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Several identities containing central binomial coefficients and derived from series expansions of powers of the arcsine function

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Abstract

In the paper, with the aid of the series expansions of the square or cubic of the arcsine function, the authors establish several possibly new combinatorial identities containing the ratio of two central binomial coefficients which are related to the Catalan numbers in combinatorial number theory.

Keywords: identity; product; ratio; central binomial coefficient; power series expansion; arcsine function; square; cubic; generating function; Catalan number
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1. Introduction

The sequence of central binomial coefficients $\binom{2n}{n}$ for $n \ge 0$ is classical, simple, and elementary. This sequence has attracted many mathematicians who have published a number of papers such as [3, 7, 10, 11, 17, 20, 27, 44, 46] and closely related references therein. It is worth to mentioning that, the integral representation

$$\binom{2n}{n} = \frac{1}{\pi}\int_0^\infty \frac{1}{(1/4+s^2)^{n+1}}\,\mathrm{d}\,s$$

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was derived in [35, Section 4.2].

In this paper, with the help of the power series expansion

$$\arcsin x = \sum_{\ell=0}^{\infty} \frac{1}{2^{2\ell}} \binom{2\ell}{\ell} \frac{x^{2\ell+1}}{2\ell+1}, \quad |x| < 1, \tag{1.1}$$

see [1, 4.4.40] and [2, p. 121, 6.41.1], the series expansion

$$(\arcsin x)^2 = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{(2x)^{2\ell}}{\ell^2 \binom{2\ell}{\ell}}, \quad |x| < 1, \tag{1.2}$$

which or its variants can be found in [2, p. 122, 6.42.1], [4, pp. 262–263, Proposition 15], [5, pp. 50–51 and p. 287], [6, p. 384], [12, Lemma 2], [15, p. 308], [16, pp. 88–90], [18, p. 61, 1.645], [24, p. 453], [33, Section 6.3], [48, p. 59, (2.56)], or [50, p. 676, (2.2)], and the power series expansion

$$(\arcsin x)^3 = 3! \sum_{\ell=0}^{\infty} [(2\ell+1)!!]^2 \left[\sum_{k=0}^{\ell} \frac{1}{(2k+1)^2}\right] \frac{x^{2\ell+3}}{(2\ell+3)!}, \quad |x| < 1, \tag{1.3}$$

which or its variants can be found in [2, p. 122, 6.42.2], [4, pp. 262–263, Proposition 15], [9, p. 188, Example 1], [15, p. 308], [16, pp. 88–90], or [18, p. 61, 1.645], we will establish several identities involving the product $\binom{2k}{k}\binom{2(n-k)}{n-k}$ or the ratio $\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}$ of two central binomial coefficients $\binom{2k}{k}$ and $\binom{2(n-k)}{n-k}$ for $0 \le k \le n$.

2. Alternative proofs of a known combinatorial identity

In this section, by means of the series expansions (1.1) and (1.2), we give two alternative proofs of a known combinatorial identity. This means that the method used in this paper is better.

Theorem 2.1 ([45, p. 77, (3.96)])**.** *For $n \ge 0$, we have*

$$\sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \binom{2(n-k)}{n-k} = \frac{2^{4n}}{(2n+1)\binom{2n}{n}}. \tag{2.1}$$

First proof. From (1.1), it follows that

$$\frac{1}{2} \arcsin(2x) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k}{k} x^{2k+1}, \quad |x| < \frac{1}{2}$$

and, by differentiation,

$$\frac{1}{\sqrt{1-4x^2}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^{2k}, \quad |x| < \frac{1}{2}.$$

Therefore, we obtain

$$\frac{\arcsin(2x)}{2x\sqrt{1-4x^2}} = \left[\sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k}{k} x^{2k}\right] \sum_{k=0}^{\infty} \binom{2k}{k} x^{2k} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \binom{2(n-k)}{n-k}\right] x^{2n}. \tag{2.2}$$

On the other hand, by virtue of the series expansion (1.2), we acquire

$$\begin{aligned} \frac{\arcsin(2x)}{2x\sqrt{1-4x^2}} &= \frac{1}{8x} \frac{\mathrm{d}}{\mathrm{d}\,x} \left([\arcsin(2x)]^2\right) = \frac{1}{8x} \frac{\mathrm{d}}{\mathrm{d}\,x} \sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!} (2x)^{2n+2} \\ &= \frac{1}{8x} \sum_{n=0}^{\infty} \frac{2^{2n+2}(n!)^2}{(2n+1)!} (2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{2^{4n}(n!)^2}{(2n+1)!} x^{2n}. \end{aligned} \tag{2.3}$$

Comparing (2.2) with (2.3) and equating coefficients of x^{2n}, we obtain

$$\sum_{k=0}^{n}\frac{1}{2k+1}\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{2^{4n}(n!)^2}{(2n+1)!}=\frac{2^{4n}}{(2n+1)\binom{2n}{n}}.$$

The identity (2.1) is thus proved. The first proof of Theorem 2.1 is complete. □

Second proof. Differentiating on both sides of (1.2) and rearranging give

$$\frac{2x\arcsin x}{\sqrt{1-x^2}}=\sum_{\ell=1}^{\infty}\frac{(2x)^{2\ell}}{\ell\binom{2\ell}{\ell}},\quad |x|<1, \tag{2.4}$$

which or its variants can also be found in [2, p. 122, 6.42.5], [6, p. 384], [8, p. 161], [24, p. 452, Theorem], and [33, Section 6.3, Theorem 21, Sections 8 and 9]. Replacing x by $2x$ in (2.4) and rearranging yield

$$\begin{aligned}\frac{\arcsin(2x)}{2x}\frac{1}{\sqrt{1-4x^2}}&=\frac{1}{8x^2}\frac{4x\arcsin(2x)}{\sqrt{1-4x^2}}=\frac{1}{8x^2}\sum_{\ell=1}^{\infty}\frac{(4x)^{2\ell}}{\ell\binom{2\ell}{\ell}}\\&=\frac{1}{8x^2}\sum_{n=0}^{\infty}\frac{(4x)^{2(n+1)}}{(n+1)\binom{2(n+1)}{n+1}}=\sum_{n=0}^{\infty}\frac{2^{4n+1}}{(n+1)\binom{2(n+1)}{n+1}}x^{2n}\end{aligned} \tag{2.5}$$

for $|x|<\frac{1}{2}$. Comparing (2.2) with (2.5) and equating coefficients of x^{2n}, we obtain

$$\sum_{k=0}^{n}\frac{1}{2k+1}\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{2^{4n+1}}{(n+1)\binom{2(n+1)}{n+1}}=\frac{2^{4n}}{(2n+1)\binom{2n}{n}}.$$

The identity (2.1) is proved again. The second proof of Theorem 2.1 is complete. □

3. Three possibly new combinatorial identities

In this section, by virtue of those three series expansions (1.1), (1.2), and (1.3), we establish three possibly new combinatorial identities involving the ratio $\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}$ in terms of the trigamma function $\psi'\bigl(n+\frac{3}{2}\bigr)$, where $\psi(x)$ is the digamma function defined by the logarithmic derivative $\psi(x)=[\ln\Gamma(x)]'=\frac{\Gamma'(x)}{\Gamma(x)}$ of the classical Euler gamma function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0.$$

This means that the method used in this paper is extensively applicable. For more information on the gamma function $\Gamma(x)$ and polygamma functions $\psi^{(k)}(x)$ for $k\ge0$, please refer to [1, pp. 255–293, Chapter 6] or the papers [32, 36] and closely related references therein.

Theorem 3.1. *For $n\ge0$, we have*

$$\sum_{k=0}^{n}\frac{1}{2^{4k}(2k+1)(n-k+1)^2}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}=\frac{3[(2n+1)!!]^2}{2^{2n+3}(2n+3)!}\biggl[\pi^2-2\psi'\biggl(n+\frac{3}{2}\biggr)\biggr], \tag{3.1}$$

$$\sum_{k=0}^{n}\frac{1}{2^{4k}(n-k+1)^2}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}=\frac{[(2n+1)!!]^2}{2^{2n+3}(2n+2)!}\biggl[\pi^2-2\psi'\biggl(n+\frac{3}{2}\biggr)\biggr], \tag{3.2}$$

and

$$\sum_{k=0}^{n}\frac{1}{2^{4k}(2k+1)(n-k+1)}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}=\frac{[(2n+1)!!]^2}{2^{2n+3}(2n+2)!}\biggl[\pi^2-2\psi'\biggl(n+\frac{3}{2}\biggr)\biggr]. \tag{3.3}$$

Proof. Differentiating on both sides of (1.3) gives

$$\frac{(\arcsin x)^2}{\sqrt{1-x^2}}=2!\sum_{n=0}^{\infty}[(2n+1)!!]^2\left[\sum_{k=0}^{n}\frac{1}{(2k+1)^2}\right]\frac{x^{2n+2}}{(2n+2)!},\quad |x|<1. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
(\arcsin x)^3&=(\arcsin x)(\arcsin x)^2\\
&=\left[\sum_{k=0}^{\infty}\frac{1}{2^{2k}}\binom{2k}{k}\frac{x^{2k+1}}{2k+1}\right]\left[\frac{1}{2}\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m^2\binom{2m}{m}}\right]\\
&=\frac{x^3}{2}\left[\sum_{k=0}^{\infty}\frac{1}{(2k+1)2^{2k}}\binom{2k}{k}x^{2k}\right]\left[\sum_{k=0}^{\infty}\frac{2^{2k}}{(k+1)^2\binom{2(k+1)}{k+1}}x^{2k}\right]\\
&=\frac{x^3}{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{(2k+1)2^{2k}}\binom{2k}{k}\frac{2^{2(n-k)}}{(n-k+1)^2\binom{2(n-k+1)}{n-k+1}}\right]x^{2n}\\
&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{2(n-2k)-1}}{(2k+1)(n-k+1)^2}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}\right]x^{2n+3},\\
\frac{(\arcsin x)^2}{\sqrt{1-x^2}}&=(\arcsin x)^2\frac{1}{\sqrt{1-x^2}}\\
&=\left[\frac{1}{2}\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m^2\binom{2m}{m}}\right]\left[\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\binom{2n}{n}x^{2n}\right]\\
&=x^2\left[\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(k+1)^2\binom{2(k+1)}{k+1}}x^{2k}\right]\left[\sum_{k=0}^{\infty}\frac{1}{2^{2k}}\binom{2k}{k}x^{2k}\right]\\
&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{2^{2k}}\binom{2k}{k}\frac{2^{2(n-k)+1}}{(n-k+1)^2\binom{2(n-k+1)}{n-k+1}}\right]x^{2n+2}\\
&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{2(n-2k)+1}}{(n-k+1)^2}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}\right]x^{2n+2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{(\arcsin x)^2}{\sqrt{1-x^2}}&=(\arcsin x)\frac{\arcsin x}{\sqrt{1-x^2}}\\
&=\left[\sum_{k=0}^{\infty}\frac{1}{2^{2k}}\binom{2k}{k}\frac{x^{2k+1}}{2k+1}\right]\left[\frac{1}{2x}\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m\binom{2m}{m}}\right]\\
&=\left[\sum_{k=0}^{\infty}\frac{1}{2^{2k}(2k+1)}\binom{2k}{k}x^{2k}\right]\left[\sum_{k=0}^{\infty}\frac{2^{2k+1}}{(k+1)\binom{2(k+1)}{k+1}}x^{2(k+1)}\right]\\
&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{2^{2k}(2k+1)}\binom{2k}{k}\frac{2^{2(n-k)+1}}{(n-k+1)\binom{2(n-k+1)}{n-k+1}}\right]x^{2n+2}\\
&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{2(n-2k)+1}}{(2k+1)(n-k+1)}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}\right]x^{2n+2},
\end{aligned}$$

where we used the power series expansions (1.1), (1.2), and (2.4). Comparing the above three power series expansions with series expansions (1.3) and (3.4) and equating coefficients of x^{2n+2} respectively reveal

$$\frac{3![(2n+1)!!]^2}{(2n+3)!}\left[\sum_{k=0}^{n}\frac{1}{(2k+1)^2}\right]=\sum_{k=0}^{n}\frac{2^{2(n-2k)-1}}{(2k+1)(n-k+1)^2}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}, \tag{3.5}$$

$$\frac{2![(2n+1)!!]^2}{(2n+2)!}\left[\sum_{k=0}^{n}\frac{1}{(2k+1)^2}\right]=\sum_{k=0}^{n}\frac{2^{2(n-2k)+1}}{(n-k+1)^2}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}, \tag{3.6}$$

and

$$\frac{2![(2n+1)!!]^2}{(2n+2)!}\left[\sum_{k=0}^{n}\frac{1}{(2k+1)^2}\right]=\sum_{k=0}^{n}\frac{2^{2(n-2k)+1}}{(2k+1)(n-k+1)}\frac{\binom{2k}{k}}{\binom{2(n-k+1)}{n-k+1}}. \tag{3.7}$$

From the formula

$$\psi'\left(\frac{1}{2}+n\right)=\frac{\pi^2}{2}-4\sum_{k=1}^{n}\frac{1}{(2k-1)^2}, \quad n\in\mathbb{N}$$

in [18, p. 914, 8.366], we derive that

$$\sum_{k=0}^{n}\frac{1}{(2k+1)^2}=\frac{1}{8}\left[\pi^2-2\psi'\left(n+\frac{3}{2}\right)\right]. \tag{3.8}$$

Substituting the formula (3.8) into (3.5), (3.6), and (3.7) and simplifying lead to three identities (3.1), (3.2), and (3.3) respectively. The proof of Theorem 3.1 is thus complete. □

4. Remarks

Finally, we list several remarks on our main results and related stuffs.

Remark 4.1. The identity (2.1) in Theorem 2.1 can be regarded as a couple of the identity

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2(n-k)}{n-k}=\binom{2n+1}{n}, \quad n\ge 0, \tag{4.1}$$

which is a special case of the identity [45, p. 77, (3.95)]. Moreover, the identity

$$\sum_{\substack{k+\ell=n,\\ k\ge0,\ell\ge0}}\frac{1}{k+1}\binom{2k}{k}\binom{2(\ell+1)}{\ell+1}=2\binom{2n+2}{n}, \quad n\ge 0, \tag{4.2}$$

which has been proved in [13] by three alternative and different methods, is an equivalence of the identity (4.1). This equivalence can be demonstrated as follows.

The identity (4.2) can be rearranged as

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2(n-k+1)}{n-k+1}=2\binom{2n+2}{n}$$

which is equivalent to

$$\sum_{k=0}^{n+1}\frac{1}{k+1}\binom{2k}{k}\binom{2(n-k+1)}{n-k+1}=2\binom{2n+2}{n}+\frac{1}{n+2}\binom{2(n+1)}{n+1}=\binom{2n+3}{n+1},$$

where we used $\binom{0}{0}=1$. Replacing $n+1$ by n in the last identity leads to the identity (4.1).

Remark 4.2. Closely related to central binomial coefficients $\binom{2n}{n}$, the Catalan numbers

$$C_n=\frac{1}{n+1}\binom{2n}{n} \tag{4.3}$$

in combinatorial number theory have attracted many mathematicians who have published several monographs [19, 23, 43, 47] and a number of papers such as [25, 26, 28, 29, 37, 38, 39, 40, 41, 42].

The second conclusion (b) in [3, Lemma 2] reads that

$$\sum_{k=0}^{n} B_k C_{n-k} = \frac{1}{2} B_{n+1}, \tag{4.4}$$

where $B_n = \binom{2n}{n}$. Rewriting the sum in (4.4) as $\sum_{k=0}^{n} B_{n-k} C_k$ and substituting $\binom{2n-2k}{n-k}$ and $\frac{1}{k+1}\binom{2k}{k}$ for B_{n-k} and C_k result in

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{2(n-k)}{n-k} C_k = \frac{1}{2}\binom{2(n+1)}{n+1} = \binom{2n+1}{n}$$

which is the same as the identity (4.1).

By the way, the combinatorial proof of the identity (4.4) in [3, Lemma 2] is longer than the combinatorial proof of the identity (4.2) in [13], while its equivalent identities (4.1) and (4.2) were proved analytically in [13] and [45, p. 77, (3.95)].

Remark 4.3. By the formula (4.3), we can rewritten the identity (4.1) and those in Theorem 2.1 and Theorem 3.1 as

$$\sum_{k=0}^{n} (n-k+1) C_k C_{n-k} = \binom{2n+1}{n},$$

$$\sum_{k=0}^{n} \frac{(k+1)(n-k+1)}{2k+1} C_k C_{n-k} = \frac{2^{4n}}{(2n+1)(n+1)C_n},$$

$$\sum_{k=0}^{n} \frac{n-k+2}{2^{4k}(k+1)(2k+1)(n-k+1)^2} \frac{C_k}{C_{n-k+1}} = \frac{3[(2n+1)!!]^2}{2^{2n}(2n+3)!} \sum_{k=0}^{n} \frac{1}{(2k+1)^2},$$

$$\sum_{k=0}^{n} \frac{n-k+2}{2^{4k}(k+1)(n-k+1)^2} \frac{C_k}{C_{n-k+1}} = \frac{[(2n+1)!!]^2}{2^{2n}(2n+2)!} \sum_{k=0}^{n} \frac{1}{(2k+1)^2},$$

and

$$\sum_{k=0}^{n} \frac{n-k+2}{2^{4k}(k+1)(2k+1)(n-k+1)} \frac{C_k}{C_{n-k+1}} = \frac{[(2n+1)!!]^2}{2^{2n}(2n+2)!} \sum_{k=0}^{n} \frac{1}{(2k+1)^2}$$

respectively. For more information on series involving the Catalan numbers C_n, please refer to the paper [34] and closely related references therein.

Remark 4.4. In [21] and its previous arXiv preprints, among other things, a nice series expansion and its applications of the function $\left(\frac{\arcsin t}{t}\right)^m$, whose value at $t=0$ is defined to be 1, were established and carried out.

Remark 4.5. This paper is a revised version of the arXiv preprints [30, 31].

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Conflict of interest

The authors declare that they have no conflict of interest.

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