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A Study of Para-Kähler-Norden Structures on Cotangent Bundle with The New Class of Metrics

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Abstract. The main purpose of the present paper is to study almost para-complex-Norden properties concerning new class of metrics on the cotangent bundle.

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1. Introduction

In this field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson, E.M., Walker, A.G. [13], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by Sekizawa, M. [19] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. Inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca, F. considered another class of metrics on cotangent bundles of Riemannian manifolds, that he called g-natural metrics [1]. Also, there are similar studies done by other authors, Salimov, A.A., Ağca, F. [2, 14], Yano, K., Ishihara, S. [22], Ocak, F., Kazimova, S. [12], Gezer, A., Altunbas, M. [10]. On the other hand, in [24] Zayatuev, B.V. introduced a generalization of the Sasaki metric on tangent bundle [18], this metric is called rescaled Sasaki metric by Wang, J. and Wang, Y. in [20], and in [7] Gezer, A. called this metric the metric deformed Sasaki metric. In [8] (resp. [9]) Gezer, A. and Altunbas, M. define the rescaled Sasaki type metric on the cotangent bundle (resp. on tensor bundles of arbitrary type).

In a previous work [23] we proposed a new class of metrics on the cotangent bundle. In this paper, we construct almost para-complex Norden structures on cotangent bundle equipped with this new class of metrics and also investigate necessary and sufficient conditions for these structures to become para-Kähler-Norden, quasi-para-Kähler-Norden. Finally we characterize some properties of almost para-complex Norden structures in context of almost product Riemannian manifolds are presented.

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2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m},\bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on T^*M, the vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M.$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $U \subset M$ of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal and the vertical lifts of X and ω are defined, respectively by

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial p_i},$$

$$\omega^V = \omega_i \frac{\partial}{\partial p_i},$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\}$, where Γ^h_{ij} are components of the Levi-Civita connection ∇ on M. (see [22] for more details).

Lemma 2.1. *[22] Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*

(1) $[\omega^V, \theta^V] = 0$,
(2) $[X^H, \theta^V] = (\nabla_X \theta)^V$,
(3) $[X^H, Y^H] = [X, Y]^H + (pR(X, Y))^V$,

for all vector fields $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Let (M, g) be a Riemannian manifold, we define the map

$$\begin{array}{ccc} \Im^0_1(M) & \to & \Im^1_0(M) \\ \omega & \mapsto & \tilde{\omega} \end{array}$$

by for all $X \in \Im^1_0(M)$, $g(\tilde{\omega}, X) = \omega(X)$.
Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\tilde{\omega} = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).
For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T^*_xM by $g^{-1}(\omega, \theta) = g(\tilde{\omega}, \tilde{\theta}) = g^{ij}\omega_i\theta_j$.
In this case we have $\tilde{\omega} = g^{-1} \circ \omega$.

3. New Class of Metrics

Definition 3.1. [23] Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. On the cotangent bundle T^*M, we define a new class of metrics noted g^f by

$$\begin{aligned} g^f(X^H, Y^H) &= g(X, Y)^V = g(X, Y) \circ \pi, \\ g^f(X^H, \theta^V) &= 0, \\ g^f(\omega^V, \theta^V) &= f g^{-1}(\omega, p) g^{-1}(\theta, p), \end{aligned}$$

where $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$.

Lemma 3.2. *[23] Let (M, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics, for all $X \in \Im^1_0(M)$ and $\omega, \theta, \eta \in \Im^0_1(M)$, we have*

(1) $X^H g^f(\theta^V, \eta^V) = \frac{1}{f} X(f) g^f(\theta^V, \eta^V) + g^f((\nabla_X \theta)^V, \eta^V) + g^f(\theta^V, (\nabla_X \eta)^V)$,
(2) $\omega^V g^f(\theta^V, \eta^V) = f g^{-1}(\omega, \theta) g^{-1}(\eta, p) + f g^{-1}(\omega, \eta) g^{-1}(\theta, p)$.

Theorem 3.3. *[23] Let (M, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the new class of metrics. If ∇ (resp ∇^f) denote the Levi-Civita connection of (M, g) (resp (T^*M, g^f)), we have:*

$$\begin{aligned}
(1)\ \nabla^f_{X^H} Y^H &= (\nabla_X Y)^H, \\
(2)\ \nabla^f_{X^H} \theta^V &= (\nabla_X \theta)^V + \frac{1}{2f} X(f)\theta^V, \\
(3)\ \nabla^f_{\omega^V} Y^H &= \frac{1}{2f} Y(f)\omega^V, \\
(4)\ \nabla^f_{\omega^V} \theta^V &= \frac{-1}{2} g^{-1}(\omega, p) g^{-1}(\theta, p)(grad\, f)^H + \frac{1}{r^2} g^{-1}(\omega, \theta)\mathcal{P}^V,
\end{aligned}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where $\mathcal{P}^V$ the canonical vertical vector field on T^*M and $r^2 = g^{-1}(p, p)$.

4. Para-Kähler-Norden Structures

An almost product structure φ on a manifold M is a $(1, 1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1, 1)$ on M). The pair (M, φ) is called an almost product manifold.

A linear connection ∇ on (M, φ) such that $\nabla\varphi = 0$ is said to be an almost product connection. There exists an almost product connection on every almost product manifold [5].

An almost para-complex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [4].

An almost para-complex Norden manifold (M^{2m}, φ, g) is a real $2m$-dimensional differentiable manifold M^{2m} with an almost para-complex structure φ and a Riemannian metric g such that:

$$g(\varphi X, Y) = g(X, \varphi Y),$$

for all $X, Y \in \Im^1_0(M)$, in this case g is called a pure metric with respect to φ or para-Norden metric (B-metric) [17].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that φ is integrable i.e. $\nabla\varphi = 0$ (B-manifold), where ∇ is the Levi-Civita connection of g [15, 17].

A Tachibana operator ϕ_φ applied to the pure metric g is given by

$$(\phi_\varphi g)(X, Y, Z) = (\varphi X)(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_Y\varphi)X, Z) + g(Y, (L_Z\varphi)X), \tag{4.1}$$

for all $X, Y, Z \in \Im^1_0(M)$ [21].

In an almost para-complex Norden manifold, a para-Norden metric g is called para-holomorphic if

$$(\phi_\varphi g)(X, Y, Z) = 0,$$

for all $X, Y, Z \in \Im^1_0(M)$ [17].

A para-holomorphic Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that g is a para-holomorphic i.e. $\phi_\varphi g = 0$.

In [17], Salimov and his collaborators proved that for an almost B-manifold,

$$\nabla\varphi = 0 \Leftrightarrow \phi_\varphi g = 0,$$

by virtue of this view, para-holomorphic Norden manifolds are similar to para-Kähler-Norden manifolds [15].

The purity conditions for a tensor field $\omega \in \Im^q_0(M)$ with respect to the almost paracomplex structure φ given by

$$\omega(\varphi X_1, X_2, \cdots, X_q) = \omega(X_1, \varphi X_2, \cdots, X_q) = \cdots = \omega(X_1, X_2, \cdots, \varphi X_q),$$

for all $X_1, X_2, \cdots, X_q \in \Im^1_0(M)$ [17].

It is well known that if (M^{2m}, φ, g) is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [17], and for all $Y, Z \in \Im^1_0(M)$

$$\begin{cases} R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) &= R(Y, Z). \end{cases} \tag{4.2}$$

Let (M, g) be a Riemannian manifold. We consider an almost para-complex structure J on T^*M defined by

$$\begin{cases} JX^H &= -X^H, \\ J\omega^V &= \omega^V \end{cases} \tag{4.3}$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$ [3].

Theorem 4.1. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J defined by (4.3). The triple (T^*M, J, g^f) is an almost para-complex Norden manifold.*

Proof. For all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, from (4.3) we have

(1) $g^f(JX^H, Y^H) = g^f(-X^H, Y^H) = g^f(X^H, -Y^H) = g^f(X^H, JY^H)$,
(2) $g^f(JX^H, \theta^V) = g^f(-X^H, \theta^V) = 0 = g^f(X^H, \theta^V) = g^f(X^H, J\theta^V)$,
(3) $g^f(J\omega^V, Y^H) = g^f(\omega^V, Y^H) = 0 = g^f(\omega^V, -Y^H) = g^f(\omega^V, JY^H)$,
(4) $g^f(J\omega^V, \theta^V) = g^f(\omega^V, \theta^V) = g^f(\omega^V, J\theta^V)$,

i.e., g^f is pure with respect to J. Hence (T^*M, J, g^f) is an almost para-complex Norden manifold. □

Proposition 4.2. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J defined by (4.3), then we get*

1. $(\phi_J g^f)(X^H, Y^H, Z^H) = 0$,
2. $(\phi_J g^f)(\omega^V, Y^H, Z^H) = 0$,
3. $(\phi_J g^f)(X^H, \theta^V, Z^H) = 2g^f((pR(X, Z))^V, \theta^V)$,
4. $(\phi_J g^f)(X^H, Y^H, \eta^V) = 2g^f((pR(X, Y))^V, \eta^V)$,
5. $(\phi_J g^f)(\omega^V, \theta^V, Z^H) = 0$,
6. $(\phi_J g^f)(\omega^V, Y^H, \eta^V) = 0$,
7. $(\phi_J g^f)(X^H, \theta^V, \eta^V) = \dfrac{-2}{f} X(f) g^f(\theta^V, \eta^V)$,
8. $(\phi_J g^f)(\omega^V, \theta^V, \eta^V) = 0$,

for all $X, Y, Z \in \Im^1_0(M)$ and $\omega, \theta, \eta \in \Im^0_1(M)$, where R denote the curvature tensor of (M, g).

Proof. We calculate Tachibana operator ϕ_J applied to the pure metric g^f. This operator is characterized by (4.1), from Lemma 3.2 we have

$$\begin{aligned} 1.\ (\phi_J g^f)(X^H, Y^H, Z^H) &= (JX^H)g^f(Y^H, Z^H) - X^H g^f(JY^H, Z^H) + g^f((L_{Y^H}J)X^H, Z^H) + g^f(Y^H, (L_{Z^H}J)X^H) \\ &= -X^H g^f(Y^H, Z^H) + X^H g^f(Y^H, Z^H) + g^f(L_{Y^H}JX^H - J(L_{Y^H}X^H), Z^H) \\ &\quad + g^f(Y^H, L_{Z^H}JX^H - J(L_{Z^H}X^H)) \\ &= -g^f([Y^H, X^H], Z^H) - g^f(J[Y^H, X^H], Z^H) - g^f(Y^H, [Z^H, X^H]) - g^f(Y^H, J[Z^H, X^H]) \\ &= 0. \end{aligned}$$

$$\begin{aligned} 2.\ (\phi_J g^f)(\omega^V, Y^H, Z^H) &= (J\omega^V)g^f(Y^H, Z^H) - \omega^V g^f(JY^H, Z^H)) + g^f((L_{Y^H}J)\omega^V, Z^H) + g^f(Y^H, (L_{Z^H}J)\omega^V) \\ &= +g^f([Y^H, \omega^V], Z^H) - g^f(J[Y^H, \omega^V], Z^H) + g^f(Y^H, [Z^H, \omega^V]) - g^f(Y^H, J[Z^H, \omega^V]) \\ &= 2g^f([Y^H, \omega^V], Z^H) + 2g^f(Y^H, [Z^H, \omega^V]) \\ &= 2g^f((\nabla_Y\omega)^V, Z^H) + 2g^f(Y^H, (\nabla_Z\omega)^V) \\ &= 0. \end{aligned}$$

$$\begin{aligned} 3.\ (\phi_J g^f)(X^H, \theta^V, Z^H) &= (JX^H)g^f(\theta^V, Z^H) - X^H g^f(J\theta^V, Z^H) + g^f((L_{\theta^V}J)X^H, Z^H) + g^f(\theta^V, (L_{Z^H}J)X^H) \\ &= -g^f([\theta^V, X^H], Z^H) - g^f(J[\theta^V, X^H], Z^H) - g^f(\theta^V, [Z^H, X^H]) - g^f(\theta^V, J[Z^H, X^H]) \\ &= -2g^f(\theta^V, [Z^H, X^H]) \\ &= -2g^f(\theta^V, (pR(Z, X))^V) \\ &= 2g^f((pR(X, Z))^V, \theta^V). \end{aligned}$$

$$
\begin{aligned}
4.\ (\phi_J g^f)(X^H, Y^H, \eta^V) &= (JX^H)g^f(Y^H, \eta^V) - X^H g^f(JY^H, \eta^V) + g^f((L_{Y^H}J)X^H, \eta^V) + g^f(Y^H, (L_{\eta^V}J)X^H) \\
&= -g^f([Y^H, X^H], \eta^V) - g^f(J[Y^H, X^H], \eta^V) - g^f(Y^H, [\eta^V, X^H]) - g^f(Y^H, J[\eta^V, X^H]) \\
&= -2g^f([Y^H, X^H], \eta^V) \\
&= 2g^f((pR(X,Y))^V, \eta^V).
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

Therefore, we have the following result.

Theorem 4.3. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J defined by (4.3). The triple (T^*M, J, g^f) is a para-Kähler-Norden manifold if and only if M is flat and f is constant.*

Proof. For all $\overline{X}, \overline{Y}, \overline{Z} \in \Im^1_0(T^*M)$ such as $\overline{X} = X^H, \omega^V$, $\overline{Y} = Y^H, \theta^V$ and $\overline{Z} = Z^H, \eta^V$,

$$
\begin{aligned}
(\phi_J g^f))(\overline{X}, \overline{Y}, \overline{Z}) = 0 &\Leftrightarrow \begin{cases} 2g^f((pR(X,Z))^V, \theta^V) = 0 \\ 2g^f((pR(X,Y))^V, \eta^V) = 0 \\ \dfrac{-2}{f} X(f) g^f(\theta^V, \eta^V) = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} pR(X,Z) = 0 \\ pR(X,Y) = 0 \\ X(f) = 0 \end{cases} \\
&\Leftrightarrow R = 0 \text{ and } f = constant.
\end{aligned}
$$
□

Now we study a quasi-para-Kähler-Norden manifold. The basis class of non-integrable almost paracomplex manifolds with para-Norden metric is the class of the quasi-para-Kähler manifolds. An almost para-complex Norden manifold (M, φ, g) is a quasi-para-Kähler-Norden manifold, if

$$
\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0,
$$

for all $X, Y, Z \in \Im^1_0(M)$, where σ is the cyclic sum by three arguments [6, 11]. It is well known that

$$
\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0 \Leftrightarrow \underset{X,Y,Z}{\sigma} (\phi_\varphi g)(X, Y, Z) = 0,
$$

which was proven in [16].

Theorem 4.4. *Let (M, g) be a Riemannian manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J defined by (4.3). The triple (T^*M, J, g^f) is a quasi-para-Kähler-Norden manifold if and only if f is constant.*

Proof. For all $\overline{X}, \overline{Y}, \overline{Z} \in \Im^1_0(T^*M)$,

$$
\underset{\overline{X},\overline{Y},\overline{Z}}{\sigma} (\phi_J g^f)(\overline{X}, \overline{Y}, \overline{Z}) = (\phi_J g^f)(\overline{X}, \overline{Y}, \overline{Z}) + (\phi_J g^f)(\overline{Y}, \overline{Z}, \overline{X}) + (\phi_J g^f)(\overline{Z}, \overline{X}, \overline{Y}).
$$

By virtue of Proposition 4.2, we have

$$
\begin{aligned}
\underset{X^H,Y^H,Z^H}{\sigma} (\phi_J g^f)(X^H, Y^H, Z^H) &= 0, \\
\underset{\omega^V,Y^H,Z^H}{\sigma} (\phi_J g^f)(\omega^V, Y^H, Z^H) &= 0, \\
\underset{\omega^V,\theta^V,Z^H}{\sigma} (\phi_J g^f)(\omega^V, \theta^V, Z^H) &= -\frac{2}{f} Z(f) g^f(\omega^V, \theta^V), \\
\underset{\omega^V,\theta^V,\eta^V}{\sigma} (\phi_J g^f)(\omega^V, \theta^V, \eta^V) &= 0.
\end{aligned}
$$

Then, to be (T^*M, J, g^f) is a quasi-para-Kähler-Norden manifold it suffices that $Z(f) = 0$, for any $Z \in \Im^1_0(M)$. i.e. f is constant. □

Now we study a generalization of the almost para-complex structure defined by (4.3).

Lemma 4.5. *Let (M^{2m}, φ) an almost para-complex manifold and define a tensor field $J_\varphi \in \Im^1_1(T^*M)$ by*

$$\begin{cases} J_\varphi X^H &= -(\varphi X)^H, \\ J_\varphi \omega^V &= \omega^V, \end{cases} \tag{4.4}$$

for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$.
Then the couple (T^*M, J_φ) is an almost para-complex manifold .

Proof. By virtue of (4.4), we have

$$\begin{cases} J^2_\varphi X^H = J_\varphi(J_\varphi X^H) = J_\varphi(-(\varphi X)^H) = (\varphi(\varphi X))^H = (\varphi^2 X)^H, \\ \\ J^2_\varphi \omega^V = J_\varphi(J_\varphi \omega^V) = J_\varphi \omega^V = \omega^V, \end{cases}$$

for any $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$. Hence $\varphi^2 = id_M$ then $J^2_\varphi = id_{T^*M}$. □

Theorem 4.6. *Let (M^{2m}, φ, g) be an almost para-complex Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J_φ defined by (4.4). The triple (T^*M, J_φ, g^f) is an almost para-complex Norden manifold.*

Proof. For all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, from (4.4) we have

$$\begin{aligned} (i)\ g^f(J_\varphi X^H, Y^H) &= g^f(-(\varphi X)^H, Y^H) = -g(\varphi X, Y) = -g(X, \varphi Y) \\ &= g^f(X^H, -(\varphi Y)^H) = g^f(X^H, J_\varphi Y^H), \\ (ii)\ g^f(J_\varphi X^H, \theta^V) &= g^f(-(\varphi X)^H, \theta^V) = 0 = g^f(X^H, \theta^V) = g^f(X^H, J_\varphi \theta^V), \\ (iii)\ g^f(J_\varphi \omega^V, \theta^V) &= g^f(\omega^V, \theta^V) = g^f(\omega^V, J_\varphi \theta^V). \end{aligned}$$

Since g is pure with respect to φ, then g^f is pure with respect to J_φ. □

Proposition 4.7. *Let (M^{2m}, φ, g) be an almost para-complex Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J_φ defined by (4.4), then we get*

1. $(\phi_{J_\varphi} g^f)(X^H, Y^H, Z^H) = -(\phi_\varphi g)(X, Y, Z)$,
2. $(\phi_{J_\varphi} g^f)(\omega^V, Y^H, Z^H) = 0$,
3. $(\phi_{J_\varphi} g^f)(X^H, \theta^V, Z^H) = g^f((pR(\varphi X + X, Z))^V, \theta^V)$,
4. $(\phi_{J_\varphi} g^f)(X^H, Y^H, \eta^V) = g^f((pR(\varphi X + X, Y))^V, \eta^V)$,
5. $(\phi_{J_\varphi} g^f)(\omega^V, \theta^V, Z^H) = 0$,
6. $(\phi_{J_\varphi} g^f)(\omega^V, Y^H, \eta^V) = 0$,
7. $(\phi_{J_\varphi} g^f)(X^H, \theta^V, \eta^V) = \dfrac{-1}{f}(\varphi X + X)(f) g^f(\theta^V, \eta^V)$,
8. $(\phi_{J_\varphi} g^f)(\omega^V, \theta^V, \eta^V) = 0$,

for all $X, Y, Z \in \Im^1_0(M)$ and $\omega, \theta, \eta \in \Im^0_1(M)$, where R denote the curvature tensor of (M, g).

Proof. We calculate Tachibana operator ϕ_{J_φ} applied to the pure metric g^f. With the same steps in the proof of Proposition 4.2. □

We get the results.

Theorem 4.8. *Let (M^{2m}, φ, g) be an almost para-complex Norden manifold, (T^*M, g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J_φ defined by (4.4). The triple (T^*M, J_φ, g^f) is a para-Kähler-Norden manifold if and only if M is flat para-Kähler-Norden manifold and f is constant.*

Proof. For all $\overline{X},\overline{Y},\overline{Z}\in\Im^1_0(T^*M)$ such as $\overline{X}=X^H,\omega^V$, $\overline{Y}=Y^H,\theta^V$ and $\overline{Z}=Z^H,\eta^V$

$$(\phi_{J_\varphi}g^f))(\overline{X},\overline{Y},\overline{Z})=0 \quad\Leftrightarrow\quad \begin{cases}(\phi_\varphi g)(X,Y,Z) & =0\\ g^f((pR(\varphi X+X,Z))^V,\theta^V) & =0\\ g^f((pR(\varphi X+X,Y))^V,\eta^V) & =0\\ \dfrac{-1}{f}(\varphi XX)(f)g^f(\theta^V,\eta^V) & =0\end{cases}$$

$$\Leftrightarrow\quad \begin{cases}(\phi_\varphi g)(X,Y,Z) & =0\\ pR(\varphi X+X,Z) & =0\\ pR(\varphi X+X,Y) & =0\\ (\varphi X+X)(f) & =0.\end{cases}$$

Since $\varphi\neq\pm id_M$ then

$$(\phi_{J_\varphi}g^f))(\overline{X},\overline{Y},\overline{Z})=0\Leftrightarrow\begin{cases}\phi_\varphi g=0\\ \quad R=0\\ \quad f=constant.\end{cases}$$ □

Theorem 4.9. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (T^*M,g^f) be its cotangent bundle equipped with the new class of metrics and the almost para-complex structure J_φ defined by (4.4). The triple (T^*M,J_φ,g^f) is a quasi-para-Kähler-Norden manifold if and only if f is constant.*

Proof. For all $\overline{X},\overline{Y},\overline{Z}\in\Im^1_0(T^*M)$,

$$\underset{\overline{X},\overline{Y},\overline{Z}}{\sigma}(\phi_{J_\varphi}g^f)(\overline{X},\overline{Y},\overline{Z}) = (\phi_{J_\varphi}g^f)(\overline{X},\overline{Y},\overline{Z})+(\phi_{J_\varphi}g^f)(\overline{Y},\overline{Z},\overline{X})+(\phi_{J_\varphi}g^f)(\overline{Z},\overline{X},\overline{Y}).$$

By virtue of Proposition 4.7 and using (4.2) we have

$$\begin{aligned}
\underset{X^H,Y^H,Z^H}{\sigma}(\phi_{J_\varphi}g^f)(X^H,Y^H,Z^H) &= -(\phi_\varphi g)(X,Y,Z)-(\phi_\varphi g)(Y,Z,X)-(\phi_\varphi g)(Z,X,Y)\\
&= 0,\\
\underset{\omega^V,Y^H,Z^H}{\sigma}(\phi_{J_\varphi}g^f)(\omega^V,Y^H,Z^H) &= g^f((pR(\varphi Y+Y,Z))^V,\omega^V)+g^f((pR(\varphi Z+Z,Y))^V,\omega^V)\\
&= g^f((pR(\varphi Y,Z)-pR(Y,\varphi Z))^V,\omega^V)\\
&= 0,\\
\underset{\omega^V,\theta^V,Z^H}{\sigma}(\phi_{J_\varphi}g^f)(\omega^V,\theta^V,Z^H) &= -\frac{1}{f}(\varphi Z+Z)(f)g^f(\omega^V,\theta^V),\\
\underset{\omega^V,\theta^V,\eta^V}{\sigma}(\phi_{J_\varphi}g^f)(\omega^V,\theta^V,\eta^V) &= 0
\end{aligned}$$

then, to be (T^*M,J_φ,g^f) is a quasi-para-Kähler-Norden manifold it suffices that $(\varphi Z+Z)(f)=0$, for any $Z\in\Im^1_0(M)$. i.e. f is constant. □

Now consider the almost product structure J_φ defined by (4.4), we define a tensor field S of type $(1,2)$ and linear connection $\widehat{\nabla}$ on T^*M by,

$$S(\overline{X},\overline{Y})=\frac{1}{2}[(\nabla^f_{J_\varphi\overline{Y}}J_\varphi)\overline{X}+J_\varphi((\nabla^f_{\overline{Y}}J_\varphi)\overline{X})-J_\varphi((\nabla^f_{\overline{X}}J_\varphi)\overline{Y})], \tag{4.5}$$

$$\widehat{\nabla}_{\overline{X}}\overline{Y}=\nabla^f_{\overline{X}}\overline{Y}-S(\overline{X},\overline{Y}), \tag{4.6}$$

for all $\overline{X},\overline{Y}\in\Im^1_0(T^*M)$, where ∇^f is the Levi-Civita connection of (T^*M,g^f) given by Theorem 3.3. $\widehat{\nabla}$ is an almost product connection on T^*M (see [5, p.151] for more details).

Lemma 4.10. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, T^*M be its cotangent bundle equipped with the new class of metrics g^f and the almost product structure J_φ defined by (4.4). Then tensor field S is as follows*

(1) $S(X^H,Y^H)=0,$

(2) $S(X^H,\theta^V) = -\dfrac{1}{2f}(\varphi X + X)(f)\theta^V$,

(3) $S(\omega^V,Y^H) = \dfrac{1}{4f}(\varphi Y + Y)(f)\omega^V$,

(4) $S(\omega^V,\theta^V) = -\dfrac{1}{4}g^{-1}(\omega,p)g^{-1}(\theta,p)(\varphi\, grad\, f + grad\, f)^H$,

for all $X,Y \in \Im^1_0(M)$ *and* $\omega,\theta \in \Im^0_1(M)$*,*

Proof. (1) Using (4.4) and (4.5), we have

$$\begin{aligned}
S(X^H,Y^H) &= \frac{1}{2}[(\nabla^f_{J_\varphi Y^H}J_\varphi)X^H + J_\varphi((\nabla^f_{Y^H}J_\varphi)X^H) - J_\varphi((\nabla^f_{X^H}J_\varphi)Y^H)]\\
&= \frac{1}{2}[\nabla^f_{(\varphi Y)^H}(\varphi X)^H + J_\varphi(\nabla^f_{(\varphi Y)^H}X^H) - J_\varphi(\nabla^f_{Y^H}(\varphi X)^H) - \nabla^f_{Y^H}X^H + J_\varphi(\nabla^f_{X^H}(\varphi Y)^H) + \nabla^f_{X^H}Y^H]\\
&= \frac{1}{2}[(\nabla_{\varphi Y}\varphi X)^H - \varphi(\nabla_{\varphi Y}X)^H + \varphi(\nabla_Y\varphi X)^H - (\nabla_Y X)^H - \varphi(\nabla_X\varphi Y)^H + (\nabla_X Y)^H].
\end{aligned}$$

Then, we have

$$S(X^H,Y^H) = 0.$$

(2) By a similar calculation to (1), we have

$$\begin{aligned}
S(X^H,\theta^V) &= \frac{1}{2}[(\nabla^f_{J_\varphi\theta^V}J_\varphi)X^H + J_\varphi((\nabla^f_{\theta^V}J_\varphi)X^H) - J_\varphi((\nabla^f_{X^H}J_\varphi)\theta^V)]\\
&= \frac{1}{2}[-\nabla^f_{\theta^V}(\varphi X)^H - J_\varphi(\nabla^f_{\theta^V}X^H) - J_\varphi(\nabla^f_{\theta^V}(\varphi X)^H) - \nabla^f_{\theta^V}X^H - J_\varphi(\nabla^f_{X^H}\theta^V) + \nabla^f_{X^H}\theta^V]\\
&= \frac{1}{2}[-\frac{1}{2f}(\varphi X)(f)\theta^V - \frac{1}{2f}X(f)\theta^V - \frac{1}{2f}(\varphi X)(f)\theta^V - \frac{1}{2f}X(f)\theta^V - (\nabla_X\theta)^V - \frac{1}{2f}X(f)\theta^V\\
&\quad +(\nabla_X\theta)^V + \frac{1}{2f}X(f)\theta^V]\\
&= \frac{1}{2}[-\frac{1}{f}(\varphi X)(f)\theta^V - \frac{1}{f}X(f)\theta^V]\\
&= -\frac{1}{2f}(\varphi X + X)(f)\theta^V.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 4.11. *Let* (M^{2m},φ,g) *be a para-Kähler-Norden manifold,* T^*M *be its cotangent bundle equipped with the new class of metrics* g^f *and the almost product structure* J_φ *defined by (4.4). Then the almost product connection* $\widehat{\nabla}$ *defined by (4.6) is as follows*

$$\begin{aligned}
(1)\ \widehat{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H,\\
(2)\ \widehat{\nabla}_{X^H}\theta^V &= (\nabla_X\theta)^V + \frac{1}{2f}(\varphi X + 2X)(f)\theta^V,\\
(3)\ \widehat{\nabla}_{\omega^V}Y^H &= -\frac{1}{4f}(\varphi Y - Y)(f)\omega^V,\\
(4)\ \widehat{\nabla}_{\omega^V}\theta^V &= \frac{1}{4}g^{-1}(\omega,p)g^{-1}(\theta,p)(\varphi\, grad\, f - grad\, f)^H + \frac{1}{r^2}g^{-1}(\omega,\theta)\mathcal{P}^V,
\end{aligned}$$

for all $X,Y \in \Im^1_0(M)$ *and* $\omega,\theta \in \Im^0_1(M)$*.*

Proof. The proof of Theorem 4.11 follows directly from Theorem 3.3, Lemma 4.10 and formula (4.6). □

Lemma 4.12. *Let* (M^{2m},φ,g) *be a para-Kähler-Norden manifold,* T^*M *be its cotangent bundle equipped with the new class of metrics* g^f *and the almost product structure* J_φ *defined by (4.4) and* $\widehat{T}$ *denote the torsion tensor of* $\widehat{\nabla}$*, then we have*

1) $\widehat{T}(X^H,Y^H) = 0$,

2) $\widehat{T}(X^H, \theta^V) = \dfrac{3}{4f}(\varphi X + X)(f)\theta^V$,

3) $\widehat{T}(\omega^V, Y^H) = -\dfrac{3}{4f}(\varphi Y + Y)(f)\omega^V$,

4) $\widehat{T}(\omega^V, \theta^V) = 0$,

for all $X, Y \in \Im^1_0(M)$ *and* $\omega, \theta \in \Im^0_1(M)$.

Proof. The proof of Lemma 4.12 follows directly from Lemma 4.10 and formula

$$\begin{aligned}\widehat{T}(\overline{X}, \overline{Y}) &= \widehat{\nabla}_{\overline{X}}\overline{Y} - \widehat{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}] \\ &= S(\overline{Y}, \overline{X}) - S(\overline{X}, \overline{Y}),\end{aligned}$$

for all $\overline{X}, \overline{Y} \in \Im^1_0(T^*M)$. □

From Lemma 4.12 we obtain:

Theorem 4.13. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold,* T^*M *be its cotangent bundle equipped with the new class of metrics* g^f *and* $\widehat{\nabla}$ *the almost product connection defined by* (4.6)*, then* $\widehat{\nabla}$ *is symmetric if and only if* f *is constant.*

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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