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Decomposition of product graphs into sunlet graphs of order eight*

Research Article

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Abstract: For any integer $k \geq 3$, we define sunlet graph of order $2k$, denoted by L_{2k}, as the graph consisting of a cycle of length k together with k pendant vertices, each adjacent to exactly one vertex of the cycle. In this paper, we give necessary and sufficient conditions for the existence of L_8-decomposition of tensor product and wreath product of complete graphs.

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Keywords: Graph decomposition, Product graphs, Corona graph, Sunlet graph

1. Introduction

All graphs considered here are finite, simple and undirected. For the standard graph-theoretic terminology the readers are referred to [7]. A *cycle* of length k is called k-cycle and it is denoted by C_k. Let K_m denotes the *complete* graph on m vertices and $K_{m,n}$ denotes the *complete bipartite* graph with m and n vertices in the parts. We denote the *complete m-partite* graph with $n_1, n_2, \ldots, n_m$ vertices in the parts by $K_{n_1,n_2,\ldots,n_m}$. For any integer $\lambda > 0$, λG denotes the graph consisting of λ edge-disjoint copies of G. The *complement* of the graph G is denoted by $\overline{G}$. The subgraph of G induced by $S \subseteq V(G)$ is denoted as $\langle S \rangle$. For any two graphs G and H of orders m and n, respectively, the *corona product* $G \odot H$ is the graph obtained by taking one copy of G, m copies of H and then joining the ith vertex of G to every vertex in the ith copy of H. We define the *sunlet graph* L_{2k} with $V(L_{2k}) = \{x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}\}$ and $E(L_{2k}) = \{x_i x_{i+1}, x_i x_{k+i} \mid i = 1, 2, ..., k$ and subscripts of the first term is taken addition modulo $k\}$. We denote it by $L_{2k} = \begin{pmatrix} x_1 & x_2 & \ldots & x_k \\ x_{k+1} & x_{k+2} & \ldots & x_{2k} \end{pmatrix}$. Clearly, $C_k \odot K_1 \cong L_{2k}$.

For two graphs G and H, their *tensor product* $G \times H$ and *lexicographic* or *wreath product* $G \otimes H$ have

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the same vertex set $V(G) \times V(H) = \{(g,h) : g \in V(G)\ and\ h \in V(H)\}$ and their edge sets are defined as follows:
$E(G \times H) = \{(g,h)(g',h') : gg' \in E(G)$ and $hh' \in E(H)\}$,
$E(G \otimes H) = \{(g,h)(g',h') : gg' \in E(G)\ or\ g = g'$ and $hh' \in E(H)\}$. It is well known that the above products are associative and distributive over edge-disjoint unions of graphs and the tensor product is commutative. It is easy to observe that $K_m \otimes \overline{K_n} \cong K_{n,n,\ldots n}(m\ times)$.

We shall use the following notation throughout the paper. Let G and H be simple graphs with vertex sets $V(G) = \{x_1, x_2, \ldots, x_n\}$ and $V(H) = \{y_1, y_2, \ldots, y_m\}$. Then for our convenience, we write $V(G) \times V(H) = \bigcup_{i=1}^{n} X_i$, where X_i stands for $x_i \times V(H)$. Further, in the sequel, we shall denote the vertices of X_i as $\left\{x_i^j | 1 \leq j \leq m\right\}$, where x_i^j stands for the vertex $(x_i, y_j) \in V(G) \times V(H)$.

A *labeling* of a graph G with n edges is an injection ρ from $V(G)$, the vertex set of G, into a subset $S \subseteq Z_{2n+1}$, the additive group Z_{2n+1}. The length of an edge $e = xy$ with end vertices x and y is defined as $l(xy) = min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. Note that the subtraction is performed in Z_{2n+1} and hence $1 \leq l(e) \leq n$. If the length of the n edges are distinct and is equal to $\{1, 2, \ldots, n\}$, then ρ is a *rosy labeling*; moreover, if $S \subseteq \{1, 2, \ldots, n\}$, then ρ is a *graceful labeling.* A graceful labeling is said to be an *α-labeling* if there exists a number α_0 with the property that for every edge $e = xy$ in G with $\alpha(x) < \alpha(y)$ it holds that $\alpha(x) \leq \alpha_0 < \alpha(y)$.

By a *decomposition* of a graph G, we mean a list of edge-disjoint subgraphs of G whose union is G. For a graph G, if $E(G)$ can be partitioned into $E_1, E_2, ..., E_k$ such that the subgraph induced by E_i is H_i, for all i, $1 \leq i \leq k$, then we say that $H_1, H_2, ..., H_k$ decompose G and we write $G = H_1 \oplus H_2 \oplus ... \oplus H_k$, since $H_1, H_2, ..., H_k$ are edge-disjoint subgraphs of G. For $1 \leq i \leq k$, if $H_i \cong H$, we say that G has a *H-decomposition.*

Study of H-decomposition of graphs is not new. Many authors around the world are working in the field of cycle decomposition [4, 8, 9, 21, 22], path decomposition [24, 25], star decompositon [19, 23, 26, 27] and Hamilton cycle decomposition [2, 3, 15, 16] problems in graphs. Here we consider the sunlet decomposition of product graphs. Anitha and Lekshmi [5, 6] proved that n-sun decomposition of complete graph, complete bipartite graph and the Harary graphs. Liang and Guo [17, 18] gave the existence spectrum of a k-sun system of order v as $k = 2, 4, 5, 6, 8$. Fu et. al. [12, 13] obtained that 3-sun decompositions of $K_{p,p,r}$, $K_n$and embed a cyclic steiner triple system of order n into a 3-sun system of order $2n-1$, for $n = 1(mod\ 6)$. Further they obtained k-sun system when $k = 6, 10, 14, 2^t$, for $t > 1$. Fu et. al. [11] obtained the existence of a 5-sun system of order v. Gionfriddo et.al. [14] obtained the spectrum for uniformly resolvable decompositions of K_v into 1-factor and h-suns. Akwu and Ajayi [1] obtained the necessary and sufficient conditions for the existence of decomposition of $K_n \otimes \overline{K_m}$ and $(K_n - I) \otimes \overline{K_m}$, where I denote the 1-factor of a complete graph into sunlet graph of order twice the prime.

In this paper, we obtain the decomposition of some product graphs into sunlet graphs of order eight which is the least even order not proved so far for product graphs, which motivate us to consider this problem. In Section 2, we obtain the necessary and sufficient conditions for the existence of L_8-decomposition of complete bipartite graphs with part size m and n. In Section 3, we obtain the necessary and sufficient conditions for the existence of L_8-decomposition of tensor product of complete graphs. In Section 4, we obtain the necessary and sufficient conditions for the existence of L_8-decomposition of complete multipartite graphs with uniform part size.

To prove our results, we state the following:

Theorem 1.1. *[20] For all $n \geq 3$, $C_n \odot K_1$ is an α-labeling.*

Theorem 1.2. *[10] Let G be a graph with n edges. If G admits a rosy labeling, then it decomposes K_{2n+1}; if G admits an α-labeling, then it decomposes K_{2np+1} for every $p > 0$.*

Theorem 1.3. *[13] Let $t \geq 2$ be an integer. An $L_{2.2^t}$-decomposition of K_n exists if and only if $n \equiv 0\ (or)\ 1\ (mod\ 2^{t+2})$.*

Remark 1.4. *If $n \equiv\ 0\ (mod\ 4)$, then $K_{4,n}$ can be decomposed as copies of $K_{4,4}$ and L_8- decomposition of $K_{4,4}$ is shown in below figure. Therefore L_8-decomposition exists in $K_{4,n}$ for n is a multiple of 4.*

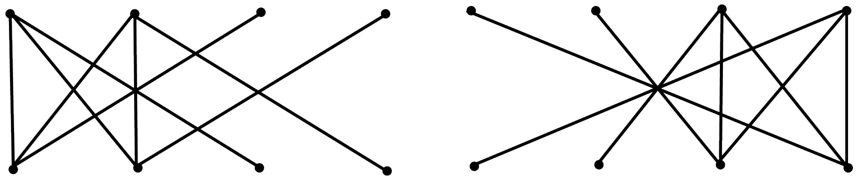

Figure 1. L_8**- decomposition of** $K_{4,4}$**.**

2. L_8- decomposition of $K_{m,n}$

Now we obtain the necessary and sufficient conditions for the existence of an L_8-decomposition of $K_{m,n}$ as follows. Let the vertices of $K_{m,n}$ be $\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\}$.

Lemma 2.1. *There exists an L_8- decomposition of $K_{8,6}$.*

Proof. We exhibit the L_8- decomposition of $K_{8,6}$ as follows:

$$\begin{pmatrix} x_1 & y_1 & x_2 & y_2 \\ y_3 & x_5 & y_4 & x_6 \end{pmatrix}, \begin{pmatrix} x_3 & y_3 & x_4 & y_4 \\ y_5 & x_2 & y_6 & x_1 \end{pmatrix}, \begin{pmatrix} x_5 & y_5 & x_6 & y_6 \\ y_3 & x_1 & y_4 & x_2 \end{pmatrix}, \begin{pmatrix} x_7 & y_5 & x_8 & y_6 \\ y_1 & x_2 & y_2 & x_1 \end{pmatrix},$$
$$\begin{pmatrix} x_7 & y_3 & x_8 & y_4 \\ y_2 & x_6 & y_1 & x_5 \end{pmatrix}, \begin{pmatrix} x_3 & y_1 & x_4 & y_2 \\ y_6 & x_6 & y_5 & x_5 \end{pmatrix}.$$

□

Lemma 2.2. *There exists an L_8- decomposition of $K_{8,7}$.*

Proof. We exhibit the L_8- decomposition of $K_{8,7}$ as follows:

$$\begin{pmatrix} x_1 & y_1 & x_2 & y_2 \\ y_5 & x_4 & y_7 & x_7 \end{pmatrix}, \begin{pmatrix} x_3 & y_3 & x_4 & y_4 \\ y_2 & x_1 & y_5 & x_2 \end{pmatrix}, \begin{pmatrix} x_5 & y_5 & x_6 & y_6 \\ y_4 & x_3 & y_7 & x_1 \end{pmatrix}, \begin{pmatrix} x_7 & y_1 & x_8 & y_7 \\ y_3 & x_6 & y_6 & x_5 \end{pmatrix},$$
$$\begin{pmatrix} x_7 & y_4 & x_8 & y_5 \\ y_6 & x_1 & y_3 & x_2 \end{pmatrix}, \begin{pmatrix} x_3 & y_6 & x_4 & y_7 \\ y_1 & x_2 & y_2 & x_1 \end{pmatrix}, \begin{pmatrix} x_5 & y_2 & x_6 & y_3 \\ y_1 & x_8 & y_4 & x_2 \end{pmatrix}.$$

□

Lemma 2.3. *There exists an L_8- decomposition of $K_{8,9}$.*

Proof. We exhibit the L_8- decomposition of $K_{8,9}$ as follows:

$$\begin{pmatrix} x_1 & y_1 & x_2 & y_2 \\ y_3 & x_6 & y_4 & x_7 \end{pmatrix}, \begin{pmatrix} x_3 & y_3 & x_4 & y_4 \\ y_8 & x_5 & y_5 & x_6 \end{pmatrix}, \begin{pmatrix} x_5 & y_5 & x_6 & y_6 \\ y_1 & x_3 & y_7 & x_4 \end{pmatrix}, \begin{pmatrix} x_7 & y_7 & x_8 & y_8 \\ y_5 & x_4 & y_6 & x_5 \end{pmatrix},$$
$$\begin{pmatrix} x_3 & y_1 & x_4 & y_9 \\ y_6 & x_8 & y_2 & x_7 \end{pmatrix}, \begin{pmatrix} x_5 & y_2 & x_6 & y_9 \\ y_7 & x_3 & y_8 & x_8 \end{pmatrix}, \begin{pmatrix} x_7 & y_3 & x_8 & y_4 \\ y_1 & x_6 & y_2 & x_5 \end{pmatrix}, \begin{pmatrix} x_1 & y_5 & x_2 & y_6 \\ y_4 & x_8 & y_9 & x_7 \end{pmatrix},$$
$$\begin{pmatrix} x_1 & y_7 & x_2 & y_8 \\ y_9 & x_3 & y_3 & x_4 \end{pmatrix}.$$

□

Lemma 2.4. *There exists an L_8- decomposition of $K_{12,6}$.*

Proof. We exhibit the L_8- decomposition of $K_{12,6}$ as follows:

$$\begin{pmatrix} x_1 & y_1 & x_2 & y_2 \\ y_3 & x_{12} & y_4 & x_{11} \end{pmatrix}, \begin{pmatrix} x_3 & y_3 & x_4 & y_4 \\ y_5 & x_2 & y_6 & x_1 \end{pmatrix}, \begin{pmatrix} x_5 & y_5 & x_6 & y_6 \\ y_1 & x_1 & y_2 & x_2 \end{pmatrix}, \begin{pmatrix} x_7 & y_1 & x_8 & y_2 \\ y_3 & x_9 & y_4 & x_{10} \end{pmatrix},$$
$$\begin{pmatrix} x_9 & y_3 & x_{10} & y_4 \\ y_6 & x_{11} & y_5 & x_{12} \end{pmatrix}, \begin{pmatrix} x_{11} & y_5 & x_{12} & y_6 \\ y_1 & x_2 & y_2 & x_1 \end{pmatrix}, \begin{pmatrix} x_3 & y_1 & x_4 & y_2 \\ y_6 & x_{10} & y_5 & x_9 \end{pmatrix}, \begin{pmatrix} x_5 & y_3 & x_6 & y_4 \\ y_2 & x_{12} & y_1 & x_{11} \end{pmatrix},$$
$$\begin{pmatrix} x_7 & y_5 & x_8 & y_6 \\ y_4 & x_9 & y_3 & x_{10} \end{pmatrix}.$$

□

Lemma 2.5. *There is no L_8- decomposition of $K_{8,5}$.*

Proof. Let A and B be the partite set of $K_{8,5}$ such that $|A| = 8$ and $|B| = 5$. In L_8, four vertices are of degree 3 and four vertices are of degree 1. Since $K_{8,5}$ is a bipartite graph, then L_8 has two vertices of degree 3 and two vertices of degree 1 in one partite set A and similarly in B. Total number of edges in $K_{8,5}$ is 40, then we have $5L_8$ in $K_{8,5}$. First we pull out $4L_8$ from $K_{8,5}$(as shown in Fig.2). Since each vertices in A has degree 5, the remaining degree of each vertices of $K_{8,5}\backslash 4L_8$in the set A is 1. Here we can't find a L_8 in $K_{8,5}\backslash 4L_8$, since we need atleast two vertices of degree 3. Hence we conclude that L_8-decomposition does not exists in $K_{8,5}$. □

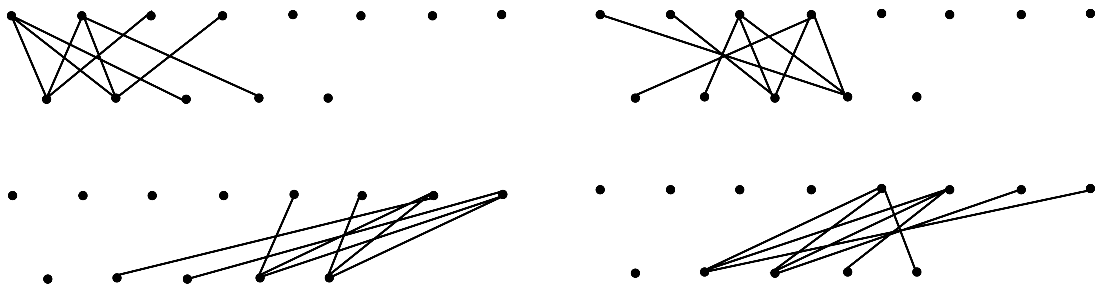

Figure 2. $4L_8$ in $K_{8,5}$

Lemma 2.6. *There is no L_8- decomposition of $K_{4,n}$ for $n \equiv 2 \ (mod\ 4)$.*

Proof. Let $n = 4s + 2$ for some $s > 0$. Suppose that $K_{4,n}$ has an L_8-decomposition, then it has $(2s + 1)L_8$. Let $A = \{x_1, x_2, x_3, x_4\}$ and $B = \{y_1, y_2, ..., y_{4s+1}, y_{4s+2}\}$ be the partite sets of $K_{4,n}$. Consider the $(2s)L_8 = \left\{L_8^1, L_8^2, ..., L_8^{2s}\right\}$ which exists in $K_{4,n-2}$. Now we have to find the last L_8 i.e., L_8^{2s+1}.

Let $(x_1y_{4s+1}x_2y_{4s+2})$ be a cycle in $K_{4,n}$. Then join y_{4s+1} to x_3 and y_{4s+2} to x_4. Now we have to find pendant edges to the vertices x_1 and x_2. Suppose there are vertices y_a and y_b which are joined to x_1 and x_2, resp, as the pendant edges in $L_8^{t_1}$ for some $t_1 \in \{1, 2, ..., 2s\}$. Then we can join these edges to the vertices x_1 and x_2 in L_8^{2s+1}. Suppose $y_a = y_{4s+1}$ and $y_b = y_{4s+2}$ or viceversa, then we can join the remaining edges x_3y_{4s+2}, x_4y_{4s+1} in $K_{4,n}$ to y_a and y_b, resp. Therefore $deg_{L_8^{t_1}}(y_{4s+1}) = deg_{L_8^{t_1}}(y_{4s+2}) = 3$ and $deg_{L_8^{2s+1}}(y_{4s+1}) = deg_{L_8^{2s+1}}(y_{4s+2}) = 3$. This implies $deg(y_{4s+1}) = deg_(y_{4s+2}) = 6$,which is a contradiction. Therefore $y_a \neq y_{4s+1}$ and $y_b \neq y_{4s+2}$ or viceversa.

Then we find the pendant edges to y_a and y_b. There exist vertices x_i and x_j, $i, j \in \{1, 2, 3, 4\}$ which are joined to y_a and y_b,resp, as the pendant edges in $L_8^{t_2}$ for some $t_1 \neq t_2 \in \{1, 2, ..., 2s\}$. Then we can join these edges to the vertices y_a and y_b in $L_8^{t_1}$. Now $x_i \neq x_3$ and $x_j \neq x_4$ or viceversa, since x_3y_a, x_3y_b, x_4y_a and x_4y_b are the edges of the cycle in $L_8^{t_1}$. Then x_i, x_j must be x_1, x_2. Again we have to find the pendant edges to x_1, x_2. Repeat the above procedure cyclically we get to find the pendant edges of x_1, x_2. Therefore we can't find the pendant edges to x_1, x_2. Hence the proof. □

Theorem 2.7. *For any $m, n \geq 4$, $K_{m,n}$ has an L_8- decomposition if and only if $mn \equiv 0 \ (mod\ 8)$ except $(m, n) = (4, 2 \ (mod\ 4))$ & $(8, 5)$.*

Proof. **Necessity.** We first observe that $K_{m,n}$ has $m + n$ vertices and mn edges. Assume that $K_{m,n}$ admits an L_8- decomposition. Then the number of edges in the graph must be divisible by 8 i.e., $8|mn$ and hence $mn \equiv 0 \ (mod\ 8)$. Further, $(m, n) \neq (4, 2 \ (mod\ 4))$ & $(8, 5)$ follows from Lemmas 2.6 and 2.5.
Sufficiency. We construct the required decomposition in two cases.
Case(1) m *(or)* $n \equiv\ 0 \ (mod\ 8)$.
Suppose we take $m \equiv\ 0 \ (mod\ 8)$. Further we divide the proof into four subcases.
Subcase(i) $m \equiv\ 0 \ (mod\ 8)$and $n \equiv\ 0 \ (mod\ 4)$.
Let $m = 8s$ and $n = 4t$ for some $s, t > 0$. Then we can write $K_{m,n} = 2stK_{4,4}$. We know that $K_{4,4}$ has an L_8-decomposition(see Fig.2).

Subcase(ii) $m \equiv\ 0\ (mod\ 8)$and $n \equiv\ 1\ (mod\ 4)$.
Let $m = 8s$ and $n = 4t + 1$ for some $s, t > 1$, since for $s = t = 1$, $K_{m,n}$ has no L_8- decomposition by Lemma 2.5. Then we can write $K_{m,n} = 2s(t-2)K_{4,4} \oplus sK_{8,9}$. Then by Lemma 2.3, we get an L_8-decomposition of $K_{m,n}$.
Subcase(iii) $m \equiv\ 0\ (mod\ 8)$and $n \equiv\ 2\ (mod\ 4)$.
Let $m = 8s$ and $n = 4t+2$ for some $s, t > 0$. Then we can write $K_{m,n} = 2s(t-1)K_{4,4} \oplus sK_{8,6}$. Then by Lemma 2.1, we get an L_8- decomposition of $K_{m,n}$.
Subcase(iv) $m \equiv\ 0\ (mod\ 8)$and $n \equiv\ 3\ (mod\ 4)$.
Let $m = 8s$ and $n = 4t+3$ for some $s, t > 0$. Then we can write $K_{m,n} = 2s(t-1)K_{4,4} \oplus sK_{8,7}$. Then by Lemma 2.2, we get an L_8- decomposition of $K_{m,n}$.
Case(2) $m \equiv\ 0\ (mod\ 4)$and $n \equiv\ 0\ (mod\ 2)$.
Subcase(i) $m \equiv\ 0\ (mod\ 4)$and $n \equiv\ 0\ (mod\ 4)$.
Let $m = 4s$ and $n = 4t$ for some $s, t > 0$. Then we can write $K_{m,n} = stK_{4,4}$. We know that $K_{4,4}$ has an L_8-decomposition.
Subcase(ii) $m \equiv\ 0\ (mod\ 4)$and $n \equiv\ 2\ (mod\ 4)$.
Let $m = 4s$ and $n = 4t + 2$ for some $s, t > 0$. For $s = 1$, $K_{4,n}$ has no L_8- decomposition by Lemma 2.6. Consider $s \geq 2$. For even s, m must be the multiple of 8. Then by case(1), result is proved for even s. It is sufficient to prove the case for odd s. Consider s is odd and $s \geq 3$. Then we can write $K_{m,n} = s(t-1)K_{4,4} \oplus K_{4(s-3),6} \oplus K_{12,6}$. Since s is odd, $s-3$ is even. Hence the results follows by the above cases and by the Lemma 2.4. $\square$

3. L_8- decomposition of $K_m \times K_n$

In this section we investigate the existence of L_8- decomposition of the tensor product of complete graphs.

Lemma 3.1. *For an even integer $k > 2$ and any graph G, there exists an L_{2k}- decomposition of $L_{2k} \times G$.*

Proof. Let L_{2k} be $\begin{pmatrix} x_1 & x_2 & \dots & x_k \\ x_{k+1} & x_{k+2} & \dots & x_{2k} \end{pmatrix}$ and $y_{j_1}y_{j_2}$ be any edge in G, then the induced subgraph $\langle L_{2k} \times \{y_{j_1}y_{j_2}\}\rangle$ of $L_{2k} \times G$ gives two L_{2k}'s as follows:
$\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & x_3^{j_1} & \dots & x_k^{j_2} \\ x_{k+1}^{j_2} & x_{k+2}^{j_1} & x_{k+3}^{j_2} & \dots & x_{2k}^{j_1} \end{pmatrix}$, $\begin{pmatrix} x_1^{j_2} & x_2^{j_1} & x_3^{j_2} & \dots & x_k^{j_1} \\ x_{k+1}^{j_1} & x_{k+2}^{j_2} & x_{k+3}^{j_1} & \dots & x_{2k}^{j_2} \end{pmatrix}$. So, for each edge in G there are two L_{2k}'s in $L_{2k} \times G$, and hence we have $2|E(G)|$ L_{2k}'s in $L_{2k} \times G$. $\square$

Lemma 3.2. *There exists an L_8- decomposition of $K_4 \times K_4$.*

Proof. The L_8- decomposition of $K_4 \times K_4$ is given below.
$\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & x_3^{j_1} & x_4^{j_2} \\ x_3^{j_2} & x_4^{j_1} & x_1^{j_2} & x_2^{j_1} \end{pmatrix}$ for $j_1 < j_2 \in \{1, 2, 3, 4\}$, $\begin{pmatrix} x_1^{j+2} & x_2^{j} & x_3^{j+2} & x_4^{j} \\ x_4^{j+1} & x_1^{j+1} & x_2^{j+1} & x_3^{j+1} \end{pmatrix}$ for $j = 1, 2$, $\begin{pmatrix} x_1^4 & x_2^1 & x_3^4 & x_4^1 \\ x_2^3 & x_3^2 & x_4^3 & x_1^2 \end{pmatrix}$. $\square$

Lemma 3.3. *There exists an L_8- decomposition of $K_4 \times K_5$.*

Proof. The L_8- decomposition of $K_4 \times K_5$ is given as follows:
$\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & x_3^{j_1} & x_4^{j_2} \\ x_3^{j_2} & x_4^{j_1} & x_1^{j_2} & x_2^{j_1} \end{pmatrix}$ for $j_1 < j_2 \in \{1, 2, 3, 4, 5\}$ except $(j_1, j_2) = (2, 4), (3, 4)$,
$\begin{pmatrix} x_1^3 & x_2^4 & x_3^3 & x_4^4 \\ x_3^4 & x_1^5 & x_1^4 & x_3^5 \end{pmatrix}$, $\begin{pmatrix} x_1^2 & x_2^4 & x_3^2 & x_4^4 \\ x_3^4 & x_3^5 & x_1^4 & x_1^5 \end{pmatrix}$, $\begin{pmatrix} x_1^3 & x_2^1 & x_3^3 & x_4^1 \\ x_2^2 & x_1^2 & x_4^2 & x_3^2 \end{pmatrix}$, $\begin{pmatrix} x_1^4 & x_2^1 & x_3^4 & x_4^1 \\ x_2^3 & x_3^2 & x_4^3 & x_1^2 \end{pmatrix}$,
$\begin{pmatrix} x_1^4 & x_2^2 & x_3^4 & x_4^2 \\ x_4^3 & x_3^3 & x_2^3 & x_1^3 \end{pmatrix}$, $\begin{pmatrix} x_1^5 & x_2^2 & x_3^5 & x_4^2 \\ x_2^1 & x_4^4 & x_4^1 & x_2^4 \end{pmatrix}$, $\begin{pmatrix} x_1^5 & x_2^3 & x_3^5 & x_4^3 \\ x_4^1 & x_4^4 & x_2^1 & x_2^4 \end{pmatrix}$. $\square$

Lemma 3.4. *There exists an L_8- decomposition of $K_4 \times K_{4,5}$.*

Proof. We can write $K_{4,5} = K_{4,4} \oplus K_{4,1}$. Now $K_4 \times K_{4,5} = (K_4 \times K_{4,4}) \oplus (K_4 \times K_{4,1})$. By Theorem 2.7 and Lemma 3.1, it is sufficient to prove the existence of L_8- decomposition of $K_4 \times K_{4,1}$. The L_8-decomposition of $K_4 \times K_{4,1}$ shown in Fig.3 gives the required decomposition. □

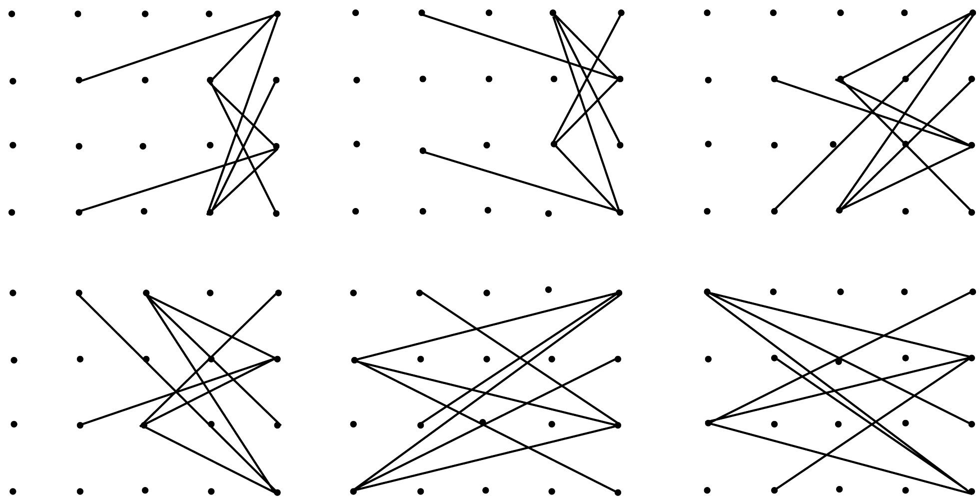

Figure 3. L_8 **decomposition of** $K_4 \times K_{4,1}$**.**

Lemma 3.5. *There exists an L_8- decomposition of $K_8 \times K_7$.*

Proof. We can write $K_8 = 3L_8 \oplus C_4$ (see Fig.4). Now $K_8 \times K_7 = 3\,(L_8 \times K_7) \oplus (C_4 \times K_7)$. To complete the proof, by Lemma 3.1, it is sufficient to prove the existence of L_8- decomposition of $C_4 \times K_7$, which is given as follows:

$$\begin{pmatrix} x_1^1 & x_2^2 & x_3^1 & x_4^2 \\ x_4^7 & x_3^7 & x_2^7 & x_1^7 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_2^1 & x_3^2 & x_4^1 \\ x_4^7 & x_1^7 & x_2^7 & x_3^7 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_2^3 & x_3^2 & x_4^3 \\ x_4^5 & x_1^6 & x_2^5 & x_3^6 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_2^2 & x_3^3 & x_4^2 \\ x_2^6 & x_1^7 & x_4^6 & x_3^7 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_2^4 & x_3^3 & x_4^4 \\ x_4^6 & x_1^7 & x_2^6 & x_3^7 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_2^3 & x_3^4 & x_4^3 \\ x_2^7 & x_3^6 & x_4^7 & x_1^6 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_2^5 & x_3^4 & x_4^5 \\ x_4^7 & x_3^7 & x_2^7 & x_1^7 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_2^4 & x_3^5 & x_4^4 \\ x_2^1 & x_3^7 & x_4^1 & x_1^7 \end{pmatrix},$$
$$\begin{pmatrix} x_1^5 & x_2^6 & x_3^5 & x_4^6 \\ x_4^1 & x_1^1 & x_2^1 & x_3^1 \end{pmatrix}, \begin{pmatrix} x_1^6 & x_2^5 & x_3^6 & x_4^5 \\ x_2^1 & x_3^1 & x_4^1 & x_1^1 \end{pmatrix}, \begin{pmatrix} x_1^6 & x_2^7 & x_3^6 & x_4^7 \\ x_4^1 & x_1^1 & x_2^1 & x_3^1 \end{pmatrix}, \begin{pmatrix} x_1^7 & x_2^6 & x_3^7 & x_4^6 \\ x_2^5 & x_3^1 & x_4^5 & x_1^1 \end{pmatrix},$$
$$\begin{pmatrix} x_1^1 & x_2^3 & x_3^1 & x_4^3 \\ x_2^5 & x_1^7 & x_4^5 & x_3^7 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_2^1 & x_3^3 & x_4^1 \\ x_2^7 & x_3^7 & x_4^7 & x_1^7 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_2^4 & x_3^2 & x_4^4 \\ x_2^6 & x_1^1 & x_4^6 & x_3^1 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_2^2 & x_3^4 & x_4^2 \\ x_2^1 & x_1^6 & x_4^1 & x_3^6 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_2^5 & x_3^3 & x_4^5 \\ x_4^7 & x_1^2 & x_2^7 & x_3^2 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_2^3 & x_3^5 & x_4^3 \\ x_4^2 & x_3^7 & x_2^2 & x_1^7 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_2^6 & x_3^4 & x_4^6 \\ x_4^1 & x_3^2 & x_2^1 & x_1^2 \end{pmatrix}, \begin{pmatrix} x_1^6 & x_2^4 & x_3^6 & x_4^4 \\ x_4^2 & x_3^1 & x_2^2 & x_1^1 \end{pmatrix},$$
$$\begin{pmatrix} x_1^5 & x_2^7 & x_3^5 & x_4^7 \\ x_2^2 & x_1^2 & x_4^2 & x_3^2 \end{pmatrix}.$$
□

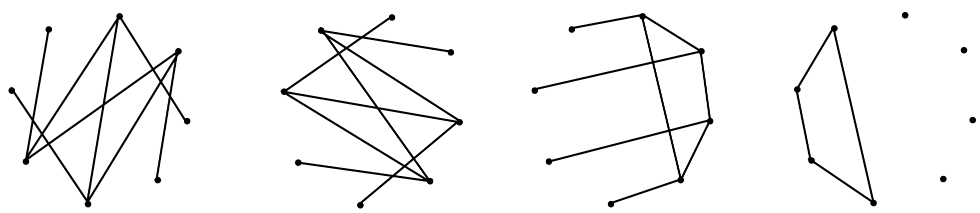

Figure 4. $K_8 = 3L_8 \oplus C_4$**.**

Lemma 3.6. *There exists an L_8-decomposition of $K_8 \times K_{4,7}$.*

Proof. We can write $K_8 \times K_{4,7} = 2(K_4 \times K_{4,4}) \oplus 2(K_4 \times K_{4,3}) \oplus (K_{4,4} \times K_{4,4}) \oplus (K_{4,4} \times K_{4,3})$. By Theorem 2.7 and Lemma 3.1, it is sufficient to prove the existence of L_8- decomposition of $K_4 \times K_{4,3}$. The L_8- decomposition of $K_4 \times K_{4,3}$ is given below.

$$\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & x_3^{1} & x_4^{j_2} \\ x_3^{j_2} & x_4^{j_1+1} & x_1^{j_2} & x_2^{j_1+1} \end{pmatrix}, \begin{pmatrix} x_1^{j_2} & x_2^{j_1} & x_3^{j_2} & x_4^{j_1} \\ x_3^{j_1+1} & x_4^{j_2} & x_1^{j_1+1} & x_2^{j_2} \end{pmatrix} \text{ for } j_1 = 3, \ j_2 \in \{5,6,7\}$$
$$\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & x_3^{j_1} & x_4^{j_2} \\ x_3^{j_2} & x_3^{j_1+2} & x_1^{j_2} & x_1^{j_1+2} \end{pmatrix}, \begin{pmatrix} x_1^{j_2} & x_2^{j_1} & x_3^{j_2} & x_4^{j_1} \\ x_4^{j_1+2} & x_4^{j_2} & x_2^{j_1+2} & x_2^{j_2} \end{pmatrix} \text{ for } j_1 = 2, \ j_2 \in \{5,6,7\}$$
$$\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & x_3^{j_1} & x_4^{j_2} \\ x_3^{j_2} & x_1^{j_1+3} & x_1^{j_2} & x_3^{j_1+3} \end{pmatrix}, \begin{pmatrix} x_1^{j_2} & x_2^{j_1} & x_3^{j_2} & x_4^{j_1} \\ x_2^{j_1+3} & x_4^{j_2} & x_4^{j_1+3} & x_2^{j_2} \end{pmatrix} \text{ for } j_1 = 1, \ j_2 \in \{5,6,7\}. \qquad \square$$

Lemma 3.7. *There exists an L_8-decomposition of $K_5 \times K_5$.*

Proof. We exhibit the L_8- decomposition of $K_5 \times K_5$ as follows:

$$\begin{pmatrix} x_1^1 & x_3^2 & x_4^1 & x_5^2 \\ x_2^5 & x_2^1 & x_1^2 & x_2^3 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_3^1 & x_4^2 & x_5^1 \\ x_2^1 & x_2^2 & x_2^4 & x_3^5 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_3^3 & x_4^2 & x_5^3 \\ x_2^3 & x_2^2 & x_2^5 & x_3^1 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_3^2 & x_4^3 & x_5^2 \\ x_2^1 & x_2^3 & x_2^5 & x_2^4 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_3^4 & x_4^3 & x_5^4 \\ x_2^5 & x_2^3 & x_2^1 & x_3^5 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_3^3 & x_4^4 & x_5^3 \\ x_2^3 & x_2^4 & x_2^5 & x_3^5 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_3^5 & x_4^4 & x_5^5 \\ x_2^5 & x_2^4 & x_2^2 & x_2^1 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_3^4 & x_4^5 & x_5^4 \\ x_2^4 & x_2^5 & x_2^1 & x_2^3 \end{pmatrix},$$
$$\begin{pmatrix} x_1^1 & x_3^5 & x_4^1 & x_5^5 \\ x_2^3 & x_2^1 & x_1^5 & x_2^4 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_3^1 & x_4^5 & x_5^1 \\ x_2^1 & x_2^5 & x_1^4 & x_2^4 \end{pmatrix}, \begin{pmatrix} x_1^1 & x_3^3 & x_4^1 & x_5^3 \\ x_4^4 & x_2^1 & x_2^5 & x_2^2 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_3^1 & x_4^3 & x_5^1 \\ x_4^1 & x_2^3 & x_1^2 & x_2^2 \end{pmatrix},$$
$$\begin{pmatrix} x_1^3 & x_3^5 & x_4^3 & x_5^5 \\ x_4^5 & x_2^3 & x_1^4 & x_3^2 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_3^3 & x_4^5 & x_5^3 \\ x_2^3 & x_5^5 & x_1^2 & x_2^5 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_3^4 & x_4^2 & x_5^4 \\ x_2^4 & x_2^2 & x_1^5 & x_2^5 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_3^2 & x_4^4 & x_5^2 \\ x_2^1 & x_2^4 & x_1^5 & x_2^5 \end{pmatrix},$$
$$\begin{pmatrix} x_1^1 & x_3^4 & x_4^1 & x_5^4 \\ x_4^5 & x_2^1 & x_1^4 & x_2^2 \end{pmatrix}, \begin{pmatrix} x_1^4 & x_3^1 & x_4^4 & x_5^1 \\ x_2^2 & x_2^4 & x_1^2 & x_2^5 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_3^5 & x_4^2 & x_5^5 \\ x_2^5 & x_2^2 & x_1^4 & x_2^3 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_3^2 & x_4^5 & x_5^2 \\ x_2^2 & x_2^5 & x_2^3 & x_3^4 \end{pmatrix},$$
$$\begin{pmatrix} x_2^1 & x_4^2 & x_2^3 & x_4^4 \\ x_5^2 & x_1^1 & x_5^1 & x_1^3 \end{pmatrix}, \begin{pmatrix} x_4^1 & x_2^2 & x_4^3 & x_2^4 \\ x_2^3 & x_5^5 & x_1^5 & x_4^5 \end{pmatrix}, \begin{pmatrix} x_3^1 & x_5^2 & x_3^3 & x_5^4 \\ x_5^5 & x_3^5 & x_2^5 & x_2^1 \end{pmatrix}, \begin{pmatrix} x_5^1 & x_3^2 & x_5^3 & x_3^4 \\ x_3^3 & x_5^4 & x_2^1 & x_5^5 \end{pmatrix},$$
$$\begin{pmatrix} x_1^1 & x_2^2 & x_1^3 & x_2^4 \\ x_4^3 & x_4^5 & x_4^2 & x_5^3 \end{pmatrix}. \qquad \square$$

Theorem 3.8. *$K_m \times K_n$ has an L_8- decomposition if and only if $mn(m-1)(n-1) \equiv 0 \ (mod\ 16)$.*

Proof. **Necessity.** Assume that $K_m \times K_n$ admits an L_8- decomposition. Then the number of edges in the graph $K_m \times K_n$ is $\frac{mn(m-1)(n-1)}{2}$ which should be divisible by 8, the number of edges in L_8 i.e., $16 | mn(m-1)(n-1)$ and hence $mn(m-1)(n-1) \equiv 0 \ (mod\ 16)$.

Sufficiency. We construct the required decomposition in the following cases.

Case(1) $m, n \equiv \ 0 \ (mod\ 4)$.

Let $m = 4s$ and $n = 4t$ for some $s, t > 0$. Then we can write $K_m \times K_n = st\,(K_4 \times K_4) \oplus 2st(s-1)(t-1)K_{4,16} \oplus 2st(s+t-2)K_{4,12}$. By Lemma 3.2 and Theorem 2.7 the graph $K_m \times K_n$ has the desired decomposition.

Case(2) $m \equiv \ 0 \ (mod\ 4), \ n \equiv \ 1 \ (mod\ 4)$.

Let $m = 4s$ and $n = 4t+1$ for some $s, t > 0$. Then we can write $K_m \times K_n = (K_{4s} \times K_{4(t-1)}) \oplus s\,(K_4 \times K_5) \oplus \left(\frac{5s(s-1)}{2}\right) K_{4,16} \oplus s(t-1)\,(K_4 \times K_{4,5}) \oplus 4s(s-1)(t-1)K_{4,20}$. By the Case (1) above, Lemmas 3.3, 3.4 and Theorem 2.7 the graph $K_m \times K_n$ has the desired decomposition.

Case(3) $m \equiv \ 0 \ (mod\ 8), \ n \equiv \ 3 \ (mod\ 4)$.

Subcase(i) *$m = 8$ and $n \equiv \ 3 \ (mod\ 4)$*

Let $n = 4t+3$ for some $t > 0$. Then we can write $K_8 \times K_n = \left(K_8 \times K_{4(t-1)}\right) \oplus (K_8 \times K_7) \oplus (t-1)\,(K_8 \times K_{4,7})$. The L_8- decomposition of all the three terms follows from Case (1) and the Lemmas 3.5, 3.6.

Subcase(ii) *$m \equiv \ 0 \ (mod\ 8), \ m > 8$ and $n \equiv \ 3 \ (mod\ 4)$*

Now, let $m = 8s$ for some $s > 1$. Then we can write $K_m \times K_n = s\,(K_8 \times K_n) \oplus \left(\frac{s(s-1)}{2}\right)(K_{8,8} \times K_n)$. $K_m \times K_n$ has the desired decomposition, by the Theorem 2.7, Lemma 3.1 and Subcase 3(i) above.

Case(4) $m \equiv \ 0 \ (mod\ 16)$.

Let $m = 16s$, for some $s > 0$. We can write $K_m \times K_n = s\,(K_{16} \times K_n) \oplus \left(\frac{s(s-1)}{2}\right)(K_{16,16} \times K_n)$. The L_8- decomposition of the terms in RHS follows from Lemma 3.1 and Theorems 1.3, 2.7.
Case(5) $m \equiv 1 \pmod{16}$.
Let $m = 16s + 1$, for some $s > 0$. Then by Theorem 1.3, we have L_8- decomposition of K_{16s+1}, for any $s > 0$. Then by Lemma 3.1, $K_m \times K_n$ has an L_8- decomposition.
Case(6) $m \equiv 1 \pmod 4$, $n \equiv 1 \pmod 4$.
Let $m = 4s + 1$ and $n = 4t + 1$ for some $s, t > 0$. Then we can write $K_m \times K_n = \left(K_{4(s-1)} \times K_{4(t-1)}\right) \oplus \left(K_{4(s-1)} \times K_5\right) \oplus \left(K_5 \times K_{4(t-1)}\right) \oplus 2(s-1)(t-1)\,(K_4 \times K_{4,5}) \oplus 4(s-1)(t-1)(s+t-4)K_{4,20} \oplus K_5 \times K_5 \oplus 5\,(s+t-2)\,K_{4,20} \oplus 2(s-1)(t-1)K_{16,25}$. Then by the Cases (1), (2) above and by Lemmas 3.4, 3.7 and Theorem 2.7, the graph $K_m \times K_n$ has the desired decomposition. □

4. L_8- decomposition of $K_m \otimes \overline{K_n}$

In this section we investigate the existence of L_8- decomposition of wreath product of complete graphs.

Lemma 4.1. *If the graph G has an L_{2k}-decomposition, then $G \otimes \overline{K_n}$ has an L_{2k}-decomposition for any* $n > 0$ and even $k > 2$.

Proof. Let G has an L_{2k}- decomposition. For each L_{2k}, $\begin{pmatrix} x_1 & x_2 & \dots & x_k \\ x_{k+1} & x_{k+2} & \dots & x_{2k} \end{pmatrix}$, in G, we exhibit the L_{2k}- decomposition of $L_{2k} \otimes \overline{K_n}$ as follows: $\begin{pmatrix} x_1^{j_1} & x_2^{j_2} & \dots & x_k^{j_2} \\ x_{k+1}^{j_2} & x_{k+2}^{j_1} & \dots & x_{2k}^{j_1} \end{pmatrix}$ for $j_1 \le j_2 \in \{1, 2, ..., n\}$, $\begin{pmatrix} x_1^{j_2} & x_2^{j_1} & \dots & x_k^{j_1} \\ x_{k+1}^{j_1} & x_{k+2}^{j_2} & \dots & x_{2k}^{j_2} \end{pmatrix}$ for $j_1 < j_2 \in \{1, 2, ..., n\}$. □

Lemma 4.2. *There exists an L_8- decomposition of $K_{4,5} \otimes \overline{K_6}$.*

Proof. The L_8- decomposition of $K_{4,5} \otimes \overline{K_6}$ is given as follows:
$\begin{pmatrix} x_1^{j_1} & x_5^{j_2} & x_2^{j_1} & x_6^{j_2} \\ x_7^{j_2} & x_3^{j_1} & x_8^{j_2} & x_4^{j_1} \end{pmatrix}$ for $j_1 \le j_2 \in \{1, 2, 3, 4, 5, 6\}$ except $(j_1, j_2) = (4, 4), (2, 3),$ $(2, 4), (3, 4),$
$\begin{pmatrix} x_1^{j_2} & x_5^{j_1} & x_2^{j_2} & x_6^{j_1} \\ x_7^{j_1} & x_3^{j_2} & x_8^{j_1} & x_4^{j_2} \end{pmatrix}$ for $j_1 < j_2 \in \{1, 2, 3, 4, 5, 6\}$ except $(j_1, j_2) = (3, 4),$ $(3, 5), (3, 6), (4, 5), (4, 6),$
$\begin{pmatrix} x_3^{j_1} & x_7^{j_2} & x_4^{j_1} & x_8^{j_2} \\ x_6^{j_2} & x_2^{j_1} & x_5^{j_2} & x_1^{j_1} \end{pmatrix}$ for $j_1 \le j_2 \in \{1, 2, 3, 4, 5, 6\}$ except $(j_1, j_2) = (2, 4), (2, 5),$
$\begin{pmatrix} x_3^{j_2} & x_7^{j_1} & x_4^{j_2} & x_8^{j_1} \\ x_6^{j_1} & x_2^{j_2} & x_5^{j_1} & x_1^{j_2} \end{pmatrix}$ for $j_1 < j_2 \in \{1, 2, 3, 4, 5, 6\}$ except $(j_1, j_2) = (3, 5), (4, 5),$
$\begin{pmatrix} x_{i_1}^2 & x_{i_2}^1 & x_{i_1}^4 & x_{i_2}^3 \\ x_{i_2}^5 & x_{i_1}^3 & x_{i_2}^4 & x_{i_1}^1 \end{pmatrix}, \begin{pmatrix} x_{i_1}^1 & x_{i_2}^2 & x_{i_1}^3 & x_{i_2}^4 \\ x_{i_2}^5 & x_{i_1}^6 & x_{i_2}^3 & x_{i_1}^2 \end{pmatrix}, \begin{pmatrix} x_{i_1}^4 & x_{i_2}^2 & x_{i_1}^5 & x_{i_2}^6 \\ x_{i_2}^5 & x_{i_1}^2 & x_{i_2}^3 & x_{i_1}^6 \end{pmatrix}$ for $i_1 \in \{1, 2\}$ & $i_2 \in \{9\}$,
$\begin{pmatrix} x_{i_1}^5 & x_{i_2}^1 & x_{i_1}^6 & x_{i_2}^5 \\ x_{i_2}^4 & x_{i_1}^1 & x_{i_2}^3 & x_{i_1}^3 \end{pmatrix}$ for $i_1 \in \{1, 2, 3, 4\}$ & $i_2 \in \{9\}$,
$\begin{pmatrix} x_1^4 & x_5^4 & x_2^4 & x_6^4 \\ x_7^4 & x_3^4 & x_8^4 & x_3^5 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_9^6 & x_2^2 & x_6^3 \\ x_7^3 & x_2^1 & x_8^3 & x_4^2 \end{pmatrix}, \begin{pmatrix} x_1^2 & x_5^4 & x_2^2 & x_6^4 \\ x_7^4 & x_3^2 & x_8^4 & x_4^3 \end{pmatrix}, \begin{pmatrix} x_1^3 & x_5^4 & x_2^3 & x_9^6 \\ x_7^4 & x_3^3 & x_8^4 & x_1^1 \end{pmatrix},$
$\begin{pmatrix} x_1^4 & x_5^3 & x_2^4 & x_6^3 \\ x_7^3 & x_1^2 & x_8^3 & x_4^4 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_5^3 & x_2^5 & x_6^3 \\ x_7^3 & x_2^2 & x_8^3 & x_4^5 \end{pmatrix}, \begin{pmatrix} x_1^6 & x_5^3 & x_2^6 & x_6^3 \\ x_7^3 & x_4^5 & x_8^3 & x_4^6 \end{pmatrix}, \begin{pmatrix} x_1^5 & x_5^4 & x_2^5 & x_6^4 \\ x_7^4 & x_3^5 & x_8^4 & x_1^3 \end{pmatrix},$
$\begin{pmatrix} x_1^6 & x_5^4 & x_2^6 & x_6^4 \\ x_7^4 & x_3^6 & x_8^4 & x_2^3 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_7^4 & x_4^2 & x_8^4 \\ x_5^3 & x_2^2 & x_5^4 & x_1^2 \end{pmatrix}, \begin{pmatrix} x_3^2 & x_7^5 & x_4^2 & x_8^5 \\ x_6^5 & x_2^2 & x_6^4 & x_1^2 \end{pmatrix}, \begin{pmatrix} x_3^5 & x_7^3 & x_4^5 & x_8^3 \\ x_6^3 & x_2^5 & x_6^4 & x_1^5 \end{pmatrix},$

$\begin{pmatrix} x_3^5 & x_7^4 & x_4^5 & x_8^4 \\ x_5^3 & x_2^5 & x_5^4 & x_1^5 \end{pmatrix}$, $\begin{pmatrix} x_3^2 & x_9^1 & x_3^4 & x_9^3 \\ x_9^5 & x_3^3 & x_5^3 & x_3^1 \end{pmatrix}$, $\begin{pmatrix} x_4^2 & x_9^1 & x_4^4 & x_9^3 \\ x_9^5 & x_4^3 & x_6^4 & x_4^1 \end{pmatrix}$, $\begin{pmatrix} x_3^1 & x_9^2 & x_3^3 & x_9^4 \\ x_9^5 & x_3^6 & x_9^3 & x_1^6 \end{pmatrix}$,
$\begin{pmatrix} x_4^1 & x_9^2 & x_4^3 & x_9^4 \\ x_9^5 & x_4^6 & x_9^3 & x_2^6 \end{pmatrix}$, $\begin{pmatrix} x_3^4 & x_9^2 & x_3^5 & x_9^6 \\ x_9^5 & x_3^2 & x_9^3 & x_3^3 \end{pmatrix}$, $\begin{pmatrix} x_4^4 & x_9^2 & x_4^5 & x_9^6 \\ x_9^5 & x_4^2 & x_9^3 & x_4^3 \end{pmatrix}$, $\begin{pmatrix} x_3^2 & x_9^4 & x_3^6 & x_9^6 \\ x_6^4 & x_3^4 & x_5^3 & x_3^1 \end{pmatrix}$,
$\begin{pmatrix} x_4^2 & x_9^4 & x_4^6 & x_9^6 \\ x_5^5 & x_4^4 & x_6^4 & x_4^1 \end{pmatrix}$.

□

Lemma 4.3. *There exists an L_8-decomposition of $K_{4,5} \otimes \overline{K_{10}}$.*

Proof. We exhibit the L_8- decomposition of $K_{4,5} \otimes \overline{K_{10}}$ as follows:
$\begin{pmatrix} x_{i_1}^1 & x_{i_2}^1 & x_{i_1}^2 & x_{i_2}^2 \\ x_{i_2}^7 & x_{i_1}^5 & x_{i_2}^8 & x_{i_1}^6 \end{pmatrix}$, $\begin{pmatrix} x_{i_1}^1 & x_{i_2}^3 & x_{i_1}^2 & x_{i_2}^4 \\ x_{i_2}^8 & x_{i_1}^6 & x_{i_2}^7 & x_{i_1}^5 \end{pmatrix}$, for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$,
$\begin{pmatrix} x_{i_1}^3 & x_{i_2}^1 & x_{i_1}^4 & x_{i_2}^2 \\ x_{i_2}^9 & x_{i_1}^7 & x_{i_2}^{10} & x_{i_1}^8 \end{pmatrix}$, $\begin{pmatrix} x_{i_1}^1 & x_{i_2}^5 & x_{i_1}^2 & x_{i_2}^6 \\ x_{i_2}^9 & x_{i_1}^9 & x_{i_2}^{10} & x_{i_1}^{10} \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$,
$\begin{pmatrix} x_{i_1}^3 & x_{i_2}^5 & x_{i_1}^4 & x_{i_2}^6 \\ x_{i_2}^8 & x_{i_1}^8 & x_{i_2}^7 & x_{i_1}^7 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$,
$\begin{pmatrix} x_{i_1}^9 & x_{i_2}^1 & x_{i_1}^{10} & x_{i_2}^2 \\ x_{i_2}^3 & x_{i_1}^6 & x_{i_2}^4 & x_{i_1}^5 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (1,9)$
$\begin{pmatrix} x_{i_1}^3 & x_{i_2}^3 & x_{i_1}^4 & x_{i_2}^4 \\ x_{i_2}^7 & x_{i_1}^5 & x_{i_2}^8 & x_{i_1}^6 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (3,7), (4,8)$
$\begin{pmatrix} x_{i_1}^5 & x_{i_2}^5 & x_{i_1}^6 & x_{i_2}^6 \\ x_{i_2}^9 & x_{i_1}^7 & x_{i_2}^{10} & x_{i_1}^8 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (3,5), (4,6)$
$\begin{pmatrix} x_{i_1}^7 & x_{i_2}^7 & x_{i_1}^8 & x_{i_2}^8 \\ x_{i_2}^2 & x_{i_1}^9 & x_{i_2}^1 & x_{i_1}^{10} \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (3,7), (4,8)$
$\begin{pmatrix} x_{i_1}^9 & x_{i_2}^9 & x_{i_1}^{10} & x_{i_2}^{10} \\ x_{i_2}^6 & x_{i_1}^2 & x_{i_2}^5 & x_{i_1}^1 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (1,5), (1,6)$
$\begin{pmatrix} x_{i_1}^5 & x_{i_2}^7 & x_{i_1}^6 & x_{i_2}^8 \\ x_{i_2}^{10} & x_{i_1}^{10} & x_{i_2}^9 & x_{i_1}^9 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (3,7), (4,8)$
$\begin{pmatrix} x_{i_1}^7 & x_{i_2}^9 & x_{i_1}^8 & x_{i_2}^{10} \\ x_{i_2}^3 & x_{i_1}^4 & x_{i_2}^4 & x_{i_1}^3 \end{pmatrix}$ for $i_1 \in \{1,2,3,4\}$ & $i_2 \in \{5,6,7,8,9\}$ except $(i_1, i_2) = (1,9), (3,7)$,
$(4,8)$
$\begin{pmatrix} x_1^9 & x_9^1 & x_1^{10} & x_9^2 \\ x_6^4 & x_1^6 & x_5^4 & x_1^5 \end{pmatrix}$, $\begin{pmatrix} x_3^3 & x_7^3 & x_3^4 & x_7^4 \\ x_7^{10} & x_1^8 & x_7^9 & x_1^7 \end{pmatrix}$, $\begin{pmatrix} x_4^3 & x_8^3 & x_4^4 & x_8^4 \\ x_8^{10} & x_1^8 & x_8^9 & x_1^7 \end{pmatrix}$, $\begin{pmatrix} x_3^5 & x_5^5 & x_3^6 & x_5^6 \\ x_5^9 & x_1^{10} & x_5^{10} & x_1^9 \end{pmatrix}$,
$\begin{pmatrix} x_4^5 & x_6^5 & x_4^6 & x_6^6 \\ x_6^9 & x_1^{10} & x_6^{10} & x_1^9 \end{pmatrix}$, $\begin{pmatrix} x_3^7 & x_7^7 & x_3^8 & x_7^8 \\ x_7^2 & x_3^3 & x_7^1 & x_3^4 \end{pmatrix}$, $\begin{pmatrix} x_4^7 & x_8^7 & x_4^8 & x_8^8 \\ x_8^2 & x_4^3 & x_8^1 & x_4^4 \end{pmatrix}$, $\begin{pmatrix} x_1^9 & x_5^9 & x_1^{10} & x_5^{10} \\ x_7^4 & x_1^2 & x_7^3 & x_1^1 \end{pmatrix}$,
$\begin{pmatrix} x_1^9 & x_6^9 & x_1^{10} & x_6^{10} \\ x_8^4 & x_1^2 & x_8^3 & x_1^1 \end{pmatrix}$, $\begin{pmatrix} x_3^5 & x_7^7 & x_3^6 & x_7^8 \\ x_7^3 & x_3^{10} & x_7^4 & x_3^9 \end{pmatrix}$, $\begin{pmatrix} x_4^5 & x_8^7 & x_4^6 & x_8^8 \\ x_8^3 & x_4^{10} & x_8^4 & x_4^9 \end{pmatrix}$, $\begin{pmatrix} x_1^7 & x_9^9 & x_1^8 & x_9^{10} \\ x_6^3 & x_1^4 & x_5^3 & x_1^3 \end{pmatrix}$,
$\begin{pmatrix} x_3^7 & x_7^9 & x_3^8 & x_7^{10} \\ x_7^3 & x_3^6 & x_7^4 & x_3^5 \end{pmatrix}$, $\begin{pmatrix} x_4^7 & x_8^9 & x_4^8 & x_8^{10} \\ x_8^3 & x_4^6 & x_8^4 & x_4^5 \end{pmatrix}$, $\begin{pmatrix} x_3^8 & x_5^3 & x_3^{10} & x_6^3 \\ x_9^3 & x_2^8 & x_7^8 & x_2^{10} \end{pmatrix}$, $\begin{pmatrix} x_3^7 & x_5^4 & x_3^9 & x_6^4 \\ x_9^4 & x_2^7 & x_7^7 & x_2^9 \end{pmatrix}$,
$\begin{pmatrix} x_3^8 & x_7^3 & x_3^{10} & x_8^3 \\ x_5^6 & x_2^8 & x_9^3 & x_2^{10} \end{pmatrix}$, $\begin{pmatrix} x_3^7 & x_7^4 & x_3^9 & x_8^4 \\ x_5^5 & x_2^7 & x_9^4 & x_2^9 \end{pmatrix}$, $\begin{pmatrix} x_4^8 & x_5^3 & x_4^{10} & x_6^3 \\ x_9^3 & x_2^{10} & x_8^8 & x_2^8 \end{pmatrix}$, $\begin{pmatrix} x_4^7 & x_5^4 & x_4^9 & x_6^4 \\ x_9^4 & x_2^9 & x_8^7 & x_2^7 \end{pmatrix}$,
$\begin{pmatrix} x_4^8 & x_7^3 & x_4^{10} & x_8^3 \\ x_6^6 & x_2^{10} & x_9^3 & x_2^8 \end{pmatrix}$, $\begin{pmatrix} x_4^7 & x_7^4 & x_4^9 & x_8^4 \\ x_6^5 & x_2^9 & x_9^4 & x_2^7 \end{pmatrix}$, $\begin{pmatrix} x_1^8 & x_9^3 & x_1^{10} & x_9^4 \\ x_5^3 & x_2^8 & x_6^3 & x_2^7 \end{pmatrix}$, $\begin{pmatrix} x_1^7 & x_9^3 & x_1^9 & x_9^4 \\ x_5^4 & x_2^{10} & x_6^4 & x_2^9 \end{pmatrix}$. □

Theorem 4.4. *$K_m \otimes \overline{K_n}$ has an L_8- decomposition if and only if $mn^2(m-1) \equiv 0$ $(mod\ 16)$.*

Proof. **Necessity.** Assume that $K_m \otimes \overline{K_n}$ admits an L_8- decomposition. Then the number of edges in the graph $K_m \otimes \overline{K_n}$ is $\frac{mn^2(m-1)}{2}$ which should be divisible by 8, the number of edges in L_8 i.e., $16|mn^2(m-1)$ and hence $mn^2(m-1) \equiv 0$ $(mod\ 16)$.
Sufficiency. We construct the required decomposition in five cases.
Case(1) $n \equiv 0$ $(mod\ 4)$.

Let $n = 4s$, for some $s > 0$. Then we can write $K_m \otimes \overline{K_n} = \left(\frac{m(m-1)}{2}\right) K_{4s,4s}$. Now, we get the desired decomposition by Theorem 2.7.
Case(2) $m \equiv 0 \ (mod\ 4),\ n \equiv 0 \ (mod\ 2)$.
Subcase(i) $m \equiv 0 \ (mod\ 4),\ n \equiv 0 \ (mod\ 4)$. Proof follows from Case (1).
Subcase(ii) $m \equiv 0 \ (mod\ 4),\ n \equiv 2 \ (mod\ 4)$.
Let $m = 4s$ and $n = 6, 10$ for some $s > 0$. Now we can write $K_m \otimes \overline{K_n} = s(K_4 \otimes \overline{K_n}) \oplus \frac{s(s-1)}{2}(K_{4,4} \otimes \overline{K_n})$ and the graphs $K_4 \otimes \overline{K_6}, K_4 \otimes \overline{K_{10}}$ can be viewed as $3K_{6,12}, 3K_{10,20}$, respectively. Therefore we get the desired decomposition, by Lemma 4.1 and the Theorem 2.7. Let $m = 4s$ and $n = 4t + 2$ for some $s > 0, t > 2$. Now we can write $K_m \otimes \overline{K_n} = \left(K_{4s} \otimes \overline{K_{4(t-1)}}\right) \oplus \left(\frac{s(s-1)}{2}\right)\left(K_{4,4} \otimes \overline{K_6}\right) \oplus 4s(4s-1)K_{4(t-1),6}$ $\oplus\ 3sK_{6,12}$. We get the desired decomposition, by the Case (1), Lemma 4.1 and Theorem 2.7.
Case(3) $m \equiv 1 \ (mod\ 4),\ n \equiv 0 \ (mod\ 2)$.
Subcase(i) $m \equiv 1 \ (mod\ 4),\ n \equiv 0 \ (mod\ 4)$. Proof follows from Case (1).
Subcase(ii) $m \equiv 1 \ (mod\ 4),\ n \equiv 2 \ (mod\ 4)$.
Let $m = 4s+1$ and $n = 6, 10$ for some $s > 0$. Now we can write $K_m \otimes \overline{K_n} = (K_{4(s-1)} \otimes \overline{K_n}) \oplus K_5 \otimes \overline{K_n}$ $\oplus\ (s-1)(K_{4,5} \otimes \overline{K_n})$and the graphs $K_5 \otimes \overline{K_6}, K_5 \otimes \overline{K_{10}}$ can be viewed as $5K_{6,12}, 5K_{10,20}$, respectively. Therefore we get the desired decomposition, by the Case(2), Lemmas 4.2, 4.3 and Theorem 2.7. Further, let $m = 4s + 1$ and $n = 4t + 2$ for some $s > 0, t > 2$. Now we can write $K_m \otimes \overline{K_n} = K_{4(s-1)} \otimes \overline{K_{4t+2}}$ $\oplus\ K_5 \otimes \overline{K_{4(t-1)}} \oplus (s-1)\left(K_{4,5} \otimes \overline{K_{4(t-1)}}\right) \oplus (s-1)\left(K_{4,5} \otimes \overline{K_6}\right) \oplus 20sK_{4(t-1),6} \oplus 5K_{6,12}$. The L_8-decomposition of 1^{st} term of the above sum follows from Case (2), 2^{nd} and 3^{rd} term follows from Case (1) and the remaining terms of the above sum follows from the Lemma 4.2 and Theorem 2.7. Hence we get the desired decomposition.
Case(4) $m \equiv 0 \ (mod\ 16)$.
Let $m = 16s$, for some $s > 0$. We can write $K_m \otimes \overline{K_n} = s\left(K_{16} \otimes \overline{K_n}\right) \oplus \left(\frac{s(s-1)}{2}\right)\left(K_{16,16} \otimes \overline{K_n}\right)$. The desired decomposition follows from Lemmma 4.1 and Theorems 1.3, 2.7.
Case(5) $m \equiv 1 \ (mod\ 16)$.
Desired decomposition follows from Theorem 1.3 and Lemma 4.1. □

5. Conclusion

In this paper, we established necessary and sufficient conditions for the decomposition of tensor/wreath product of graphs into sunlet graphs of order 8. Further, research on the existence of such decomposition of product graphs into sunlet graphs of higher order $r > 8$ is under progress.

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References

[1] A. D. Akwu, D. D. A. Ajayi, Decomposing certain equipartite graphs into Sunlet graphs of length $2p$, AKCE Int. J. Graphs Combin. 13 (2016) 267–271.
[2] B. Alspach, The wonderful Walecki construction, Bull. Inst. Combin. Appl. 52 (2008) 7–20.
[3] B. Alspach, J. C. Bermond, D. Sotteau, Decomposition into cycles I: Hamilton decompositions, In: Cycles and rays (Montreal, PQ, 1987), Kluwer Academic Publishers, Dordrecht, (1990) 9–18.
[4] B. Alspach, H. Gavlas, Cycle decompositions of K_n and K_{n-1}, J. Combin. Theory Ser. B 81(1)

(2001) 77–99.
[5] R. Anitha, R. S. Lekshmi, N-sun decomposition of complete graphs and complete bipartite graphs, World Acad. Sci. Eng. Tech. 27 (2007) 262-266.
[6] R. Anitha, R.S. Lekshmi, N-sun decomposition of complete, complete bipartite and some Harary graphs, Int. J. Comput. Math. Sci. 2 (2008).
[7] J. A. Bondy, U. R. S. Murty, Graph theory with applications, The Macmillan Press Ltd, New York (1976).
[8] D. Bryant, Cycle decompositions of complete graphs, in Surveys in Combinatorics 2007, A. Hilton and J. Talbot (Editors), London Mathematical Society Lecture Note Series 346, Proceedings of the 21st British Combinatorial Conference, Cambridge University Press (2007) 67–97.
[9] D. Bryant, C. A. Rodger, Cycle decompositions, C.J. Colbourn, J.H. Dinitz (Eds.), The CRC Handbook of Combinatorial Designs (2nd edition), CRC Press, Boca Raton (2007) 373–382.
[10] D. Froncek, Decomposition of complete graphs into small graphs, Opuscula Math. 30(3) (2010) 277–280.
[11] C. M. Fu, M. H. Huang, Y. L. Lin, On the existence of 5-sun systems, Discrete Math. 313(24) (2013) 2942–2950.
[12] C. M. Fu, N. H. Jhuang, Y. L. Lin, H. M. Sung, From steiner triple systems to 3-sun systems, Taiwanese J. Math. 16(2) (2012) 531–543.
[13] C. M. Fu, N. H. Jhuang, Y. L. Lin, H. M. Sung, On the existence of k-sun systems, Discrete Math. 312 (2012) 1931–1939.
[14] M. Gionfriddo, G. Lo Faro, S. Milici, A. Tripodi, On the existence of uniformly resolvable decompositions of K into 1-factors and h-suns, Utilitas Mathematica 99 (2016) 331–339.
[15] A. J. W. Hilton, Hamiltonian decompositions of complete graphs, J. Combin.Theory B 36(2) (1984) 125–134.
[16] A. J. W. Hilton, C. A. Rodger, Hamiltonian decompositions of complete regular s-partite graphs, Discrete Math. 58 (1986) 63–78.
[17] Z. Liang, J. Guo, Decomposition of complete multigraphs into Crown graphs, J. Appl. Math. Comput. 32 (2010) 507–517.
[18] Z. Liang, J. Guo, J. Wang, On the Crown graph decompositions containing odd cycle, Int. J. Comb. Graph Theory Appl. 4 (2019) 1–23.
[19] C. Lin, T-W Shyu, A necessary and sufficient condition for the star decomposition of complete graphs, J. Graph Theory, 23 (1996) 361–364.
[20] R. Frucht, Graceful numbering of wheels and related graphs, Ann. New York Acad. Sci 319 (1979) 219–229.
[21] M. Sajna, Cycle decompositions III: Complete graphs and fixed length cycles, J. Combin. Des. 10 (2002) 27–78.
[22] D. Sotteau, Decomposition of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Combin. Theory Ser. B 30(1) (1981) 75–81.
[23] M. Tarsi, Decomposition of complete multigraphs into stars, Discrete Mathematics 26(3) (1979) 273–278.
[24] M. Tarsi, Decomposition of complete multigraph into simple paths: Nonbalanced Handcuffed designs, J. Combin. Theory Ser. A 34 (1983) 60–70.
[25] M. Truszczynski, Note on the decomposition of $\lambda K_{m,n}(\lambda K^*_{m,n})$ into paths, Discrete Math. 55 (1985) 89–96.
[26] K. Ushio, S. Tazawa, S. Yamamoto, On claw-decomposition of complete multipartite graphs, Hiroshima Math. J. 8(1) (1978) 207–210.
[27] S. Yamamoto, H. Ikeda, S. Shige-Eda, K. Ushio, N. Hamada, On claw decomposition of complete graphs and complete bipartite graphs, Hiroshima Math. J. 5(1) (1975) 33–42.