) - z(x) D_q y(x), \]
where \( x \in [0, \infty) \).

3. Main Results

Consider the \( q \)-Sturm-Liouville equation
\[ L(y) := \frac{1}{q} D_{q-1} D_q y(x) + r(x) y(x) = \lambda y(x), \]
satisfying the conditions
\[ y(0, \lambda) \cos \beta + D_{q-1} y(0, \lambda) \sin \beta = 0, \]
\[ y(q^{-n}, \lambda) \cos \alpha + D_{q-1} y(q^{-n}, \lambda) \sin \alpha = 0, \]
where \( \lambda \in \mathbb{C} \), \( r \) is a real-valued function defined on \([0, \infty)\), continuous at zero and \( r \in L_{q,loc}^1(0, \infty) \).

Let \( \varphi(x, \lambda) \) and \( \theta(x, \lambda) \) be the solutions of the Eq. (3) satisfying the following conditions
\[ \varphi(0, \lambda) = \sin \beta, \quad D_{q-1} \varphi(0, \lambda) = -\cos \beta, \]
\[ \theta(0, \lambda) = \cos \beta, \quad D_{q-1} \theta(0, \lambda) = \sin \beta. \]

**Lemma 1** (9). Let \( \lambda \notin \mathbb{R} \) and let
\[ \chi_{q^{-n}}(x, \lambda) = \theta(x, \lambda) + l(\lambda, q^{-n}) \varphi(x, \lambda) \in L_q^2(0, \infty), \]
where \( n \in \mathbb{N} \). Then we have
\[ \chi_{q^{-n}}(x, \lambda) \rightarrow \chi(x, \lambda), \]
\[ \int_0^\infty |\chi_{q^{-n}}(qt, \lambda)|^2 d_q x \rightarrow \int_0^\infty |\chi(x, \lambda)|^2 d_q x, \quad n \rightarrow \infty. \]
Putting
\[ G_{q^{-n}}(x, t, \lambda) = \begin{cases} 
\chi_{q^{-n}}(x, \lambda) \varphi(t, \lambda), & t \leq x \\
\varphi(x, \lambda) \chi_{q^{-n}}(t, \lambda), & t > x,
\end{cases} \]
\[ y(x, \lambda) := (R_{q^{-n}} f)(x, \lambda) = \int_0^{q^{-n}} G_{q^{-n}}(x, t, \lambda) f(t) \, dq_t, \quad (\lambda \in \mathbb{C}, \ \text{Im} \lambda \neq 0), \]  
where \( f \in L_q^2[0, q^{-n}] \). Now, we shall show that the equality (7) satisfies the equation
\[ L(y) - \lambda y(x) = f(x), \ x \in (0, q^{-n}) \ (\lambda \in \mathbb{C}, \ \text{Im} \lambda \neq 0) \]  
and the boundary conditions (4)-(5). From (7), we get
\[ y(x, \lambda) = q \chi_{q^{-n}}(x, \lambda) \int_0^x \varphi(qt, \lambda) f(qt) \, dq_t + q \varphi(x, \lambda) \int_x^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t. \]  
From (8), it follows that
\[ D_q y(x, \lambda) = q D_q \chi_{q^{-n}}(x, \lambda) \int_0^x \varphi(qt, \lambda) f(qt) \, dq_t + q D_q \varphi(x, \lambda) \int_x^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t, \]
and
\[ D_{q^{-1}} D_q y(x, \lambda) = q D_{q^{-1}} D_q \chi_{q^{-n}}(x, \lambda) \int_0^x \varphi(qt, \lambda) f(qt) \, dq_t + q D_{q^{-1}} D_q \varphi(x, \lambda) \int_x^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t \]
\[ + q W_q(\chi_{q^{-n}}, \varphi) f(x). \]
Hence, by \( W_q(\varphi, \chi_{q^{-n}}) = 1 \ (n \in \mathbb{N}) \), we deduce that
\[ -\frac{1}{q} D_{q^{-1}} D_q y(x, \lambda) = (\lambda - r(x)) q \chi_{q^{-n}}(x, \lambda) \int_0^x \varphi(qt, \lambda) f(qt) \, dq_t + (\lambda - r(x)) q \varphi(x, \lambda) \int_x^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t + f(x) \]
\[ = (\lambda - r(x)) y(x, \lambda) + f(x), \]
i.e., the function $y(x, \lambda)$ satisfies the equation $L(y) - \lambda y(x) = f(x)$, $x \in (0, q^{-n})$.

Moreover,

$$y(0, \lambda) = q \varphi(0, \lambda) \int_0^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t$$

$$= q \cos \beta \int_0^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t,$$

$$D_{q^{-1}} y(0, \lambda) = q D_{q^{-1}} \varphi(0, \lambda) \int_0^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t$$

$$= -q \sin \beta \int_0^{q^{-n}} \chi_{q^{-n}}(qt, \lambda) f(qt) \, dq_t,$$

i.e., $y(x, \lambda)$ satisfies (4). Similarly, we may infer that $y(x, \lambda)$ satisfies (5).

Note that the problem (3)-(5) has a purely discrete spectrum [10].

Let $\lambda_{m,q^{-n}}$ be the eigenvalues of the problem (3)-(5). Let $\varphi_{m,q^{-n}}$ be the corresponding eigenfunctions and

$$\alpha_{m,q^{-n}} := \| \varphi_{m,q^{-n}} \|^2 = \left( \int_0^{q^{-n}} \varphi_{m,q^{-n}}^2(x) \, dq_x \right)^{\frac{1}{2}},$$

where $\varphi_{m,q^{-n}}(x) := \varphi_{m,q^{-n}}(x, \lambda_{m,q^{-n}})$ and $m \in \mathbb{N}$.

Then we have the following Parseval equality (see [8])

$$\int_0^{q^{-n}} |f(x)|^2 \, dq_x = \sum_{m=1}^{\infty} \frac{1}{\alpha^2_{m,q^{-n}}} \left\{ \int_0^{q^{-n}} f(x) \varphi_{m,q^{-n}}(x) \, dq_x \right\}^2,$$  \hspace{1cm} (9)

where $f(.) \in L^2_q[0, q^{-n}]$.

Now, let us define the nondecreasing step function $\varphi_{q^{-n}}$ on $[0, \infty)$ by

$$\varphi_{q^{-n}}(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_{m,q^{-n}} < 0} \frac{1}{\alpha^2_{m,q^{-n}}} \varphi_{m,q^{-n}}(x) & \text{for } \lambda \leq 0, \\ \sum_{0 \leq \lambda < \lambda_{m,q^{-n}} < \lambda} \frac{1}{\alpha^2_{m,q^{-n}}} \varphi_{m,q^{-n}}(x) & \text{for } \lambda > 0. \end{cases}$$

It follows from (9) that

$$\int_0^{q^{-n}} |f(x)|^2 \, dq_x = \int_{-\infty}^{\infty} F^2(\lambda) \, d\varphi_{q^{-n}}(\lambda),$$  \hspace{1cm} (10)

where

$$F(\lambda) = \int_0^{q^{-n}} f(x) \varphi(x, \lambda) \, dq_x.$$
Lemma 2. Let $\kappa > 0$. Then the following relation holds
\[
\bigvee_{-\kappa}^{\kappa} \{ q_{\kappa}^{-n} (\lambda) \} = \sum_{-\kappa \leq \lambda_{m,q^{-n}} < \kappa} \frac{1}{\alpha_{m,q^{-n}}^2} = q_{\kappa}^{-n} (\kappa) - q_{\kappa}^{-n} (-\kappa) < \Upsilon,
\]
where $\Upsilon = \Upsilon (\kappa)$ is a positive constant not depending on $q^{-n}$.

Proof. Let $\sin \beta \neq 0$. Since $\varphi (x, \lambda)$ is continuous at zero, by condition $\varphi (0, \lambda) = \sin \beta$, there exists a positive number $h$ and nearby 0 such that
\[
|\varphi (x, \lambda)| > \frac{1}{\sqrt{2}} |\sin \beta|, \quad 0 \leq x \leq h
\]
and
\[
\left( \frac{1}{h} \int_0^h \varphi (x, \lambda) \, dq \right)^2 > \left( \frac{1}{\sqrt{2h}} \sin \beta \int_0^h dq \right)^2 = \frac{1}{2} \sin^2 \beta. \quad (12)
\]
Let us define $f_h (x)$ by
\[
f_h (x) = \begin{cases} 
0, & x > h \\
\frac{1}{h}, & 0 \leq x \leq h.
\end{cases}
\]
It follows from (10) and (12) that
\[
\int_0^h f_h^2 (x) \, dq = \frac{1}{h} = \int_{-\infty}^{\infty} \left( \frac{1}{h} \int_0^h \varphi (x, \lambda) \, dq \right)^2 \, dq_{\kappa}^{-n} (\lambda)
\]
\[
\geq \int_{-\kappa}^{\kappa} \left( \frac{1}{h} \int_0^h \varphi (x, \lambda) \, dq \right)^2 \, dq_{\kappa}^{-n} (\lambda)
\]
\[
> \frac{1}{2} \sin^2 \beta \{ q_{\kappa}^{-n} (\kappa) - q_{\kappa}^{-n} (-\kappa) \},
\]
which proves the inequality (11).
Let $\sin \beta = 0$ and
\[
f_h (x) = \begin{cases} 
0, & x > h \\
\frac{1}{h^2}, & 0 \leq x \leq h.
\end{cases}
\]
By (10), we can get the desired result. \qed

We now return to the formula (7), whose right-hand side has been called the resolvent. The resolvent is known to exist for all $\lambda$ which are not eigenvalues of the problem (3)-(5). Now, we will get the expansion of the resolvent.

Since the function $y (x, \lambda)$ satisfies the equation $L(y) - \lambda y (x) = f(x), x \in (0, q^{-n})$ ($\lambda \in \mathbb{C}, \lambda \neq \lambda_{m,q^{-n}}, m \in \mathbb{N}$) and the boundary conditions (4), (5), via the $q$-integration by parts, we find (the operator $A$ generated by the expression $L$ and the boundary conditions (4), (5) is a self-adjoint (see [10]))
\[
(Ay, \varphi_{m,q^{-n}})
\]
The set of all eigenfunctions \( \varphi_{m,q^{-n}}(x) \) (\( m \in \mathbb{N} \)) of the self-adjoint operator \( A \) form an orthonormal basis for \( L^2_q(0, q^{-n}) \) (see [10]). Then, the function \( y(\cdot, \lambda) \in L^2_q(0, q^{-n}) \) (\( \lambda \in \mathbb{C}, \lambda \neq \lambda_{m,q^{-n}}, m \in \mathbb{N} \)) can be expanded into Fourier series of eigenfunctions \( \varphi_{m,q^{-n}}(x) \) (\( m \in \mathbb{N} \)) of the problem (3)-(5) (or of the operator \( A \)). Then we have

\[
y(x, \lambda) = \sum_{m=1}^{\infty} t_m(\lambda) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}},
\]

where \( t_m(\lambda) \) is the Fourier coefficient, i.e.,

\[
t_m(\lambda) = \int_0^{q^{-n}} y(x, \lambda) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} dx, \quad m \in \mathbb{N}.
\]

Since \( y(x, \lambda) \) (\( \lambda \in \mathbb{C}, \lambda \neq \lambda_{m,q^{-n}}, m \in \mathbb{N} \)) satisfies the equation

\[
-\frac{1}{q} D_{q^{-1}} D_q y(x, \lambda) + (r(x) - \lambda) y(x, \lambda) = f(x), \quad x \in (0, q^{-n}),
\]

we get

\[
a_m : = \int_0^{q^{-n}} f(x) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} dx
\]

\[
= \int_0^{q^{-n}} \left[ -\frac{1}{q} D_{q^{-1}} D_q y(x, \lambda) + (r(x) - \lambda) y(x, \lambda) \right] \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} dx
\]

\[
= \int_0^{q^{-n}} \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{m,q^{-n}}(x) + (r(x) - \lambda) \varphi_{m,q^{-n}}(x) \right] \frac{y(x, \lambda)}{\alpha_{m,q^{-n}}} dx
\]

\[
= \int_0^{q^{-n}} \left[ \lambda_{m,q^{-n}} \varphi_{m,q^{-n}}(x) - \lambda \varphi_{m,q^{-n}}(x) \right] \frac{y(x, \lambda)}{\alpha_{m,q^{-n}}} dx
\]

\[
= \lambda_{m,q^{-n}} t_m(\lambda) - \lambda t_m(\lambda), \quad m \in \mathbb{N}.
\]
Thus, we have
\[ t_m (\lambda) = \frac{a_m}{\lambda_{m,q^{-n}} - \lambda}, \]
and
\[ y (x, \lambda) = \int_0^{q^{-n}} G_{q^{-n}} (x, t, \lambda) f (t) d_q t \]
\[ = \sum_{m=1}^{\infty} \frac{a_m}{\lambda_{m,q^{-n}} - \lambda} \varphi_{m,q^{-n}} (x) \ (\lambda \in \mathbb{C}, \ \lambda \neq \lambda_{m,q^{-n}}, \ m \in \mathbb{N}). \]
Then
\[ y (x, z) = (R_{q^{-n}} f) (x, z) \]
\[ = \sum_{m=1}^{\infty} \frac{\varphi_{m,q^{-n}} (x)}{\alpha_{m,q^{-n}} (\lambda_{m,q^{-n}} - z)} \int_0^{q^{-n}} f (t) \varphi_{m,q^{-n}} (t) d_q t \]
\[ = \int_{-\infty}^{\infty} \frac{\varphi (x, \lambda)}{\lambda - z} \left\{ \int_0^{q^{-n}} f (t) \varphi_{m,q^{-n}} (t, \lambda) d_q t \right\} d_q q^{-n} (\lambda). \quad (13) \]

**Lemma 3.** The following formula holds
\[ \int_{-\infty}^{\infty} \left| \frac{\varphi (x, \lambda)}{\lambda - z} \right|^2 d_q q^{-n} (\lambda) < K, \]
where \( x \) is a fixed number and \( z \) is a non-real number.

**Proof.** Let \( f (t) = \frac{\varphi_{m,q^{-n}} (t)}{\alpha_{m,q^{-n}}} \). By (13), we conclude that
\[ \frac{1}{\alpha_{m,q^{-n}}} \int_0^{q^{-n}} G_{q^{-n}} (x, t, z) \varphi_{m,q^{-n}} (t) d_q t = \frac{\varphi_{m,q^{-n}} (x)}{\alpha_{m,q^{-n}} (\lambda_{m,q^{-n}} - z)}. \quad (15) \]
Under (15) and (9), we see that
\[ \int_0^{q^{-n}} |G_{q^{-n}} (x, t, z)|^2 d_q t = \sum_{m=1}^{\infty} \frac{|\varphi_{m,q^{-n}} (x)|^2}{\alpha_{m,q^{-n}} (\lambda_{m,q^{-n}} - z)^2} \]
\[ = \int_{-\infty}^{\infty} \left| \frac{\varphi (x, \lambda)}{\lambda - z} \right|^2 d_q q^{-n} (\lambda). \]
It follows from Lemma 1 that the last integral is convergent. The proof is complete.

Now, we present below for the convenience of the reader.
Theorem 4 ([19]). Let \((w_n)_{n \in \mathbb{N}}\) be a uniformly bounded sequence of real non-decreasing function on a finite interval \([a, b]\). Then

(i) there exists a subsequence \((w_{n_k})_{k \in \mathbb{N}}\) and a non-decreasing function \(w\) such that

\[
\lim_{k \to \infty} w_{n_k}(\lambda) = w(\lambda),
\]

where \(a \leq \lambda \leq b\).

(ii) suppose

\[
\lim_{n \to \infty} w_n(\lambda) = w(\lambda),
\]

where \(a \leq \lambda \leq b\). Then, we have

\[
\lim_{n \to \infty} \int_a^b f(\lambda) \, dw_n(\lambda) = \int_a^b f(\lambda) \, dw(\lambda),
\]

where \(f \in C[a, b]\).

By Lemma 2 and Theorem 4, one can find a sequence \(\{q^{-n_k}\}\) such that

\[
\lim_{k \to \infty} \theta_{q^{-n_k}}(\lambda) = q(\lambda),
\]

where \(q(\lambda)\) is a monotone function.

Lemma 5. Let \(z \notin \mathbb{R}\). Then we have

\[
\int_{-\infty}^{\infty} \left| \frac{\varphi(x, \lambda)}{\lambda - z} \right|^2 \, d\varphi(\lambda) \leq K,
\]

where \(x\) is a fixed number.

Proof. Let \(\eta > 0\). It follows from (16) that

\[
\int_{-\eta}^{\eta} \left| \frac{\varphi(x, \lambda)}{\lambda - z} \right|^2 \, d\theta_{q^{-n}}(\lambda) < K.
\]

Then

\[
\int_{-\infty}^{\infty} \left| \frac{\varphi(x, \lambda)}{\lambda - z} \right|^2 \, d\varphi(\lambda) = \lim_{n \to \infty} \int_{-\eta}^{\eta} \left| \frac{\varphi(x, \lambda)}{\lambda - z} \right|^2 \, d\theta_{q^{-n}}(\lambda) < K.
\]

Lemma 6. Let \(\eta > 0\). Then we have

\[
\int_{-\infty}^{0} \frac{d\varphi(\lambda)}{\lambda - z} < \infty, \quad \int_{0}^{\infty} \frac{d\varphi(\lambda)}{\lambda - z} < \infty
\]

(17)

Proof. Let \(\sin \beta \neq 0\). From (16), we deduce that

\[
\int_{-\infty}^{\infty} \frac{d\varphi(\lambda)}{\lambda - z} < \infty.
\]
Let $\sin \beta = 0$. Hence we see that

$$
\frac{1}{\alpha_{m,q^{-n}}} \int_0^{q^{-n}} \varphi_{m,q^{-n}}(t) D_{q,x} \left[ G_{q^{-n}}(x, t, z) \right] d_q t = \frac{D_{q,x} \varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}} (\lambda_{m,q^{-n}} - z)}.
$$

It follows from (9) that

$$
\int_0^{q^{-n}} \left| D_{q,x} \left[ G_{q^{-n}}(x, t, z) \right] \right|^2 d_q t = \int_{-\infty}^{\infty} \left| \frac{D_{q,x} \varphi(x, \lambda)}{\lambda - z} \right|^2 d\varphi_{q^{-n}}(\lambda).
$$

Proceeding similarly, we can get the desired result. \hfill \Box

**Lemma 7.** Let

$$
\begin{align*}
G(x, t, z) &= \begin{cases} 
\chi(x, z) \varphi(t, z), & x \geq t \\
\varphi(x, z) \chi(t, z), & x < t, 
\end{cases} 
\end{align*}
$$

and let $f(.) \in L_2^q[0, \infty)$. Then we have

$$
\int_0^{\infty} \left| (Rf)(x, z) \right|^2 d_q x \leq \frac{1}{\nu^2} \int_0^{\infty} \left| f(x) \right|^2 d_q x,
$$

where

$$
(Rf)(x, z) = \int_0^{\infty} G(x, t, z) f(t) d_q t,
$$

and $z = u + iv$.

**Proof.** See [9]. \hfill \Box

Now we shall state the main result of this paper.

**Theorem 8.** The following relation holds

$$
(Rf)(x, z) = \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda)}{\lambda - z} F(\lambda) d\varphi(\lambda),
$$

where $f(.) \in L_2^q[0, \infty)$,

$$
F(\lambda) = \lim_{\xi \to \infty} \int_0^{q^{-\xi}} f(x) \varphi(x, \lambda) d_q x,
$$

and $z \notin \mathbb{R}$.

**Proof.** Define the function $f_\xi(x)$ as

$$
\begin{align*}
f_\xi(x) &= \begin{cases} 
f_{\xi}(x), & x \in [0, q^{-\xi}] \\
0, & x \notin [0, q^{-\xi}], \quad \left(q^{-\xi} < q^{-n}\right)
\end{cases}
\end{align*}
$$

such that $f_\xi(x)$ satisfies [4]. By [14], we conclude that

$$
\begin{align*}
(R_{q^{-n}}f_\xi)(x, z) &= \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda)}{\lambda - z} F_\xi(\lambda) d\varphi_{q^{-n}}(\lambda) = \int_{-\infty}^{-a} \frac{\varphi(x, \lambda)}{\lambda - z} F_\xi(\lambda) d\varphi_{q^{-n}}(\lambda)
\end{align*}
$$
\[ + \int_{-a}^{a} \frac{\varphi(x, \lambda)}{\lambda - z} F_\xi(\lambda) \, d\vartheta_{q^{-n}}(\lambda) + \int_{a}^{\infty} \frac{\varphi(x, \lambda)}{\lambda - z} F_\xi(\lambda) \, d\vartheta_{q^{-n}}(\lambda) \]

\[ = I_1 + I_2 + I_3, \quad \text{(19)} \]

where

\[ F_\xi(\lambda) = \int_{0}^{q^{-\tau}} f(x) \varphi(x, \lambda) \, dq x, \]

and \( a > 0 \).

It follows from (13) that

\[ |I_1| = \left| \int_{-\infty}^{-a} \frac{\varphi(x, \lambda)}{\lambda - z} F_\xi(\lambda) \, d\vartheta_{q^{-n}}(\lambda) \right| \]

\[ \leq \sum_{\lambda_{k,q^{-n}} < -a} \frac{|\varphi_{k,q^{-n}}(x)|}{\alpha_{k,q^{-n}}^2} \left| \int_{0}^{q^{-\tau}} f_\xi(t) \varphi_{k,q^{-n}}(t) \, dq t \right| \]

\[ \leq \left( \sum_{\lambda_{k,q^{-n}} < -a} \frac{\varphi_{k,q^{-n}}^2(x)}{\alpha_{k,q^{-n}}^2} \right)^{1/2} \]

\[ \times \left( \sum_{\lambda_{k,q^{-n}} < -a} \frac{1}{\alpha_{k,q^{-n}}^2} \left[ \int_{0}^{q^{-\tau}} f_\xi(t) \varphi_{k,q^{-n}}(t) \, dq t \right]^2 \right)^{1/2}. \quad \text{(20)} \]

Using the \( q \)-integration-by-parts formula in the integral below, we have

\[ \int_{0}^{q^{-\tau}} f_\xi(x) \varphi_{k,q^{-n}}(x) \, dq x \]

\[ = \frac{1}{\lambda_{k,q^{-n}}} \int_{0}^{q^{-\tau}} f_\xi(x) \left\{ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{k,q^{-n}}(x) + r(x) \varphi_{k,q^{-n}}(x) \right\} \, dq x \]

\[ = \frac{1}{\lambda_{k,q^{-n}}} \int_{0}^{q^{-\tau}} \left\{ -\frac{1}{q} D_{q^{-1}} D_q f_\xi(x) + r(x) f_\xi(x) \right\} \varphi_{k,q^{-n}}(x) \, dq x. \quad \text{(21)} \]

From Lemma 3, we get

\[ |I_1| \leq \frac{K^{1/2}}{a} \left( \sum_{\lambda_{k,q^{-n}} < -a} \frac{1}{\alpha_{k,q^{-n}}^2} \right)^{1/2}. \]
Application of Bessel inequality yields
\[
|I_1| \leq \frac{K^{1/2}}{a} \left[ \int_0^{q^{-1}} \left\{ -\frac{1}{q} D_{q^{-1}} D_q f_\xi(x) + r(x)f_\xi(x) \right\}^2 d_q x \right]^{1/2} = \frac{C}{a}.
\]
Likewise, we show that \( |I_3| \leq \frac{C}{a} \). Then \( I_1, I_3 \to 0 \), as \( a \to \infty \), uniformly in \( q^{-n} \). By virtue of (19) and Theorem 4, we see that
\[
(Rf_\xi)(x, z) = \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda)}{\lambda - z} F_\xi(\lambda) d\varphi(\lambda).
\]
We can find a sequence \( \{f_\xi(x)\}_{\xi=1}^\infty \) which satisfies the previous conditions and tends to \( f(x) \) as \( \xi \to \infty \), since \( f(.) \in L_q^2[0, \infty) \). It follows from (9) that the sequence of Fourier transform converges to the transform of \( f(x) \). Using Lemmas 5 and 7, one can pass to the limit \( \xi \to \infty \) in (22).

**Remark 9.** The following formula holds.
\[
\int_0^\infty (Rf)(x, z) g(x) d_q x = \int_{-\infty}^{\infty} \frac{F(\lambda) G(\lambda)}{\lambda - z} d\varphi(\lambda),
\]
where
\[
G(\lambda) = \lim_{\xi \to \infty} \int_0^{q^{-\xi}} g(x) \varphi(x, \lambda) d_q x,
\]
and
\[
F(\lambda) = \lim_{\xi \to \infty} \int_0^{q^{-\xi}} f(x) \varphi(x, \lambda) d_q x.
\]

Now, we will study some properties of the resolvent operator. We give the following definition and theorems.

**Definition 10.** Let \( M(x, t) \) be a complex-valued function, where \( x, t \in (a, b) \). If
\[
\int_a^b \int_a^b |M(x, t)|^2 d_q x d_q t < +\infty,
\]
then \( M(x, t) \) is called the \( q \)-Hilbert-Schmidt kernel.

**Theorem 11 (22).** Let us define the operator \( A \) as
\[
A\{x_i\} = \{y_i\},
\]
where
\[
y_i = \sum_{k=1}^{\infty} a_{ik} x_k, \ i \in \mathbb{N}.
\]
If
\[
\sum_{i,k=1}^{\infty} |a_{ik}|^2 < +\infty
\]
then \( A \) is a compact operator in the sequence space \( l^2 \).
Theorem 12. Let the limit circle case holds for Eq. (3) and
\[ G(x, t) = G(x, t, 0) = \begin{cases} \varphi(x) \chi(t), & x < t \\ \chi(x) \varphi(t), & x \geq t. \end{cases} \tag{26} \]
Then the function \( G(x, t) \) defined by (26) is a \( q \)-Hilbert-Schmidt kernel.

**Proof.** It follows from (26) that
\[ \int_0^\infty d_q x \int_0^x |G(x, t)|^2 d_q t < +\infty, \]
and
\[ \int_0^\infty d_q x \int_x^{\infty} |G(x, t)|^2 d_q t < +\infty, \]
since the integrals
\[ \int_0^\infty |G(x, t)|^2 d_q x \]
and
\[ \int_0^\infty |G(x, t)|^2 d_q t \]
exist and are a linear combination of the products \( \varphi(x) \chi(t) \), and these products belong to \( L^2_q[0, \infty) \times L^2_q[0, \infty) \). Then
\[ \int_0^\infty \int_0^{\infty} |G(x, t)|^2 d_q x d_q t < +\infty. \tag{27} \]

Theorem 13. Let us define the operator \( R \) as
\[ (Rf)(x) = \int_0^\infty G(x, t) f(t) d_q t \]
Under the assumptions of Theorem 12, \( R \) is a compact operator.

**Proof.** Let \( \phi_i = \phi_i(t) \) \((i \in \mathbb{N})\) be a complete, orthonormal basis of \( L^2_q[0, \infty) \). By Theorem 12, we can define
\[ x_i = (f, \phi_i) = \int_0^\infty \overline{\phi_i(t)} f(t) d_q t, \]
\[ y_i = (g, \phi_i) = \int_0^\infty \overline{\phi_i(t)} g(t) d_q t, \]
\[ a_{ik} = \int_0^\infty \int_0^\infty \overline{\phi_k(t)\phi_i(x)} G(x, t) d_q x d_q t, \]
where \( i, k \in \mathbb{N} \). Then, \( L^2_q[0, \infty) \) is mapped isometrically \( l^2 \). Therefore, the operator \( R \) transforms into \( A \) defined by (24) in \( l^2 \) by this mapping, and (27) is translated into (25). It follows from Theorem 11 that \( A \) is compact operator. Consequently, \( R \) is a compact operator. \( \square \)
Now, we will give some auxiliary lemmas.

**Lemma 14.** The following equalities hold.

\[
\lim_{x \to \infty} W_q (\chi (x, \lambda), \chi (x, \lambda')) = 0,
\]

\[
\int_{0}^{\infty} \chi (x, \lambda) \chi (x, \lambda') \, d_q x = \frac{m (\lambda) - m (\lambda')}{\lambda - \lambda'},
\]

where \(\lambda\) and \(\lambda'\) are any fixed nonreal numbers.

*Proof.* See [9].

Using (29) and setting \(\lambda = u + i \delta\) and \(\lambda' = \bar{\lambda}\), we obtain

\[
\int_{0}^{\infty} |\chi (x, \lambda)|^2 \, d_q x = - \frac{\text{Im} \{m (\lambda)\}}{\nu}.
\]

**Lemma 15.** For fixed \(u_1\) and \(u_2\), we have

\[
\int_{u_1}^{u_2} - \text{Im} \{m (u + i \delta)\} \, du = O (1), \quad \text{as } \delta \to 0.
\]

*Proof.* Let \(\sin \beta \neq 0\). It follows from [9] and (18) that

\[
\int_{0}^{\infty} |\chi (t, z)|^2 \, d_q t = \int_{-\infty}^{\infty} \frac{d \varphi (\lambda)}{(u - \lambda)^2 + \nu^2}.
\]

where \(z = u + iv\).

Let \(\sin \beta = 0\). If the equality (15) is \(q\)-differentiated throughout with respect to \(x\), and the limit is taken as \(n \to \infty\), then we can get the desired result.

By virtue of (30) and (32), we conclude that

\[- \text{Im} \{m (u + i \delta)\} = \delta \int_{-\infty}^{\infty} \frac{d \varphi (\lambda)}{(u - \lambda)^2 + \delta^2}.
\]

Then we have

\[- \int_{u_1}^{u_2} \text{Im} \{m (u + i \delta)\} \, du = \delta \int_{u_1}^{u_2} du \int_{-\infty}^{\infty} \frac{d \varphi (\lambda)}{(u - \lambda)^2 + \delta^2}.
\]

Let \((a, b)\) be a finite interval where \(a < u_1\) and \(b > u_2\). From (17), we see that

\[
\delta \int_{u_1}^{u_2} du \int_{-\infty}^{a} \frac{d \varphi (\lambda)}{(u - \lambda)^2 + \delta^2} = O (1),
\]

\[
\delta \int_{u_1}^{u_2} du \int_{b}^{\infty} \frac{d \varphi (\lambda)}{(u - \lambda)^2 + \delta^2} = O (1).
\]

Hence, we get

\[
\delta \int_{u_1}^{u_2} du \int_{a}^{b} \frac{d \varphi (\lambda)}{(u - \lambda)^2 + \delta^2} = \int_{a}^{b} d \varphi (\lambda) \int_{\frac{u_2 - \lambda}{u_1 - \lambda}}^{u_2 - \lambda} \frac{dv}{1 + v^2} = O (1).
\]

\(\square\)
Assume that \( \sigma (\lambda) = \sigma_1 (\lambda) + i \sigma_2 (\lambda) \) is a complex bounded variation on the entire line. Set
\[
\varphi (z) = \int_{-\infty}^{\infty} \frac{d\sigma (\lambda)}{\lambda - z}, \quad \psi (\sigma, \tau) = \frac{\text{sgn} \varphi (z) - \varphi (z)}{2i}
\]
\[
= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\tau| d\sigma (\lambda)}{(\lambda - \sigma)^2 + \tau^2}, \quad z = \sigma + i\tau.
\]

**Theorem 16 (20).** Let the points \( a, b \) are points of continuity of \( \sigma (\lambda) \). Then we obtain
\[
\sigma (b) - \sigma (a) = \lim_{\tau \to 0} \int_{a}^{b} -\psi (\sigma, \tau) d\sigma.
\]

**Theorem 17.** Let the endpoints of \( \Delta = (\lambda, \lambda + \Delta) \) be the points of continuity of \( \varphi (\lambda) \). Then, we deduce that
\[
\varphi (\lambda + \Delta) - \varphi (\lambda) = \frac{1}{\pi} \lim_{\delta \to 0} \int_{\Delta} -\text{Im} \{ m (u + i\delta) \} du. \quad (33)
\]

*Proof.* Let \( f (.) , g (.) \in L_q [0, \infty) \) vanish outside a finite interval. By (23), we deduce that
\[
y (\lambda) = \int_{0}^{\infty} (Rf) (x, z) g (x) d_{q} x
\]
\[
= \int_{-\infty}^{\infty} \frac{F (\lambda) G (\lambda)}{\lambda - z} d\rho (\lambda) = \int_{-\infty}^{\infty} \frac{d\rho (\lambda)}{\lambda - z},
\]
where
\[
\rho (\Delta) = \int_{\Delta} F (\lambda) G (\lambda) d\theta (\lambda).
\]

It follows from Theorem 16 that
\[
\rho (\Delta) = -\frac{1}{\pi} \lim_{\delta \to 0} \int_{\Delta} \text{Im} \{ \psi (u + i\delta) \} du. \quad (34)
\]

Furthermore, we have
\[
\text{Im} \{ \psi (u + i\delta) \} = \int_{0}^{\infty} g (x) d_{q} x
\]
\[
\times \{ \int_{0}^{x} [\theta (x, u + i\delta) + m (u + i\delta) \varphi (x, u + i\delta)] \varphi (t, u + i\delta) f (t) d_{q} t
\]
\[
+ \int_{x}^{\infty} [\theta (t, u + i\delta) + m (u + i\delta) \varphi (t, u + i\delta)] \varphi (x, u + i\delta) f (t) d_{q} t \},
\]
where \( \theta (x, u), \ \varphi (x, u), g (x) \) and \( f (x) \) are real-valued functions. It follows from (34) and Lemma 15 that
\[
\rho (\Delta) = \frac{1}{\pi} \lim_{\delta \to 0} \int_{\Delta} -\text{Im} \{ m (u + i\delta) \} G (u) F (u) du. \quad (35)
\]
If we choose \( g(x) \) and \( f(x) \) conveniently, we can make \( G(u) \) and \( F(u) \) differ as little from unity in the fixed interval \( \Delta \). From Lemma 15 and (33), we get the desired result.

**Theorem 18.** Let \( z \notin \mathbb{R} \). Then we have

\[
m(z) = -\cot \beta + \int_{-\infty}^{\infty} \frac{d\phi(\lambda)}{\lambda - z}.
\]  

(36)

**Proof.** It follows from (18) that

\[
G(x, t, z) = \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda) \varphi(t, \lambda) d\phi(\lambda)}{\lambda - z},
\]

(37)

since \( f(x) \) is an arbitrary function. By definition, we get

\[
G(x, t, z) = \begin{cases} 
\theta(t, z) + m(z) \varphi(t, z), & t > x \\
\theta(x, z) + m(z) \varphi(x, z), & t \leq x.
\end{cases}
\]

By virtue of (6) and (37), we conclude that

\[
G(0, 0, z) = \sin \beta \{\cos \beta + m(z) \sin \beta\}
\]

\[
= \int_{-\infty}^{\infty} \frac{\sin^2 \beta d\phi(\lambda)}{\lambda - z},
\]

i.e.,

\[
m(z) = -\cot \beta + \int_{-\infty}^{\infty} \frac{d\phi(\lambda)}{\lambda - z}.
\]

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