ON SOME WEAKER HESITANT FUZZY OPEN SETS

Hariwan Z. IBRAHIM
Department of Mathematics, Faculty of Education, University of Zakho,
Zakho, IRAQ

Abstract. The purpose of this paper is to define and study some new types of hesitant fuzzy open sets namely, hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open in hesitant fuzzy topological space. Some properties and the relationships between these hesitant fuzzy sets are investigated. Furthermore, some relationships between them in hesitant fuzzy subspace are introduced.

1. Introduction

Hesitant fuzzy sets are very useful to deal with group decision making problems when experts have a hesitation among several possible memberships for an element to a set. During the evaluating process in practice, however, these possible memberships may be not only crisp values in [0, 1], but also interval values. Then hesitant fuzzy set theory has many applications in various fields like decision making problems, decision support systems, clustering algorithms, algebras, etc. After that time, hesitant fuzzy set theory has been developed rapidly by some scholars in theory and practice. In 1965, Zadeh [16] introduced the concept of a fuzzy set as a generalization of a crisp set. Chang [3] defined initially the notion of fuzzy topological spaces. In 2010, Torra [14] introduced the notion of a hesitant fuzzy set as an extension of a fuzzy set. In 2011, Xia and Xu [15] applied a hesitant fuzzy set to decision making by defining "hesitant fuzzy information aggregation". Jun et al. [5] studied hesitant fuzzy bi-ideals in semigroups. Divakaran and John [4] introduced a basic version of hesitant fuzzy rough sets through hesitant fuzzy relations. On the other hand, Jun and Ahn [6] applied hesitant fuzzy sets to BCK/BCI-algebras. Kim et al. [7] gave characterizations of a hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal, and a hesitant fuzzy commutative ideal, respectively.

2020 Mathematics Subject Classification. Primary: 54A05, 03E72; Secondary: 54A40.

Keywords. Hesitant fuzzy sets, hesitant fuzzy topology, hesitant fuzzy interior, hesitant fuzzy closure, hesitant fuzzy subspaces.

hariwan_math@yahoo.com
0000-0001-9417-2695.
in BCK-algebras. Recently, Lee and Hur \cite{10} defined a hesitant fuzzy topology and introduced the concepts of a hesitant fuzzy neighborhood, closure, interior, hesitant fuzzy subspace and obtained some of their properties. Also, they defined a hesitant fuzzy continuous mapping and investigated some of its properties. In 1965, Njastad \cite{13} defined the class of \(\alpha\)-open sets in topological spaces. In 1982, Mashhour et al \cite{12} introduced the concept of preopen sets. The study of semiopen sets and their properties was initiated by Levine \cite{11}. In 1996, Andrijevic' \cite{2} introduced hesitant fuzzy sets and all preopen sets. Especially, we will denote the set of all hesitant fuzzy sets in \(X\) by \(HS(X)\). Then, a mapping \(h : X \to P[0,1] \) is called a hesitant fuzzy set in \(X\).

Definition 1. \cite{14} Let \(X\) be a reference set, and \(P[0,1]\) denote the power set of \([0,1]\). Then, a mapping \(h : X \to P[0,1]\) is called a hesitant fuzzy set in \(X\).

The hesitant fuzzy empty (resp. whole) set, denoted by \(h^0\) (resp. \(h^1\)), is a hesitant fuzzy set in \(X\) defined as \(h^0(x) = \emptyset\) (resp. \(h^1(x) = [0,1]\)), for each \(x \in X\). Especially, we will denote the set of all hesitant fuzzy sets in \(X\) as \(HS(X)\).

Definition 2. Assume that \(X\) is a nonempty set and \(h, h_i \in HS(X)\) for \(i\) belong to the set of natural numbers \(N\). Then,

1. \(h_1\) is a subset of \(h_2\), denoted by \(h_1 \subseteq h_2\), if \(h_1(x) \subseteq h_2(x)\), for each \(x \in X\).
2. \(h_1\) is equal to \(h_2\), denoted by \(h_1 = h_2\), if \(h_1(x) = h_2(x)\) and \(h_2(x) = h_1(x)\), for each \(x \in X\).
3. the intersection of \(h_1\) and \(h_2\), denoted by \(\bigcap h_1 \cap h_2\), is a hesitant fuzzy set in \(X\) defined as: for each \(x \in X\),
\[
(\bigcap h_1 \cap h_2)(x) = h_1(x) \cap h_2(x).
\]
4. the union of \(h_1\) and \(h_2\), denoted by \(h_1 \cup h_2\), is a hesitant fuzzy set in \(X\) defined as: for each \(x \in X\),
\[
(\bigcup h_1 \cup h_2)(x) = h_1(x) \cup h_2(x).
\]
5. the complement of \(h\), denoted by \(h^c\), is a hesitant fuzzy set in \(X\) defined as: for each \(x \in X\),
\[
h^c(x) = \bar{h(x)} = [0,1] \setminus h(x).
\]
6. the intersection of \(\{h_i\}_{i \in N}\), denoted by \(\bigcap_{i \in N} h_i\), is a hesitant fuzzy set in \(X\) defined as: for each \(x \in X\),
\[
(\bigcap_{i \in N} h_i)(x) = \bigcap_{i \in N} h_i(x).
\]
7. the union of \(\{h_i\}_{i \in N}\), denoted by \(\bigcup_{i \in N} h_i\), is a hesitant fuzzy set in \(X\) defined as: for each \(x \in X\),
\[
(\bigcup_{i \in N} h_i)(x) = \bigcup_{i \in N} h_i(x).
\]

Definition 3. \cite{9} Let \(h \in HS(X)\). Then, \(h\) is called a hesitant fuzzy point with the support \(x \in X\) and the value \(\delta\), denoted by \(x_\delta\), if \(x_\delta : X \to P[0,1]\) is the mapping
given by: for each \( y \in X \),
\[
x_\delta(y) = \begin{cases} 
\delta \subseteq [0, 1] & \text{if } y = x, \\
\phi & \text{otherwise.}
\end{cases}
\]
In particular, \( H_P(X) \) is called the set of all hesitant fuzzy points in \( X \). If \( \delta \subseteq h(x) \), then \( x_\delta \) is said to belong to \( h \), denoted by \( x_\delta \in h \). It is obvious that \( h = \bigcup_{x_\delta \in h} x_\delta \).

**Definition 4.** [10] Let \( X \) be a nonempty set, and \( \tau \subseteq HS(X) \). Then, \( \tau \) is called a hesitant topology \((HFT)\) on \( X \), if it satisfies the following axioms:

1. \( h^0, h^1 \in \tau \).
2. For any \( h^1, h^2 \in \tau \), we have \( h^1 \cap h^2 \in \tau \).
3. For each \( h_i \in \tau \), we have \( \bigcup_{i \in N} h_i \in \tau \).

The pair \((X, \tau)\) is called a hesitant fuzzy topological space. Each member of \( \tau \) is called a hesitant fuzzy open set (HFOS) in \( X \). A hesitant fuzzy set \( h \) in \( X \) is called a hesitant fuzzy closed set (HFCS) in \((X, \tau)\), if \( h^c \in \tau \). The set of all hesitant fuzzy closed sets is denoted by \( HFC(X) \).

**Definition 5.** [10] Let \((X, \tau)\) be a hesitant fuzzy topological space, and \( h_A \in HS(X) \). Then:

1. \( int_H(h_A) = \bigcup\{ h_U \in \tau : h_U \subseteq h_A \} \).
2. \( cl_H(h_A) = \bigcap\{ h_F \in HFC(X) : h_A \subseteq h_F \} \).

3. **Weaker hesitant fuzzy open sets**

**Definition 6.** Let \((X, \tau)\) be a hesitant fuzzy topological space. A subset \( h \) of \( HS(X) \) is called:

1. hesitant fuzzy \( \alpha \)-open if \( h \subseteq int_H(cl_H(int_H(h))) \).
2. hesitant fuzzy preopen if \( h \subseteq int_H(cl_H(h)) \).
3. hesitant fuzzy semiopen if \( h \subseteq cl_H(int_H(h)) \).
4. hesitant fuzzy b-open if \( h \subseteq int_H(cl_H(h)) \cup cl_H(int_H(h)) \).
5. hesitant fuzzy \( \beta \)-open if \( h \subseteq cl_H(int_H(cl_H(h))) \).

**Theorem 1.** Let \((X, \tau)\) be a hesitant fuzzy topological space, then the following statements are hold:

1. Every hesitant fuzzy open set is hesitant fuzzy \( \alpha \)-open.
2. Every hesitant fuzzy \( \alpha \)-open set is hesitant fuzzy preopen.
3. Every hesitant fuzzy \( \alpha \)-open set is hesitant fuzzy semiopen.
4. Every hesitant fuzzy preopen set is hesitant fuzzy b-open.
5. Every hesitant fuzzy semiopen set is hesitant fuzzy b-open.
6. Every hesitant fuzzy b-open set is hesitant fuzzy \( \beta \)-open.

**Proof.**

(1) If \( h_A \) is hesitant fuzzy open, then \( h_A = int_H(h_A) \subseteq int_H(cl_H(h_A)) = int_H(cl_H(int_H(h_A))) \). Thus, \( h_A \) is hesitant fuzzy \( \alpha \)-open.
Remark 1. The concepts of hesitant fuzzy preopen and hesitant fuzzy semiopen are independent.

Remark 2. The converse of the Theorem 7 need not be true as shown by the following examples.

Example 1. Consider the hesitant fuzzy sets in \( X = \{a, b, c\} \) given by:
\[
\begin{align*}
& h_1(a) = [0.7, 1], h_1(b) = \{0.2, 0.5, 0.8\}, h_1(c) = [0.7, 1], \\
& h_2(a) = [0.5, 1], h_2(b) = \{0.2, 0.5, 0.7\}, h_2(c) = (0.7, 1], \\
& h_3(a) = [0.7, 1], h_3(b) = \{0.2, 0.5\}, h_3(c) = (0.7, 1), \text{ and} \\
& h_4(a) = [0.5, 1], h_4(b) = \{0.2, 0.5, 0.7, 0.8\}, h_4(c) = [0.7, 1].
\end{align*}
\]
Then, \( \tau = \{h^0, h^1, h_1, h_2, h_3, h_4\} \) a hesitant topology on \( X \). If \( h_A \) is the hesitant fuzzy set in \( X \) given by:
\[
\begin{align*}
& (1) \ h_A(a) = [0.6, 1], h_A(b) = \{0.2, 0.5, 0.6, 0.8, 0.9\}, h_A(c) = [0.3, 1], \\
& \quad \text{then} \ h_A \text{ is hesitant fuzzy } \alpha\text{-open but } h_A \text{ is not hesitant fuzzy open.}
\end{align*}
\]
\[
\begin{align*}
& (2) \ h_A(a) = [0, 1], h_A(b) = \phi, h_A(c) = \phi, \\
& \quad \text{then} \ h_A \text{ is both hesitant fuzzy preopen and hesitant fuzzy } b\text{-open but } h_A \text{ is neither hesitant fuzzy } \alpha\text{-open nor hesitant fuzzy semiopen.}
\end{align*}
\]

Example 2. Consider the hesitant fuzzy sets in \( X = \{a, b, c\} \) given by:
\[
\begin{align*}
& h_1(a) = \{0.4\}, h_1(b) = \{0.1\}, h_1(c) = \{0.8\}, \\
& h_2(a) = \{0.3\}, h_2(b) = \{0.2\}, h_2(c) = \{0.7\}, \text{ and} \\
& h_3(a) = \{0.3, 0.4\}, h_3(b) = \{0.1, 0.2\}, h_3(c) = \{0.7, 0.8\}.
\end{align*}
\]
Then, \( \tau = \{h^0, h^1, h_1, h_2, h_3\} \) a hesitant topology on \( X \). If \( h_A \) is the hesitant fuzzy set in \( X \) given by:
\[
\begin{align*}
& h_A(a) = \{0.4, 0.6\}, h_A(b) = \{0.1, 0.6\}, h_A(c) = \{0.6, 0.8\}, \\
& \quad \text{then} \ h_A \text{ is both hesitant fuzzy semiopen and hesitant fuzzy } b\text{-open but } A \text{ is neither hesitant fuzzy } \alpha\text{-open nor hesitant fuzzy preopen.}
\end{align*}
\]

Example 3. Consider the hesitant fuzzy sets in \( X = \{a\} \) given by:
\[
\begin{align*}
& h_1(a) = \{0.1\},
\end{align*}
\]
Theorem 2. Let \( (X, \tau) \) be a hesitant fuzzy topological space and \( h_A \in HS(X) \). Then:

(1) \( cl_H(h_A) \cap h_G \subseteq cl_H(h_A \cap h_G) \), for every hesitant fuzzy open set \( h_G \).
(2) \( int_H(h_A \cup h_F) \subseteq int_H(h_A) \cup h_F \), for every hesitant fuzzy closed set \( h_F \).

Proof. (1) Let \( x_\delta \in cl_H(h_A) \cap h_G \), then \( x_\delta \in cl_H(h_A) \) and \( x_\delta \in h_G \). If \( h_V \) is a hesitant fuzzy open set containing \( x_\delta \), then \( h_V \cap h_G \) is also hesitant fuzzy open set containing \( x_\delta \). Since \( x_\delta \in cl_H(h_A) \) implies \( h_V \cap h_G \cap h_A \neq h^0 \) and hence \( h_V \cap h_G \cap h_A \neq h^0 \). This is true for every \( h_V \) containing \( x_\delta \), so \( x_\delta \in cl_H(h_G \cap h_A) \). Therefore hesitant fuzzy \( cl_H(h_A) \cap h_G \subseteq cl_H(h_A \cap h_G) \).

(2) Follows from (1) and so it is obvious.

\[ \square \]

Theorem 3. If \( \{h_i : i \in \mathbb{N}\} \) is a collection of hesitant fuzzy b-open (resp. hesitant fuzzy \( \alpha \)-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy \( \beta \)-open) sets of a hesitant fuzzy topological space \( (X, \tau) \), then \( \bigcup_{i \in \mathbb{N}} h_i \) is a hesitant fuzzy b-open (resp. hesitant fuzzy \( \alpha \)-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy \( \beta \)-open) set.

Proof. We prove only the first case since the other cases are similarly shown. Since \( h_i \subseteq int_H(cl_H(h_i)) \cup cl_H(int_H(h_i)) \) for every \( i \in \mathbb{N} \), we have

\[ \bigcup_{i \in \mathbb{N}} h_i \subseteq \bigcup_{i \in \mathbb{N}} [int_H(cl_H(h_i)) \cup cl_H(int_H(h_i))] \]
Theorem 4. Let \((\tilde{U}, \tau)\) be a hesitant fuzzy topological space, \(h_U \in \tau\) and \(h_A \in HS(X)\).

1. If \(h_A\) is hesitant fuzzy preopen, then \(h_U \cap h_A\) is hesitant fuzzy preopen.

2. If \(h_A\) is hesitant fuzzy semiopen, then \(h_U \cap h_A\) is hesitant fuzzy semiopen.

Proof. (1) Since \(h_A\) is hesitant fuzzy preopen and \(h_U\) is hesitant fuzzy open, then \(h_A \subseteq \text{int}_H(\text{cl}_H(h_A))\) and \(\text{int}_H(h_U) = h_U\) and so by Theorem 2 (1),

\[
\begin{align*}
\text{cl}_H(h_U \cap h_A) \subseteq \text{int}_H(h_U) \cap \text{int}_H(\text{cl}_H(h_A)) = \text{int}_H(h_U \cap \text{cl}_H(h_A)) \subseteq \text{int}_H(h_U \cap h_A).
\end{align*}
\]

Therefore, \(h_U \cap h_A\) is hesitant fuzzy preopen.

(2) Since \(h_A\) is hesitant fuzzy semiopen, then by Theorem 2 (1),

\[
\begin{align*}
\text{cl}_H(h_U \cap h_A) \subseteq \text{int}_H(\text{cl}_H(h_A)) \subseteq \text{cl}_H(h_U \cap \text{cl}_H(h_A)) = \text{cl}_H(h_U \cap \text{cl}_H(h_A)).
\end{align*}
\]

Therefore, \(h_U \cap h_A\) is hesitant fuzzy semiopen.

\[\blacksquare\]

Theorem 5. Let \((\tilde{U}, \tau)\) be a hesitant fuzzy topological space, \(h_U \in \tau\) and \(h_A \in HS(X)\). If \(h_A\) is hesitant fuzzy \(\beta\)-open, then \(h_U \cap h_A\) is hesitant fuzzy \(\beta\)-open.

Proof. Since \(h_A\) is hesitant fuzzy \(\beta\)-open, then

\[
\begin{align*}
h_U \cap h_A \subseteq h_U \cap \text{cl}_H(\text{int}_H(h_A)) \\
\subseteq \text{cl}_H(h_U \cap \text{int}_H(h_A)) \\
= \text{cl}_H(\text{int}_H(h_U) \cap \text{int}_H(h_A)) \\
= \text{cl}_H(\text{int}_H(h_U \cap h_A)) \\
\subseteq \text{cl}_H(h_U \cap \text{cl}_H(h_A)).
\end{align*}
\]

This shows that \(h_U \cap h_A\) is hesitant fuzzy \(\beta\)-open.

\[\blacksquare\]

Theorem 6. Let \((\tilde{U}, \tau)\) be a hesitant fuzzy topological space, \(h_U \in \tau\) and \(h_A \in HS(X)\). If \(h_A\) is hesitant fuzzy b-open, then \(h_U \cap h_A\) is hesitant fuzzy b-open.

Proof. Since \(h_A\) is hesitant fuzzy b-open, then

\[
\begin{align*}
h_U \cap h_A \subseteq h_U \cap [\text{int}_H(\text{cl}_H(h_A))] \cap \text{cl}_H(\text{int}_H(h_A)) \\
= [h_U \cap \text{int}_H(\text{cl}_H(h_A))] \cap [h_U \cap \text{cl}_H(\text{int}_H(h_A))] \\
= [\text{int}_H(h_U) \cap \text{int}_H(h_A)] \cap [h_U \cap \text{cl}_H(\text{int}_H(h_A))] \\
\subseteq [\text{int}_H(h_U) \cap \text{cl}_H(h_A)] \cap [h_U \cap \text{int}_H(h_A)].
\end{align*}
\]

\[\blacksquare\]
This shows that \( h_U \cap h_A \) is hesitant fuzzy b-open. \( \square \)

**Remark 4.** We note that the intersection of two hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy \( \beta \)-open) sets need not be hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy \( \beta \)-open) as can be seen from the following examples:

**Example 4.** Consider the hesitant fuzzy sets in \( X = \{ a \} \) given by \( h(a) = \{ 0.3, 0.6 \} \). Then, \( \tau = \{ h^0, h^1, h \} \) is a hesitant topology on \( X \). If \( h_A(a) = \{ 0.1, 0.3 \} \) and \( h_B(a) = \{ 0.1, 0.6 \} \), then \( h_A \) and \( h_B \) are hesitant fuzzy preopen (resp. hesitant fuzzy b-open and hesitant fuzzy \( \beta \)-open), but \( h_A \cap h_B = \{ 0.1 \} = h_C \) which is not hesitant fuzzy preopen (resp. hesitant fuzzy b-open and hesitant fuzzy \( \beta \)-open).

**Example 5.** From Example 3 if \( h_A \) is the hesitant fuzzy set in \( X \) given by:
\[
\begin{align*}
  h_A(a) &= \{ 0.4, 0.6 \}, \\
  h_A(b) &= \{ 0.1, 0.6 \}, \\
  h_A(c) &= \{ 0.6, 0.8 \},
\end{align*}
\]
and \( h_B \) is the hesitant fuzzy set in \( X \) given by:
\[
\begin{align*}
  h_B(a) &= \{ 0.3, 0.6 \}, \\
  h_B(b) &= \{ 0.2, 0.6 \}, \\
  h_B(c) &= \{ 0.7, 0.9 \},
\end{align*}
\]
then \( h_A \) and \( h_B \) are hesitant fuzzy semiopen, but \( h_A \cap h_B = h_C \) which is not hesitant fuzzy semiopen, where \( h_C \) is the hesitant fuzzy set in \( X \) given by:
\[
\begin{align*}
  h_C(a) &= \{ 0.6 \}, \\
  h_C(b) &= \{ 0.6 \}, \\
  h_C(c) &= \varnothing.
\end{align*}
\]

**Remark 5.** From Remark 4 we notice that the family of all hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy \( \beta \)-open) sets need not be a topology in general.

**Theorem 7.** Let \((X, \tau)\) be a hesitant fuzzy topological space. If \( h_A \) and \( h_B \) are hesitant fuzzy \( \alpha \)-open, then \( h_B \cap h_A \) is also hesitant fuzzy \( \alpha \)-open.

**Proof.** Since \( h_A \) and \( h_B \) are hesitant fuzzy \( \alpha \)-open, then
\[
\begin{align*}
  h_B \cap h_A \subseteq & \text{int}_H(\text{cl}_H(\text{int}_H(h_B))) \cap \text{int}_H(\text{cl}_H(\text{int}_H(h_A))) \\
  \subseteq & \text{int}_H[\text{cl}_H(\text{int}_H(h_B)) \cap \text{int}_H(\text{cl}_H(\text{int}_H(h_A)))] \\
  \subseteq & \text{int}_H \text{cl}_H[\text{int}_H(h_B) \cap \text{int}_H(h_A)] \\
  \subseteq & \text{int}_H \text{cl}_H \text{cl}_H[\text{int}_H(h_B) \cap \text{int}_H(h_A)] \\
  \subseteq & \text{int}_H \text{cl}_H \text{cl}_H \text{cl}_H[\text{int}_H(h_B) \cap \text{int}_H(h_A)].
\end{align*}
\]
Thus, \( h_B \cap h_A \) is hesitant fuzzy \( \alpha \)-open. \( \square \)

**Remark 6.** From the Theorems 3 and 7 we notice that the family of all hesitant fuzzy \( \alpha \)-open is a topology.
Theorem 8. Let \((X, \tau)\) be a hesitant fuzzy topological space and \(h_A \in HS(X)\). If \(h_A\) is both hesitant fuzzy semiopen and hesitant fuzzy preopen, then \(h_A\) is hesitant fuzzy \(\alpha\)-open.

Proof. By assumption, \(h_A \subseteq cl_H(int_H(h_A))\) and \(h_A \subseteq int_H(cl_H(h_A))\). Then, \(h_A \subseteq int_H(cl_H(int_H(h_A))) \subseteq int_H(cl_H(cl_H(int_H(h_A)))) = int_H(cl_H(int_H(h_A)))\). Therefore, \(h_A\) is hesitant fuzzy \(\alpha\)-open. \(\Box\)

Theorem 9. Let \((X, \tau)\) be a hesitant fuzzy topological space and \(h_A\) be hesitant fuzzy \(\alpha\)-open.

1. If \(h_B\) is hesitant fuzzy semiopen, then \(h_A \cap h_B\) is hesitant fuzzy semiopen.
2. If \(h_B\) is hesitant fuzzy preopen, then \(h_A \cap h_B\) is hesitant fuzzy preopen.

Proof. (1) By assumption, \(h_A \subseteq int_H(cl_H(int_H(h_A)))\) and \(h_B \subseteq cl_H(int_H(h_B))\), then by Theorem 2 (1), we have that
\[
\begin{align*}
h_A \cap h_B \subseteq cl_H(int_H(int_H(h_A))) \cap cl_H(int_H(h_B)) \\
\subseteq cl_H(int_H(cl_H(int_H(h_A)))) \cap int_H(h_B) \\
\subseteq cl_H(cl_H(int_H(h_A))) \cap int_H(h_B) \\
\subseteq cl_H(int_H(h_A)) \cap int_H(h_B) \\
= cl_H(int_H(h_A \cap h_B)).
\end{align*}
\]
Therefore, \(h_A \cap h_B\) is hesitant fuzzy semiopen.

(2) By assumption, \(h_A \subseteq int_H(cl_H(int_H(h_A)))\) and \(h_B \subseteq int_H(cl_H(h_B))\), then
\[
\begin{align*}
h_A \cap h_B \subseteq int_H(cl_H(int_H(h_A))) \cap int_H(cl_H(h_B)) \\
= int_H(int_H(cl_H(int_H(h_A))) \cap int_H(cl_H(h_B))) \\
\subseteq int_H(cl_H(int_H(h_A)) \cap int_H(cl_H(h_B))) \\
\subseteq int_H(int_H(cl_H(int_H(h_A))) \cap int_H(cl_H(h_B))) \\
\subseteq int_H(cl_H(int_H(h_A)) \cap int_H(cl_H(h_B))) \\
= int_H(cl_H(h_A \cap h_B)).
\end{align*}
\]
Therefore, \(h_A \cap h_B\) is hesitant fuzzy preopen. \(\Box\)

Theorem 10. Let \((X, \tau)\) be a hesitant fuzzy topological space. If \(h_A\) is hesitant fuzzy preopen and \(h_B\) is hesitant fuzzy semiopen, then \(h_A \cap h_B\) is hesitant fuzzy \(\beta\)-open.

Proof. By assumption, \(h_A \subseteq int_H(cl_H(h_A))\) and \(h_B \subseteq cl_H(int_H(h_B))\), then by Theorem 2 (1), we have that
\[
\begin{align*}
h_A \cap h_B \subseteq int_H(cl_H(h_A)) \cap cl_H(int_H(h_B)) \\
\subseteq cl_H(int_H(cl_H(h_A)) \cap int_H(h_B))
\end{align*}
\]
Therefore, \( h_A \bar{\cap} h_B \) is hesitant fuzzy \( \beta \)-open. \( \square 

**Theorem 11.** Let \((X, \tau)\) be a hesitant fuzzy topological space and \( h_A, h_B \in HS(X) \). Then,

1. \( h_A \) is hesitant fuzzy semiopen if and only if there exists a hesitant fuzzy open set \( h_U \) such that \( h_U \subseteq h_A \subseteq cl_H(h_U) \).
2. \( h_B \) is hesitant fuzzy semiopen if \( h_A \) is hesitant fuzzy semiopen and \( h_A \subseteq h_B \subseteq cl_H(h_A) \).
3. \( h_A \) is hesitant fuzzy semiopen if and only if \( cl_H(h_A) = cl_H(int_H(h_A)) \).

**Proof:**

(1) Let \( h_A \) be hesitant fuzzy semiopen, then \( h_A \subseteq cl_H(int_H(h_A)) \). Take \( h_U = int_H(h_A) \), then \( h_U \) is hesitant fuzzy open such that \( h_U = int_H(h_A) \subseteq h_A \subseteq cl_H(int_H(h_A)) = cl_H(h_U) \).

Conversely, since \( h_U \subseteq h_A \) implies that \( h_U = int_H(h_U) \subseteq int_H(h_A) \) and so \( h_A \subseteq cl_H(h_U) = cl_H(int_H(h_U)) \subseteq cl_H(int_H(h_A)) \). Thus, \( h_A \) is hesitant fuzzy semiopen.

(2) Since \( h_A \) is hesitant fuzzy semiopen, then by (1) there exists a hesitant fuzzy open set \( h_U \) such that \( h_U \subseteq h_A \subseteq cl_H(h_U) \). Since \( h_A \subseteq h_B \), so \( h_U \subseteq h_B \). But \( cl_H(h_A) \subseteq cl_H(h_U) \), then \( h_B \subseteq cl_H(h_U) \). Hence, \( h_U \subseteq h_B \subseteq cl_H(h_U) \). Thus, \( h_B \) is hesitant fuzzy semiopen.

(3) Let \( h_A \) be hesitant fuzzy semiopen, then \( h_A \subseteq cl_H(int_H(h_A)) \) which implies that \( cl_H(h_A) \subseteq cl_H(int_H(h_A)) \subseteq cl_H(h_A) \) and hence \( cl_H(h_A) = cl_H(int_H(h_A)) \).

Conversely, since by Theorem 1 \( int_H(h_A) \) is hesitant fuzzy semiopen such that \( int_H(h_A) \subseteq h_A \subseteq cl_H(h_A) = cl_H(int_H(h_A)) \) and therefore \( h_A \) is hesitant fuzzy semiopen. \( \square 

**Definition 7.** \([10]\) Let \((X, \tau)\) be a hesitant fuzzy topological space and \( h \in HS(X) \). Then, the collection \( \tau_h = \{ U \cap h : U \in \tau \} \) is called a hesitant fuzzy subspace topology or hesitant fuzzy relative topology on \( h \). The pair \((h, \tau_h)\) is called a hesitant fuzzy subspace, and each member of \( \tau_h \) is called a hesitant fuzzy open set in \( h \).

**Proposition 1.** \([10]\) Let \((X, \tau)\) be a hesitant fuzzy topological space, \( h, h_A \in HS(X) \) and \( h_A \subseteq h \). Then, \( cl_{\tau_h}(h_A) = h \bar{\cap} cl_H(h_A) \), where \( cl_{\tau_h}(h_A) \) denotes the closure of \( h_A \) in \((h, \tau_h)\).

**Definition 8.** Let \((X, \tau)\) be a hesitant fuzzy topological space, \( h, h_A \in HS(X) \) and \( h_A \subseteq h \). Then, \( int_{\tau_h}(h_A) = \bigcup\{ h_U \in \tau_h : h_U \subseteq h_A \} \).
Theorem 12. Let $(X, \tau)$ be a hesitant fuzzy topological space and $h_A, h_B \in HS(X)$. If $h_A$ is hesitant fuzzy preopen in $X$ and $h_B$ is hesitant fuzzy semiopen in $X$, then

1. $h_A \overline{\cap} h_B$ is hesitant fuzzy semiopen in $h_A$.
2. $h_A \overline{\cap} h_B$ is hesitant fuzzy preopen in $h_B$.

Proof. By assumption, $h_A \subseteq int_H(cl_H(h_A))$ and $h_B \subseteq cl_H(int_H(h_B))$.

1. Then,

$$h_A \overline{\cap} h_B \subseteq int_H(cl_H(h_A)) \overline{\cap} cl_H(int_H(h_B))$$
$$\subseteq cl_H(int_H(cl_H(h_A)) \overline{\cap} int_H(h_B))$$
$$\subseteq cl_H(cl_H(h_A) \overline{\cap} int_H(h_B))$$
$$= cl_H[h_A \overline{\cap} int_H(h_B)].$$

Hence, $h_A \overline{\cap} h_B \subseteq cl_H(h_A \overline{\cap} int_H(h_B))$ and so $h_A \overline{\cap} h_B \subseteq cl_H(h_A \overline{\cap} int_H(h_B)) \overline{\cap} h_A = cl_{\tau \overline{\cap} a}(h_A \overline{\cap} int_H(h_B))$. Since $h_A \overline{\cap} int_H(h_B)$ is a hesitant fuzzy open set in $h_A$, so $h_A \overline{\cap} h_B \subseteq cl_{\tau \overline{\cap} a}(h_A \overline{\cap} int_H(h_B)) \subseteq cl_{\tau \overline{\cap} a}(int_{\tau \overline{\cap} a}(h_A \overline{\cap} h_B))$. Therefore, $h_A \overline{\cap} h_B$ is hesitant fuzzy semiopen in $h_A$.

2. Now,

$$h_A \overline{\cap} h_B \subseteq int_H(cl_H(h_A)) \overline{\cap} h_B$$
$$= int_{\tau \overline{\cap} b}[int_H(cl_H(h_A)) \overline{\cap} h_B]$$
$$\subseteq int_{\tau \overline{\cap} b}[int_H(cl_H(h_A)) \overline{\cap} cl_H(int_H(h_B))]$$
$$\subseteq int_{\tau \overline{\cap} b}[cl_H(int_H(cl_H(h_A)) \overline{\cap} int_H(h_B))]$$
$$\subseteq int_{\tau \overline{\cap} b}[cl_H(cl_H(h_A) \overline{\cap} int_H(h_B))]$$
$$\subseteq int_{\tau \overline{\cap} b}[cl_H(cl_H(h_A) \overline{\cap} h_B)]$$
$$= int_{\tau \overline{\cap} b}(cl_H(h_A \overline{\cap} h_B)).$$

Since $int_{\tau \overline{\cap} b}(cl_H(h_A \overline{\cap} h_B))$ is hesitant fuzzy open in $h_B$, then

$$int_{\tau \overline{\cap} b}(cl_H(h_A \overline{\cap} h_B)) \overline{\cap} h_B = int_{\tau \overline{\cap} b}(cl_H(h_A \overline{\cap} h_B)) \overline{\cap} h_B,$$

and hence $h_A \overline{\cap} h_B \subseteq int_{\tau \overline{\cap} b}(cl_H(h_A \overline{\cap} h_B)) \overline{\cap} h_B = int_{\tau \overline{\cap} b}(cl_{\tau \overline{\cap} b}(h_A \overline{\cap} h_B)).$

Therefore, $h_A \overline{\cap} h_B$ is hesitant fuzzy preopen in $h_B$. \qed

Theorem 13. Let $(X, \tau)$ be a hesitant fuzzy topological space, $h_A, h_B \in HS(X)$, $h_A \subseteq h_B$ and $h_B$ be hesitant fuzzy semiopen in $X$. Then, $h_A$ is hesitant fuzzy semiopen in $X$ if and only if $h_A$ is hesitant fuzzy semiopen in $h_B$.

Proof. Let $h_A$ be hesitant fuzzy semiopen in $X$, then there is a hesitant fuzzy open set $h_U$ such that $h_U \subseteq h_A \subseteq cl_H(h_U)$ implies that $h_U \subseteq h_A \subseteq h_B$. Hence,
Let \((X, \tau)\) be a hesitant fuzzy topological space, \(h_A, h_B \in HS(X)\), \(h_A \subseteq h_B\) and \(h_B\) be hesitant fuzzy preopen in \(X\). Then, \(h_A\) is hesitant fuzzy preopen in \(X\) if and only if \(h_A\) is hesitant fuzzy preopen in \(h_B\).

**Proof.** Suppose that \(h_A\) is hesitant fuzzy preopen in \(X\), then \(h_A = h_A \cap h_B \subseteq int_H(cl_H(h_A)) \cap h_B\). Since \(int_H(cl_H(h_A)) \cap h_B\) is hesitant fuzzy open in \(h_B\), then \(h_A \subseteq int_H(cl_H(h_A)) \cap h_B \subseteq int_{\tau_B}[int_H(cl_H(h_A)) \cap h_B] \subseteq int_{\tau_B}[cl_H(h_A) \cap h_B] = int_{\tau_B}(cl_{\tau_B}(h_A))\). Hence, \(h_A\) is hesitant fuzzy preopen in \(h_B\).

Conversely, assume that \(h_A\) is hesitant fuzzy preopen in \(h_B\). Then, \(h_A \subseteq int_{\tau_B}(cl_{\tau_B}(h_A))\). Since \(int_{\tau_B}(cl_{\tau_B}(h_A))\) is hesitant fuzzy open in \(h_B\), so there a hesitant fuzzy open set \(h_U\) in \(X\) such that \(int_{\tau_B}(cl_{\tau_B}(h_A)) = h_U \cap h_B\). By Theorem 9 (2), \(int_{\tau_B}(cl_{\tau_B}(h_A))\) is hesitant fuzzy preopen in \(X\). Therefore,

\[
h_A \subseteq int_{\tau_B}(cl_{\tau_B}(h_A)) \\
\subseteq int_H(cl_H(int_{\tau_B}(cl_{\tau_B}(h_A)))) \\
= int_H(cl_H(int_{\tau_B}[cl_H(h_A) \cap h_B])) \\
\subseteq int_H(cl_H(cl_H(h_A))) \\
\subseteq int_H(cl_H(h_A)).
\]

This shows that \(h_A\) is hesitant fuzzy preopen in \(X\). □

**Declaration of Competing Interests** The author of this paper declare that there are no conflicts of interest about publication of the paper.

**References**