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AN IMPROVED APPROACH FOR SOLUTIONS OF SYSTEMS OF LINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, a numerical matrix method is used to solve the systems of high-order linear Fredholm integro-differential equations with variable coefficients under mixed conditions. The technique consists of collocation points and the Morgan-Voyce polynomials. The residual error functions of numerical solutions of the method are also presented. Firstly, the approximate solutions are formed and secondly, an error problem is constituted in favor of the residual error function. The numerical solutions are computed for this error problem by using the present method. The approximate solutions of the original problem and the error problem are the corrected Morgan-Voyce polynomial solutions. When the exact solutions of the problem are not known, the absolute errors can be approximately constructed through the approximate solutions of the error problem. Numerical examples are included to demonstrate the validity and the applicability of the technique, and also the results are compared with the different methods. All numerical computations have been performed using MATLAB v7.11.0 (R2010b).

Keywords: Morgan-Voyce Polynomials, Systems of Linear Fredholm Integro-Differential Equations, Collocation Points, Residual Error

LİNEER FREDHOLM İNTEGRO-DİFERANSİYEL DENKLEM SİSTEMLERİNİN ÇÖZÜMLERİ İÇİN GELİŞTİRİLMİŞ BİR YAKLAŞIM

Özet

Bu çalışmada, karışık koşullar altında değişken katsayılı yüksek mertebeli doğrusal Fredholm integro-diferansiyel denklem sistemlerini çözmek için sayısal bir matris yöntemi kullanılmıştır. Bu yöntem, sıralama noktalarına ve Morgan-Voyce polinomlarına dayanmaktadır. Yöntemin sayısal çözümlerinin artık hata fonksiyonları da verilmiştir. İlk olarak yaklaşık çözümler elde edilir, ikinci olarak artık hata fonksiyonu ile bir hata problemi oluşturulur ve bu hata problemi mevcut yöntem kullanılarak çözülür. Orijinal problemin ve hata probleminin yaklaşık çözümleri toplanarak, düzeltilmiş Morgan-Voyce polinom çözümleri elde edilir. Problemin kesin çözümleri bilinmediğinde, mutlak hatalar yaklaşık olarak hata probleminin yaklaşık çözümleri ile hesaplanabilir. Yöntemin geçerliliğini ve uygulanabilirliğini göstermek için sayısal örnekler verilmiş ve ayrıca sonuçlar farklı yöntemlerle karşılaştırılmıştır. Tüm sayısal hesaplamalar için MATLAB v7.11.0 (R2010b) programı kullanılmıştır.

Anahtar Kelimeler: Morgan-Voyce Polinomları, Lineer Fredholm İntegro-Diferansiyel Denklem Sistemleri, Sıralama Noktaları, Rezidüel Hata

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1. Introduction

Systems of high-order linear integro-differential equations have an essential role in science and engineering, such as the glass-farming process [1], dropwise condensation [2], wind ripple in the desert [3]. Thus, it is important to solve the equations. However, these equations can be solved using numerical methods. There are several numerical methods [4-10].

Since 1994 the Taylor, Bessel, Pell-Lucas, Morgan-Voyce, Dickson, Bernstein, orthoexponential, and Bernoulli matrix methods have been applied to the linear differential, integro-differential equations, fractional differential equations, and nonlinear differential equations, delay differential equations [11-28].

Some properties of Morgan-Voyce polynomials of the first kind are defined [29],

1. $B_{n+1}B_{n-1} - B_n^2 = -1$
2. $B_n'(x) = nB_{n-1}(x) + B_{n-2}'(x)$
3. $B_n^{(m)}(x) = nB_{n-1}^{(m)}(x) + B_{n-2}^{(m)}(x)$
4. $B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$

$n = 2, 3, \ldots$, with, $B_0(x) = 1, B_1(x) = 2 + x$

In this paper, we introduced a method using the methods mentioned above. Firstly, we obtained the matrix relations between the Morgan-Voyce polynomials and their derivates, and using these relations; we find the approximate solutions. This method is known as the Morgan-Voyce collocation method for solutions of a system of high-order linear Fredholm differential equations with variable coefficients in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x)y_j^{(n)}(x) = g_i(x) + \int_a^b \sum_{j=1}^{k} K_{i,j}(x,t)y_j(t)dt$$

$i = 1,2,\ldots k,\ 0 \le a \le x \le b.$ (1)

with the mixed conditions

$$\sum_{j=0}^{m-1} a_{i,j}^{n}\, y_j^{(n)}(a) + b_{i,j}^{n} y_j^{(n)}(b) = \lambda_{n,i},$$

$i = 1,2,\ldots,m-1, \quad n = 1,2,\ldots,k$ (2)

where, $y_j^{(0)}(x) = y_j(x)$ will be found, the given functions $P_{i,j}^{n}(x)$, $g_i(x)$, $K_{i,j}(x,t)$ are defined in the interval $a \le x,t \le b$, the functions $K_{i,j}(x,t)$ for $i,j = 1,2,\ldots k$ can be written in Maclaurin series and also $a_{i,j}^{n}, b_{i,j}^{n}$ and $\lambda_{n,i}$ are propers numbers.

We aim to solve the problem (1) and (2) expressed in the truncated Morgan-Voyce series form

$$y_N(x) = \sum_{n=0}^{N} a_n B_n(x) \quad (3)$$

Here, a_n, $n = 0,1,\ldots,N$ are defined Morgan-Voyce coefficients; N is an arbitrary positive integer $(N \ge 2)$. We will calculate a_n. $B_n(x)$, $n = 0,1,\ldots,N$ are the Morgan-Voyce polynomials defined by [29],

$$B_n(x) = \sum_{n=0}^{N} \binom{n+k+1}{n-k} x^k, \quad n \in \mathbb{N}$$

In this paper, we obtain the fundamental matrix relations of systems of linear Fredholm-integro-differential equations and introduce the method in section 2. We describe residual error analysis in Section 3. In Section 4, we give numerical examples to support our method. Section 5 concludes this paper.

2. Method for Solution

The matrix form of Morgan-Voyce polynomials is as the following,

$$\boldsymbol{B^T}(x) = \boldsymbol{R}\boldsymbol{X^T}(x) \Leftrightarrow \boldsymbol{B}(x) = \boldsymbol{X}(x)\boldsymbol{R^T} \quad (4)$$

where

$$\boldsymbol{B}(x) = [B_0(x) \quad B_1(x) \quad \ldots \quad B_N(x)]$$

$$\boldsymbol{X}(x) = [1 \quad x^1 \quad x^2 \quad \ldots \quad x^N]$$

$$\boldsymbol{R} = \begin{bmatrix} \binom{1}{0} & 0 & 0 & \cdots & 0 \\ \binom{2}{1} & \binom{3}{0} & 0 & \cdots & 0 \\ \binom{3}{2} & \binom{4}{1} & \binom{5}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \binom{n+1}{n} & \binom{n+2}{n-1} & \binom{n+3}{n-2} & \cdots & \binom{2n+1}{0} \end{bmatrix}$$

The matrix form of the desired solutions $y_j(x)$ of Equation (1)

$$[y_j(x)] = \boldsymbol{B}(x)\boldsymbol{A}_j, \qquad j = 1,2,\ldots,k$$

where

$\boldsymbol{A}_j = [a_{j,0} \quad a_{j,1} \quad a_{j,2} \quad \ldots \quad a_{j,N}]^T$ or from Equation (4)

$$[y_j(x)] = \boldsymbol{X}(x)\boldsymbol{R^T}\boldsymbol{A}_j, \qquad j = 1,2,\ldots,k \quad (5)$$

Otherwise, $\boldsymbol{X}^{(1)}(x)$ is known as the derivative of the matrix $\boldsymbol{X}(x)$. So, $\boldsymbol{X}^{(1)}(x)$ can be written in terms of the matrix $\boldsymbol{X}(x)$ as the following,

$$\boldsymbol{X}^{(1)}(x) = \boldsymbol{X}(x)\boldsymbol{T^T},\ \boldsymbol{X}^{(0)}(x) = \boldsymbol{X}(x) \quad (6)$$

here

$$\boldsymbol{T^T} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & N \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

We derive from Equation (6) and Equation (4)

$$\boldsymbol{X}^{(0)}(x) = \boldsymbol{X}(x)$$

$$\boldsymbol{X}^{(1)}(x) = \boldsymbol{X}(x)\boldsymbol{T^T}$$

$$\boldsymbol{X}^{(2)}(x) = \boldsymbol{X}^{(1)}(x)\boldsymbol{T^T} = \boldsymbol{X}(x)(\boldsymbol{T^T})^2$$

$$\vdots \qquad\qquad \vdots \qquad (7)$$

$$\boldsymbol{X}^{(k)}(x) = \boldsymbol{X}^{(k-1)}(x)(\boldsymbol{T^T})^{k-1} = \boldsymbol{X}(x)(\boldsymbol{T^T})^{k}$$

and so

$$\boldsymbol{B}^{(k)}(x) = \boldsymbol{X}^{(k)}(x)\boldsymbol{R^T} = \boldsymbol{X}(x)(\boldsymbol{T^T})^{k}\boldsymbol{R^T} \quad (8)$$

Here, $(\boldsymbol{T}^{\boldsymbol{T}})^{\boldsymbol{0}} = [I]_{(N+1)\times(N+1)}$ is the unit matrix.

We derive from Equation (5), Equation (7), and Equation (8) the matrix equation

$$y_j^{(i)}(x) = \boldsymbol{B}^{(i)}(x)\boldsymbol{A}_j = \boldsymbol{X}^{(i)}(x)\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{A}_j = \boldsymbol{X}(x)(\boldsymbol{T}^{\boldsymbol{T}})^{\boldsymbol{k}}\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{A}_j$$

$$i = 0,1,\dots,m,\ \ j = 0,1,\dots,k \qquad (9)$$

So, we can write the matrices $y^{(i)}(x),\ i = 0,1,\dots,m$ as

$$y^{(i)}(x) = \overline{\boldsymbol{X}}(x)(\overline{\boldsymbol{T}})^{\boldsymbol{i}}(x)\overline{\boldsymbol{R}}\boldsymbol{A},\quad i = 1,2,\dots,m \qquad (10)$$

where

$$\boldsymbol{y}^{\boldsymbol{i}}(x) = \begin{bmatrix} y_1^{(i)}(x) \\ y_2^{(i)}(x) \\ \vdots \\ y_k^{(i)}(x) \end{bmatrix},\ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_3 \end{bmatrix},$$

$$\overline{\boldsymbol{X}} = \begin{bmatrix} \boldsymbol{X}(x) & 0 & \cdots & 0 \\ 0 & \boldsymbol{X}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{X}(x) \end{bmatrix}_{k\times k}$$

$$\overline{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{R}^{\boldsymbol{T}} & 0 & \cdots & 0 \\ 0 & \boldsymbol{R}^{\boldsymbol{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{R}^{\boldsymbol{T}} \end{bmatrix},\ \overline{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{T}} & 0 & \cdots & 0 \\ 0 & \boldsymbol{T}^{\boldsymbol{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{T}^{\boldsymbol{T}} \end{bmatrix}_{k\times k}$$

The system (1) can be written in the matrix form

$$\sum_{j=0}^{m} \boldsymbol{P}_i(x)\boldsymbol{y}^{(i)}(x) = \boldsymbol{g}(x) + \boldsymbol{I}(x) \qquad (11)$$

in here

$$\boldsymbol{P}_i(x) = \begin{bmatrix} P_{1,1}^{(i)}(x) & P_{1,2}^{(i)}(x) & \cdots & P_{1,k}^{(i)}(x) \\ P_{2,1}^{(i)}(x) & P_{2,2}^{(i)}(x) & \cdots & P_{2,k}^{(i)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{k,1}^{(i)}(x) & P_{k,2}^{(i)}(x) & \cdots & P_{k,k}^{(i)}(x) \end{bmatrix}$$

$$\boldsymbol{y}^{\boldsymbol{i}}(x) = \begin{bmatrix} y_1^{(i)}(x) \\ y_2^{(i)}(x) \\ \vdots \\ y_k^{(i)}(x) \end{bmatrix}, \boldsymbol{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix},\ \boldsymbol{I}(x) = \begin{bmatrix} I_1(x) \\ I_2(x) \\ \vdots \\ I_k(x) \end{bmatrix},$$

$\boldsymbol{I}(x) = \int_a^b \boldsymbol{K}(x,t)\boldsymbol{y}(t)dt,$

where

$$\boldsymbol{K}(x,t) = \begin{bmatrix} K_{1,1}(x,t) & K_{1,2}(x,t) & \cdots & K_{1,k}(x,t) \\ K_{2,1}(x,t) & K_{2,2}(x,t) & \cdots & K_{2,k}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{k,1}(x,t) & K_{k,2}(x,t) & \cdots & K_{k,k}(x,t) \end{bmatrix},$$

and

$$\boldsymbol{I}_i(x) = \int_a^b \sum_{j=1}^{k} \boldsymbol{K}_{i,j}(x,t)\boldsymbol{y}_j(t)dt \qquad (12)$$

We can find the kernel function $K_{i,j}(x,t)$ with the truncated Morgan-Voyce series and the truncated Taylor series,

$$\boldsymbol{K}_{i,j}(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}^{M}k_{mn}^{ij} B_m(x)B_n(t) \qquad (13)$$

where, $\boldsymbol{K}_M^{ij} = [{}^{M}k_{mn}^{ij}]$ and

$$\boldsymbol{K}_{i,j}(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}^{t}k_{mn}^{ij}\, x^m t^n,\ \ \boldsymbol{K}_t^{ij} = [\,{}^{t}k_{mn}^{ij}] \qquad (14)$$

in here

$${}^{t}k_{mn}^{ij} = \frac{1}{m!\,n!}\frac{\partial^{m+n}K(0,0)}{\partial x^m \partial t^n};\qquad m,n = 0,1,\dots N,$$

$$i,j = 0,1,\dots,k$$

We convert Equation (13) and Equation (14) to the matrix form and then equate them, so we have the relation

$$[\boldsymbol{K}_{i,j}(x,t)] = \boldsymbol{B}(x)\boldsymbol{K}_M^{ij}\boldsymbol{B}^{\boldsymbol{T}}(t) = \boldsymbol{X}(x)\boldsymbol{K}_t^{ij}\boldsymbol{X}^{\boldsymbol{T}}(t) \qquad (15)$$

So, we derive from Equation (15) the following relation

$$\boldsymbol{K}_M^{ij} = (\boldsymbol{R}^{-1})^T\boldsymbol{K}_t^{ij}\boldsymbol{R}^{-1}$$

Equation (5) and Equation (13) are substituted into Equation (12), the matrix equation is obtained following as

$$\boldsymbol{I}_i(x) = \int_a^b \sum_{j=1}^{k} \boldsymbol{B}(x)\boldsymbol{K}_M^{ij}\boldsymbol{B}^{\boldsymbol{T}}(t)\boldsymbol{X}(t)\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{A}_j dt$$

$$= \sum_{j=1}^{k}\int_a^b \boldsymbol{B}(x)\boldsymbol{K}_M^{ij}\boldsymbol{B}^{\boldsymbol{T}}(t)\boldsymbol{X}(t)\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{A}_j dt$$

$$= \sum_{j=1}^{k} \boldsymbol{B}(x)\boldsymbol{K}_M^{ij}\boldsymbol{Q}_{ij}\,\boldsymbol{A}_j \qquad (16)$$

from here

$$[\boldsymbol{Q}_{ij}] = \int_a^b \boldsymbol{B}^{\boldsymbol{T}}(t)\boldsymbol{X}(t)\boldsymbol{R}^{\boldsymbol{T}}dt$$

$$= \int_a^b \boldsymbol{R}\boldsymbol{X}^{\boldsymbol{T}}(t)\boldsymbol{X}(t)\boldsymbol{R}^{\boldsymbol{T}}dt = \boldsymbol{R}\boldsymbol{H}\boldsymbol{R}^{\boldsymbol{T}}$$

in here

$$\boldsymbol{H} = \int_a^b \boldsymbol{X}^{\boldsymbol{T}}(t)\boldsymbol{X}(t)dt = [h_{rs}];\ \ h_{rs} = \frac{b^{r+s+1} - a^{r+s+1}}{r+s+1},$$

$$r,s = 0,1,2,\dots,N$$

The matrix form of (4) is substituted into Equation (16), the matrix relation is obtained as,

$$[\boldsymbol{I_i}\,(x)] = \sum_{j=0}^{k} \boldsymbol{X}(x)\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{K}_M^{ij}\boldsymbol{Q}_{ij}\,\boldsymbol{A}_j \qquad (17)$$

We define the collocation points following as

$$x_s = a + \frac{b-a}{N}, \quad s = 0,1,2,\dots,N \qquad (18)$$

and the collocation points are placed in Equation (11), so we have the system of the matrix equations

$$\sum_{j=0}^{m} \boldsymbol{P}_i\,(x_s)\boldsymbol{y}^{(i)}(x_s) = \boldsymbol{g}(x_s) + \boldsymbol{I}(x_s)$$

or concisely the fundamental matrix equation

$$\sum_{j=0}^{m} \boldsymbol{P}_i\,\boldsymbol{Y}^{(i)} = \boldsymbol{G} + \boldsymbol{I} \qquad (19)$$

where

$$\boldsymbol{P}_i = \begin{bmatrix} \boldsymbol{P}_i(x_0) & 0 & \cdots & 0 \\ 0 & \boldsymbol{P}_i(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{P}_i(x_N) \end{bmatrix}, \boldsymbol{Y}^{(i)} = \begin{bmatrix} \boldsymbol{y}^{(i)}(x_0) \\ \boldsymbol{y}^{(i)}(x_1) \\ \vdots \\ \boldsymbol{y}^{(i)}(x_N) \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}(x_0) \\ \boldsymbol{g}(x_1) \\ \vdots \\ \boldsymbol{g}(x_N) \end{bmatrix}, \ \boldsymbol{I} = \begin{bmatrix} \boldsymbol{I}(x_0) \\ \boldsymbol{I}(x_1) \\ \vdots \\ \boldsymbol{I}(x_N) \end{bmatrix}$$

By placing in Equation (10) the collocation points (18), we get the matrix relation as the following

$$y^{(i)}(x_s) = \overline{\boldsymbol{X}}(x_s)\left(\overline{\boldsymbol{T}}\right)^{i}\overline{\boldsymbol{R}}\boldsymbol{A} \qquad (20)$$

$$s = 1,2,\dots,N, \ i = 1,2,\dots,m$$

briefly, we can write Equation (20) following as

$$\boldsymbol{Y}^{(i)} = \overline{\boldsymbol{X}}\left(\overline{\boldsymbol{T}}\right)^{i}\overline{\boldsymbol{R}}\boldsymbol{A},$$

where

$$\boldsymbol{X} = \begin{bmatrix} \overline{\boldsymbol{X}}(x_0) \\ \overline{\boldsymbol{X}}(x_1) \\ \vdots \\ \overline{\boldsymbol{X}}(x_N) \end{bmatrix}, \ \overline{\boldsymbol{X}}(x_s) = \begin{bmatrix} \boldsymbol{X}(x_s) & 0 & \cdots & 0 \\ 0 & \boldsymbol{X}(x_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{X}(x_s) \end{bmatrix}_{k\times k}$$

$$s = 1,2,\dots,N, \ i = 1,2,\dots,m$$

we substitute the collocation points (18) in the matrix relation (17), and so the relation is as

$$[\boldsymbol{I_i}\,(x_s)] = \sum_{j=0}^{k} \boldsymbol{X}(x_s)\boldsymbol{R}^{\boldsymbol{T}}\boldsymbol{K}_M^{ij}\boldsymbol{Q}_{ij}\,\boldsymbol{A}_j \qquad (21)$$

$$s = 1,2,\dots,N, i = 1,2,\dots,m$$

and substituting Equation (18) into the matrix $\boldsymbol{I}(x)$ given in Equation (11) and using Equation (21), we have

$$\boldsymbol{I}(x_s) = \begin{bmatrix} I_1(x_s) \\ I_2(x_s) \\ \vdots \\ I_k(x_s) \end{bmatrix} = \overline{\boldsymbol{X}}(x_s)\overline{\boldsymbol{R}}\boldsymbol{K}_f\overline{\boldsymbol{O}}\boldsymbol{A}, \qquad (22)$$

where

$$\overline{\boldsymbol{X}}(x_s) = \begin{bmatrix} \boldsymbol{X}(x_s) & 0 & \cdots & 0 \\ 0 & \boldsymbol{X}(x_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{X}(x_s) \end{bmatrix}_{k\times k},$$

$$\overline{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{R}^{\boldsymbol{T}} & 0 & \cdots & 0 \\ 0 & \boldsymbol{R}^{\boldsymbol{T}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{R}^{\boldsymbol{T}} \end{bmatrix}, \ \overline{\boldsymbol{Q}} = \begin{bmatrix} \boldsymbol{Q} & 0 & \cdots & 0 \\ 0 & \boldsymbol{Q} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{Q} \end{bmatrix},$$

$$\boldsymbol{K}_f = \begin{bmatrix} \boldsymbol{K}_M^{11} & \boldsymbol{K}_M^{12} & \cdots & \boldsymbol{K}_M^{1k} \\ \boldsymbol{K}_M^{21} & \boldsymbol{K}_M^{22} & \cdots & K_M^{2k} \\ \vdots & \vdots & \ddots & \vdots \\ K_M^{k1} & K_M^{k2} & \cdots & K_M^{kk} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_3 \end{bmatrix}$$

So, by using Equation (22) and the matrix of the Morgan-Voyce coefficients, we can write Equation (19) in matrix form as the follows

$$\boldsymbol{I} = \begin{bmatrix} \boldsymbol{I}(x_0) \\ \boldsymbol{I}(x_1) \\ \vdots \\ \boldsymbol{I}(x_N) \end{bmatrix} = \boldsymbol{X}\overline{\boldsymbol{R}}\boldsymbol{K}_f\overline{\boldsymbol{Q}}\boldsymbol{A} \qquad (23)$$

where

$$\boldsymbol{X} = \begin{bmatrix} \overline{\boldsymbol{X}}(x_0) \\ \overline{\boldsymbol{X}}(x_1) \\ \vdots \\ \overline{\boldsymbol{X}}(x_N) \end{bmatrix}$$

If we substitute Equation (20) and Equation (23) into Equation (19), we have the fundamental matrix equation

$$\sum_{i=0}^{m} \left\{\boldsymbol{P}_i\boldsymbol{X}\left(\overline{\boldsymbol{T}}\right)^{i}\overline{\boldsymbol{R}} - \boldsymbol{X}\overline{\boldsymbol{R}}\boldsymbol{K}_f\overline{\boldsymbol{Q}}\right\}\boldsymbol{A} = \boldsymbol{G} \qquad (24)$$

Therefore, Equation (24) equal to Equation (1) can be expressed in the form

$$\boldsymbol{W}\boldsymbol{A} = \boldsymbol{G} \quad \text{or} \quad [\boldsymbol{W};\boldsymbol{G}] \qquad (25)$$

This form refers to a linear system of $k(N+1)$ algebraic equation with $k(N+1)$ the unknown Morgan-Voyce coefficients, so

$$\boldsymbol{W} = \sum_{i=0}^{m} \mathbf{P}_i\mathbf{X}\left(\overline{\mathbf{T}}\right)^{i}\overline{\mathbf{R}} - \mathbf{X}\overline{\mathbf{R}}\mathbf{K}_f\overline{\mathbf{Q}} = [\mathrm{w}_{p,q}],$$

$$\mathrm{p,q} = 1,2,\dots,\mathrm{k(N+1)}$$

Now, we build the matrix symbols of the conditions. When we use the conditions (2), we get

$$\sum_{j=0}^{m-1}\left[a_{i,j}^{1}y_1^{(j)}(a)+b_{i,j}^{1}y_1^{(j)}(b)\right]=\lambda_{1,i}$$

$$\sum_{j=0}^{m-1}\left[a_{i,j}^{2}y_2^{(j)}(a)+b_{i,j}^{2}y_2^{(j)}(b)\right]=\lambda_{2,i}$$

$$\vdots$$

$$\sum_{j=0}^{m-1}\left[a_{i,j}^{k}y_k^{(j)}(a)+b_{i,j}^{k}y_k^{(j)}(b)\right]=\lambda_{k,i}$$

or

$$\sum_{j=0}^{m-1}\left[a_{i,j}^{1}y_1^{(j)}(a)+b_{i,j}^{1}y_1^{(j)}(b)\right]=\lambda_{1}$$

$$\sum_{j=0}^{m-1}\left[a_{i,j}^{2}y_2^{(j)}(a)+b_{i,j}^{2}y_2^{(j)}(b)\right]=\lambda_{2}$$

$$\vdots$$

$$\sum_{j=0}^{m-1}\left[a_{i,j}^{k}y_k^{(j)}(a)+b_{i,j}^{k}y_k^{(j)}(b)\right]=\lambda_{k}$$

in that equation

$$\lambda_i=\begin{bmatrix}\lambda_{i,0}\\ \lambda_{i,1}\\ \vdots\\ \lambda_{i,m-1}\end{bmatrix}_{m\times1},\ \mathrm{a}_j^i=\begin{bmatrix}\mathrm{a}_{0,j}^i\\ \mathrm{a}_{1,j}^i\\ \vdots\\ \mathrm{a}_{m-1,j}^i\end{bmatrix}_{m\times1},\ \mathrm{b}_j^i=\begin{bmatrix}\mathrm{b}_{0,j}^i\\ \mathrm{b}_{1,j}^i\\ \vdots\\ \mathrm{b}_{m-1,j}^i\end{bmatrix}_{m\times1}$$

$i=1,2,\dots,k$

or briefly

$$\sum_{j=0}^{m-1}\left[\mathbf{a}_j y^{(j)}(a)+\mathbf{b}_j y^{(j)}(b)\right]=\lambda \tag{26}$$

where

$$\mathbf{a}_j=\begin{bmatrix}a_j^1 & 0 & \cdots & 0\\ 0 & a_j^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_j^k\end{bmatrix}_{k\times k},\ \mathbf{b}_j=\begin{bmatrix}b_j^1 & 0 & \cdots & 0\\ 0 & b_j^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b_j^k\end{bmatrix}_{k\times k},$$

$$\boldsymbol{\lambda}=\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_k\end{bmatrix}_{k\times1}$$

We substitute the points a and b in Equation (10), we have

$$y^{(i)}(a)=\overline{\boldsymbol{X}}(a)\left(\boldsymbol{T}\right)^{i}\overline{\boldsymbol{R}}\boldsymbol{A},\quad i=1,2,\dots,m$$

$$y^{(i)}(b)=\overline{\boldsymbol{X}}(b)\left(\boldsymbol{T}\right)^{i}\overline{\boldsymbol{R}}\boldsymbol{A},\quad i=1,2,\dots,m \tag{27}$$

The matrix relations in Equation (27), which depend on the matrix of Morgan-Voyce coefficients, are substituted in Equation (16), and the equation is simplified, we get

$$\sum_{\mathrm{i}=0}^{\mathrm{m}}\left[\mathbf{a}_{\mathrm{j}}\overline{\mathbf{X}}(a)+\mathbf{b}_{\mathrm{j}}\overline{\mathbf{X}}(b)\right]\left(\mathbf{T}\right)^{\mathrm{i}}\overline{\mathbf{R}}\mathbf{A}=\boldsymbol{\lambda} \tag{28}$$

Now, we define $\boldsymbol{U}$ following as,

$$\boldsymbol{U}=\sum_{\mathrm{i}=0}^{\mathrm{m}}\left[\mathbf{a}_{\mathrm{j}}\overline{\mathbf{X}}(a)+\mathbf{b}_{\mathrm{j}}\overline{\mathbf{X}}(b)\right]\left(\mathbf{T}\right)^{\mathrm{i}}\overline{\mathbf{R}}$$

so, for the conditions, the matrix relation becomes as

$$\boldsymbol{UA}=\boldsymbol{\lambda}\quad\text{or}\quad[\boldsymbol{U};\boldsymbol{\lambda}] \tag{29}$$

Finally, we replace the rows of the matrix $\boldsymbol{U}$ and $\boldsymbol{\lambda}$, by the rows of matrix $\boldsymbol{W}$ and $\boldsymbol{G}$, respectively, we get

$$\overline{\boldsymbol{W}}\boldsymbol{A}=\overline{\boldsymbol{G}} \tag{30}$$

By replacing the last mk rows of the matrix $\boldsymbol{W}$, we obtain the augmented matrix following as,

$$[\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{G}}]=\begin{bmatrix}
w_{11} & w_{12} & \cdots & w_{1,k(N+1)} & ; & g_1(x_0)\\
w_{21} & w_{22} & \cdots & w_{2,k(N+1)} & ; & g_2(x_0)\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
w_{k2} & w_{k2} & \cdots & w_{k,k(N+1)} & ; & g_k(x_0)\\
w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & g_1(x_1)\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
w_{k,(N-m+1),1} & w_{k,(N-m+1),2} & \cdots & w_{k,(N-m+1),k(N+1)} & ; & g_k(x_{N-m})\\
v_{11} & v_{12} & \cdots & v_{1,k(N+1)} & ; & \lambda_{1,0}\\
v_{21} & v_{22} & \cdots & v_{2,k(N+1)} & ; & \lambda_{1,1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
v_{k1} & v_{k2} & \cdots & v_{k,k(N+1)} & ; & \lambda_{1,m-1}\\
v_{k+1,1} & v_{k+1,2} & \cdots & v_{k+1,k(N+1)} & ; & \lambda_{2,0}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
v_{mk,1} & v_{mk,2} & \cdots & v_{mk,k(N+1)} & ; & \lambda_{k,m-1}
\end{bmatrix} \tag{31}$$

But, the last rows haven't to be replaced. For instance, if the matrix $\boldsymbol{W}$ is singular, then the rows with the same factor or all zeros are returned.

If $rank\,\widetilde{\boldsymbol{W}}=rank\left[\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{G}}\right]=N+1$, it can be written

$\boldsymbol{A}=(\widetilde{\boldsymbol{W}})^{-1}\widetilde{\boldsymbol{G}}$. Thereby, the unknown coefficients $a_0,a_1,\dots,a_N$ can be uniquely determined. Thus, the problem (1) and (2) has a unique solution, and this solution is expressed by Morgan-Voyce series solution (3).

3. Error Estimation

In this section, with the residual error function [30] for the Morgan-Voyce polynomial solutions (3), an error estimation is obtained. With the help of the residual error function, the Morgan-Voyce polynomial solution (3) is improved. Firstly, the residual function of the Morgan-Voyce collocation method can be constructed as

$$R_{i,N}(x)=L\left[y_{i,N}(x)\right]-g_i(x) \tag{32}$$

Here $y_{i,N}(x)$ is the Morgan-Voyce polynomial solution of Equation (1) with condition (2). Thus, $y_{i,N}(x)$ satisfies the problem

$$\begin{cases} L[y_i(x)] = \sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) y_{i,N}^{(n)}(x) \\ -\int_a^b \sum_{j=1}^{k} K_{i,j}(x,t) y_{i,N}(t)dt = g_i(x) + R_{i,N}(x) \\ i = 1,2,\dots,k, \quad 0 \le a \le x \le b \\ \sum_{j=0}^{m-1} (a_{i,j}^{n} y_{n,N}^{(j)}(a) + b_{i,j}^{n} y_{n,N}^{(j)}(b)) = \lambda_n, \quad i = 1,2,\dots,m-1, \\ n = 1,2,\dots,k \end{cases}$$

The error functions of approximate solutions $y_{i,N}(x)$ to $y_i(x)$ are

$$e_{i,N}(x) = y_i(x) - y_{i,N}(x) \tag{33}$$

where $y_i(x)$ is the exact solution of problem (1) and (2).

From Equation (1), Equation (2), Equation (32) and Equation (33) the error differential equation is

$$L[e_{i,N}(x)] = L[y_i(x)] - L[y_{i,N}(x)] = -R_{i,N}(x)$$

with the conditions

$$\sum_{j=0}^{m-1} (\boldsymbol{a_{i,j}^{n} e_{n,N}^{(j)}(a)} + \boldsymbol{b_{i,j}^{n} e_{n,N}^{(j)}(b)}) = \boldsymbol{0}$$

$$i = 0,1,2,\dots,m-1, n = 1,2,\dots,k$$

or clearly, the error problem is

$$\sum_{n=0}^{m}\sum_{j=1}^{k} P_{i,j}^{n}(x) e_{i,N}^{(n)}(x) - \int_a^b \sum_{j=1}^{k} K_{i,j}(x,t) e_{i,N}(t)dt = -R_{i,N}(x)$$

$$i = 1,2,\dots,k, \quad 0 \le a \le x \le b$$

$$\sum_{j=0}^{m-1} (a_{i,j}^{n} e_{n,N}^{(j)}(a) + b_{i,j}^{n} e_{n,N}^{(j)}(b)) = 0, \quad i = 1,2,\dots,m-1$$

$$n = 1,2,\dots,k$$

(34)

If we solve problem (34) by using the method given in section (2), we have the approximation solutions,

$$e_{i,N,M}(x) = \sum_{n=0}^{M} a_n^{*} B_n(x), \quad M \ge N$$

to $e_{i,N}(x)$.

Consequently, utilizing the polynomials $y_{i,N}(x)$ and $e_{i,N,M}(x)$, $(M \ge N)$, the correct Morgan-Voyce polynomial solutions are obtained as $y_{i,N,M}(x) = y_{i,N}(x) + e_{i,N,M}(x)$. Also, we construct the error function as $e_{i,N}(x) = y_i(x) - y_{i,N}(x)$, the correct error function solution $|E_{i,N,M}(x)| = |e_{i,N}(x) - e_{i,N,M}(x)| = |y_i(x) - y_{i,N,M}(x)|$ and the estimated error function $e_{i,N,M}(x)$.

If we have not the exact solution of Equation (1), we cannot find the absolute errors $|e_{i,N}(x_i)| = |y_i(x_i) - y_{i,N}(x_i)|$, $(0 \le x_i \le b)$. Otherwise, we can approximately calculate the absolute errors using the estimated absolute error function $|e_{i,N,M}(x)|$. This method is an important and very useful tool for the solution of the problem.

4. Illustrative examples

This section investigates the proposed method's accuracy and efficiency. We present two numerical examples which compare errors $|e_{i,N}(x)|$, $|E_{i,N,M}(x)|$ and $e_{i,N,M}(x)$. These comparisons have been given in Tables and Figures at the specified points of the given interval.

Example 1. With the exact solutions $y_1(x) = e^{-x}$ and $y_2(x) = e^{x}$ and the initial conditions $y_1(0) = 1$, $y_1'(0) = -1$, $y_2(0) = 1$, $y_2'(0) = 1$, $y_1''(0) = 1$, $y_2''(0) = 1$, for $0 \le x \le 1$, consider the equations

$$-y_1'''(x) - x^2 y_2''(x) - x y_1'(x) + x^2 y_2(x) = g_1(x) + \int_0^1 [(xe^{t}) y_1(t) + (e^{-t} sinx) y_2(t)]dt \tag{35}$$

$$y_2'''(x) + y_1''(x) - y_2'(x) - x y_1(x) = g_2(x) + \int_0^1 [(e^{t} cosx) y_1(t) + (x+t) y_2(t)]dt$$

where $g_1(x) = e^{-x}(x+1) - x - sinx$ and $g_2(x) = e^{-x}(1-x) - cosx - 1$.

The approximate solutions $y_i(x)$ by the truncated Morgan-Voyce series are

$$y_{i,6}(x) = \sum_{n=0}^{6} a_{i,n} B_n(x), \quad i = 1,2$$

Now we can use the method to get the approximate solutions. Firstly we construct the matrix relation, and then by using the collocation points and Equation (35) we can write the fundamental matrix equation as the following,

$$\left\{\boldsymbol{P_0 X \overline{R}} + \boldsymbol{P_1 X \overline{T} \overline{R}} + \boldsymbol{P_2 X (\overline{T})^2 \overline{R}} + \boldsymbol{P_3 X (\overline{T})^3 \overline{R}} - \boldsymbol{X \overline{R} K_f \overline{Q}}\right\} \boldsymbol{A} = \boldsymbol{G}$$

Hence, from the method given in Section 2, for $i = 1,2$ and $N = 6$ by the Morgan-Voyce polynomials the approximate solutions of the problem respectively are as the following,

$$y_{1,6}(x) = 1 - x + 0.5x^2 - 0.166666666667x^3$$
$$+(0.41625019037e-1)x^4 - (0.815499085386e-3)x^5$$
$$+(0.10848283106e-3)x^6$$
$$y_{2,6}(x) = 1 + x + 0.5x^2 + 0.166664620327x^3$$
$$+(0.41727698665e-1)x^4 - (0.807449560784e-2)x^5$$
$$+(0.17920672461e-2)x^6$$

Now, let us find the corrected Morgan-Voyce polynomial solutions for $M = 9$ with the method introduced in Sections 2 and 3. For this purpose, firstly, we find the estimated absolute error functions as follows

$$e_{1,6,9}(x) = (-7.346e-40) - (1.836e-40)x$$
$$-(4.591e-40)x^2 + (0.388e-40)x^3 + (0.416e-9)x^4$$
$$-(0.178e-3)x^5 + (0.303e-3)x^6 - (0.197e-3)x^7$$
$$+(0.237e-4)x^8 - (0.196e-5)x^9$$
$$e_{2,6,9}(x) = (7.34e-40) - (1.836e-40)x^2 + (0.204e-5)x^3$$
$$-(0.611e-4)x^4 + (0.258e-3)x^5 - (0.403e-3)x^6$$
$$+(0.199e-3)x^7 + (0.231e-4)x^8 - (0.388e-5)x^9$$

Besides, we have the corrected Morgan-Voyce polynomial solutions

$$y_{1,6,9}(x) = 0.99999 - x + 0.49999x^2 - 0.16666x^3$$
$$+0.0416x^4 - 0.00833x^5 + 0.00138x^6 - (0.197e-3)x^7$$
$$+(0.237e-4)x^8 - (0.196e-5)x^9$$
$$y_{2,6,9}(x) = 0.99999 + 0.99999x + 0.49999x^2$$
$$+0.16666x^3 + 0.0416x^4 + 0.00833x^5 + 0.00138x^6$$
$$+(0.199e-3)x^7 + (0.231e-4)x^8 + (0.388e-5)x^9$$

In Table 1 and Table 2, we compare the errors $e_{1,N}(x_i)$ and $e_{1,N,M}(x_i)$ ($e_{2,N}(x_i)$ and $e_{2,N,M}(x_i)$) for $N = 6,9$ and $M = 9,12,15$ and also compare the error functions $e_{1,N}(x)$ and the $e_{1,N,M}(x)$ for $N = 6,9$ and $M = 9,12,15$ in Fig. 1a and Fig. 1b. These results show that the difference between the estimated absolute errors and the actual absolute errors is very small. Table 3 and Table 4 illustrate the errors $|E_{1,N,M}(x_i)|$ and $|E_{2,N,M}(x_i)|$ for $N = 6,9$ and $M = 9,12,15$, respectively. We give the error functions $|E_{1,N,M}(x_i)|$ and $|E_{2,N,M}(x_i)|$ in Fig. 1c. We see from Tables 1, 2, 3 and 4 and Fig. 1a, Fig. 1b, and Fig. 1c that the errors decrease when N and M are increased.

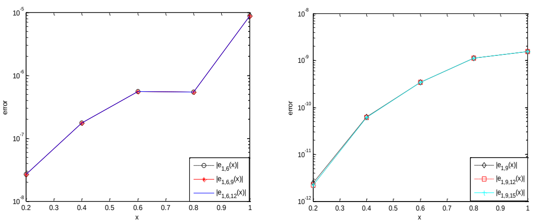

Figure 1a. Comparison of $e_{1,N}(x)$ and the $e_{1,N,M}(x)$ for $N = 6,9, M = 9,12,15$ of Equation (35).

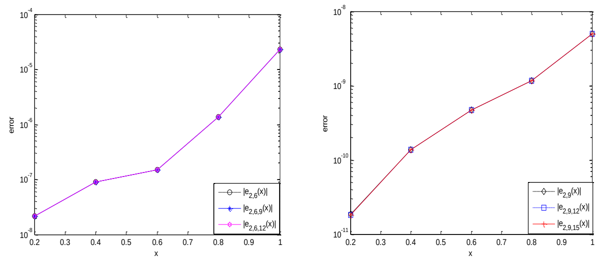

Figure 1b. Comparison of $e_{2,N}(x)$ and $e_{2,N,M}(x)$ for $N = 6,9, M = 9,12,15$ of Equation (35).

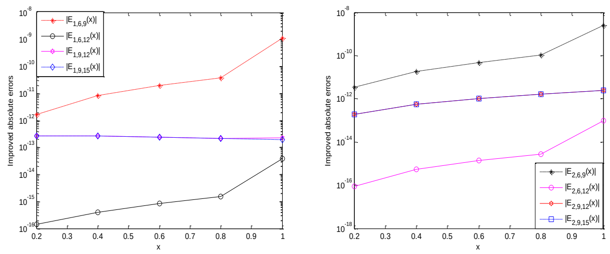

Figure 1c. Comparison of $E_{1,N,M}(x)$ and $E_{2,N,M}(x)$ for $N = 6,9,\ M = 9,12,15$ of Equation (35).

Example 2. With the exact solutions $y_1(x) = \sin(-5x)$ and $y_2(x) = e^{-3x}$, for $0 \le x \le 1$, consider equations

$$y_1(x) = g_1(x) + \int_0^1 [-xt^2 y_1(t) + xty_2(t)]dt$$
$$y_2(x) = g_2(x) + \int_0^1 [x(t+1)y_1(t) + x^2 t y_2(t)]dt \quad (36)$$

where

$$g_1(x) = -\sin(5x) - x\left(-\frac{23}{125}\cos(5) + \frac{2}{25}\sin(5) - \frac{2}{125}\right)$$
$$-x\left(-\frac{4}{9}e^{-3} + \frac{1}{9}\right)$$
$$g_2(x) = e^{-3x} - x\left(\frac{2}{5}\cos(5) - \frac{1}{25}\sin(5) - \frac{1}{5}\right)$$
$$-x^2\left(-\frac{4}{9}e^{-3} + \frac{1}{9}\right)$$

When the presented method is applied to this system, the approximate solutions are obtained for different values of N and M. In Table 5 and Table 6, the absolute errors by the present method, the Bessel Collocation method (BCM) [22], and Modified Homotopy Perturbation Method (MHPM) [8] are compared. In Fig. 2, the absolute error functions are compared.

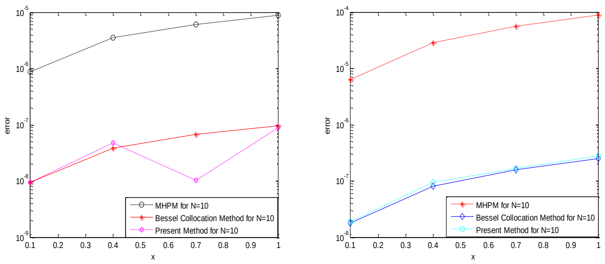

Figure 2. Comparison of $e_{1,10}(x)$ and $e_{2,10}(x)$ of Eqn. (5.2).

5. Conclusions

In cases where the high-order linear integro-differential equations system is complicated to solve analytically, solutions should be approximated. In this article, a new technique using the Morgan-Voyce polynomials to numerically solve high-order linear Fredholm differential equations systems is proposed. This technique is related to the residual error function. Besides, an error estimation is introduced using the residual error function. Moreover, suppose the exact solution to the problem is unknown. In that case, the absolute errors $|e_{i,N}(x_i)| = |y_i(x_i) - y_{i,N}(x_i)|$, $(a \leq x_i \leq b)$ can be estimated by the approximation $|e_{i,N,M}(x)|$. As a result of numerical approaches, it is seen that the proposed method is a useful method for solutions of a system of linear integro-differential equations. The crucial benefit of the technique is that approximate solutions can be computed very easily and quickly by using MATLAB v7.11.0 or MAPLE 15 and Morgan-Voyce polynomials can be applied to this method. This method can be improved and applied to the nonlinear differential equations, nonlinear integral and integrodifferential equation, fractional differential equations, and also systems of the partial differential equation, but it is required some modifications. Moreover, the convergence of the approximation can be investigated.

6. References

[1] Wang, H., Fu, H.M., Zhang, H.F., "A practical thermodynamic method to calculate the best glass-forming composition for bulk metallic glasses", *Int J Nonlinear Sci Numer Simul*, 8.2., 171-8, 2007.

[2] Sun, F.Z., Gao, M., Lei, S.H., "The fractal dimension of the fractal model of dropwise condensation and its experimental study", *Int J Nonlinear Sci Numer Simul*, 8. 2., 211-22, 2007.

[3] Bo, T.L., Xie, L., Zheng, X.L., "Numerical approach to wind ripple in desert", *Int J Nonlinear Sci Numer Simul*, 8. 2., 223-8, 2007.

[4] Bellomo, N., Firmani, B., Guerri, L., "Bifurcation analysis for a nonlinear system of integro-differential equations modelling tumor-immune cells competition", *App. Math. Lett*, 12., 39-44, 1999.

[5] Wazwaz, A.M., "The existence of noise terms for systems of inhomogeneous differential and integral equations", *Appl. Math. Comput.*, 16., 81-92, 2003.

[6] Yusufoğlu, E., "A homotopy perturbation algorithm to solve a system of Fredholm-Volterra type integral equations", *Math. Comput. Modelling*, 47., 1099-1107, 2008.

[7] Yusufoğlu, E., (Agadjanov), "Numerical expansion methods for solving systems of linear integral equations using interpolation and quadrature rules", *Int. J. Comput. Math.*, 84. 1., 133-149, 2007.

[8] Javidi, M., "Modified homotopy perturbation method for solving system of linear Fredholm integral equations", *Math. Comput. Modelling*, 50., 159-165, 2009.

[9] Aziz, I., Al-Fhaid, A. S. "An improved method based on Haar wavelets for numerical solution of nonlinear integral and integro-differential equations of first and higher orders.", *Journal of Computational and Applied Mathematics*, 260: 449-469, 2014.

[10] Mirzaee, F., Hoseini, S. F., "Solving systems of linear Fredholm integro-differential equations with Fibonacci polynomials.", *Ain Shams Engineering Journal*, 5(1): 271-283, 2014.

[11] Gülsu, M., Sezer, M., "Taylor collocation method for solution of systems of high-order linear Fredholm-Volterra integro-differential equations", *Int. J. Comput. Math.* 83., 429-448, 2006.

[12] İlhan, Ö., "An Improved Morgan-Voyce Collocation Method for Numerical Solution of Generalized Pantograph Equations", *Journal of Scientific and Engineering Research,* 4., 10., 320-332, 2017.

[13] İlhan, Ö., Şahin, N., "On Morgan-Voyce Polynomials Approximation For Linear Differential Equations", *Turkish Journal of Mathematics and Computer Science*, 1., 1., 2014.

[14] İlhan, Ö., "An improved Morgan-voyce collocation method for numerical solution of multi-pantograph equations", *New Trends in Mathematical Science*, 4., 5., 248-260, 2017.

[15] Özel, M., Kürkçü, Ö.K., Sezer M., "Morgan-Voyce Matrix Method For Generalized Functional Integro-Differential Equations Of Volterra-Type", *Journal of Science and Arts,* 2 (47): 295-310, 2019 .

[16] Kürkçü, K. Ö., Aslan, E., Sezer, M., İlhan, Ö., "A Numerical Approach Technique for Solving Generalized Delay

Integro-Differential Equations with Functional Bounds by Means of Dickson Polynomials", *International Journal of Computational Methods*, 15., 5., 2018.

[17] Gökmen, E., Sezer, M., "Taylor collocation approach for delayed Lotka–Volterra predator–prey system.", *Journal of the Frankin Institute.,* 345(8): 839-850, 2008.

[18] Karamete, A., Işık, O. R., Sezer, M., "A Taylor collocation method for the solution of linear integro-differential equations.", *Appl. Math. and Comput*., 268: 671-684, (2015).

[19] Kürkçü, Ö.K., Sezer M., "A directly convergent numerical method based on orthoexponential polynomials for solving integro-differential-delay equations with variable coefficients and infinite boundary on half-line", *Journal of Computational and Applied Mathematics,* 386: 113250, 2021.

[20] Gülsu, M., Öztürk, Y., Anapalı, A., "Numerical solution the fractional Bagley–Torvik equation arising in fluid mechanics", *Int. J. Comput. Math*., 94., 1., 173-184, 2017.

[21] Biçer, G. G., Öztürk, Y., Gülsu, M., "Numerical approach for solving linear Fredholm integro-differential equation with piecewise intervals by Bernoulli polynomials",", *Int. J. Comput. Math*., 95., 10., 2100-2111, 2018.

[22] Yüzbaşı, Ş., Şahin, N., Sezer, M., "Numerical solutions of systems of Linear Fredholm integro-differential equations with Bessel polynomial bases". *Computers and Mathematics with Applications*. 61., 10., 3079-96, 2011.

[23] Demir, D. D., Lukonde, A. P., Kürkçü, Ö.K., Sezer M., "Pell–Lucas series approach for a class of Fredholm-type delay integro-differential equations with variable delays.", *Math Sci.,* doi:10.1007/s40096-020-00370-5, 2021.

[24] Kürkçü, K. Ö., Aslan, E., Sezer, M., "A numerical approach with error estimation to solve general integro-differential–difference equations using Dickson polynomials", *Applied Mathematics and Computation*, 276: 324-339, 2016.

[25] Kürkçü, K. Ö., Aslan, E., Sezer, M., "A numerical method for solving some model problems arising in science and convergence analysis based on residual function", *Applied Numerical Mathematics*, 121: 134-148, 2017.

[26] Türkyılmaz, B., Gürbüz, B., Sezer, M., "Morgan-Voyce Polynomial Approach for Solution of High-Order Linear Differential-Difference Equations with Residual Error Estimation." *Düzce Üniversitesi Bilim ve Teknoloji Dergisi*, 4(1), 2016.

[27] Gürbüz, B., Sezer, M., "An hybrid numerical algorithm with error estimation for a class of functional integro-differential equations." *Gazi University Journal of Science*, 29(2): 419-434, 2016.

[28] Gürbüz, B., "Hybrid approximation for solutions of high-order integro-differential equations including variable delay.", *In Journal of Physics: Conference Series,* 1641(1): p. 012062. IOP Publishing, 2020.

[29] Swamy, M. N. S., Bhattacharryya, B. B., "A Study of Recurrent Ladders Using the Polynomials Defined by Morgan-Voyce", *İEEE Transactions on Circuit Theory*, ct-14., No.3, 260-264, 1967.

[30] Çelik, İ., "Collocation method and residual correction using Chebyshev series", *Appl. Math. Comput*., 174., 910-920, 2006.

Table 1. Comparison of the absolute error functions of $y_1(x)$ Equation of (35)

	Absolute errors for Morgan-Voyce polynomial solutions		Estimated absolute errors for Morgan-Voyce polynomial solutions			
x_i	$\lvert e_{1,6}(x_i)\rvert$	$\lvert e_{1,9}(x_i)\rvert$	$\lvert e_{1,6,9}(x_i)\rvert$	$\lvert e_{1,9,12}(x_i)\rvert$	$\lvert e_{1,6,12}(x_i)\rvert$	$\lvert e_{1,9,15}(x_i)\rvert$
0.0	7.1734e-17	2.5849e-13	7.3468e-40	8.5522e-40	4.9622e-40	5.8775e-03
0.2	2.6549e-08	2.4862e-12	2.6547e-08	2.2191e-12	2.6549e-08	2.2190e-01
0.4	1.7586e-07	6.1814e-11	1.7586e-07	6.1553e-11	1.7586e-07	6.1552e-11
0.6	5.5197e-07	3.4135e-10	5.5195e-07	3.4111e-10	5.5197e-07	3.4111e-10
0.8	5.3582e-07	1.1055e-09	5.3579e-07	1.1053e-09	5.3582e-07	1.1052e-09
1.0	8.7487e-06	1.5305e-09	8.7498e-06	1.5303e-09	8.7487e-06	1.5303e-09

Table 2. Comparison of the absolute error functions of $y_2(x)$ Equation of (35)

	Absolute errors for Morgan-Voyce polynomial solutions		Estimated absolute errors for Morgan-Voyce polynomial solutions			
x_i	$\|e_{2,6}(x_i)\|$	$\|e_{2,9}(x_i)\|$	$\|e_{2,6,9}(x_i)\|$	$\|e_{2,9,12}(x_i)\|$	$\|e_{2,6,12}(x_i)\|$	$\|e_{2,9,15}(x_i)\|$
0.0	9.0937e-18	1.0165e-13	7.3468e-40	1.4694e-39	3.2806e-40	2.2041e-39
0.2	2.1651e-08	1.8350e-11	2.1655e-08	1.8157e-11	2.1651e-08	1.8157e-11
0.4	9.0288e-08	1.3697e-10	9.0307e-08	1.3640e-10	9.0288e-08	1.3640e-10
0.6	1.5082e-07	4.6763e-10	1.5086e-07	4.6659e-10	1.5082e-07	4.6659e-10
0.8	1.3471e-06	1.1724e-09	1.3470e-06	1.1708e-09	1.3471e-06	1.1708e-09
1.0	2.2947e-05	5.0040e-09	2.2944e-05	5.0015e-09	2.2947e-05.	5.0016e-09

Table 3. Numerical results of the corrected error functions of $y_1(x_i)$ of Equation (35)

x_i	$\|E_{1,6,9}(x_i)\|$	$\|E_{1,6,12}(x_i)\|$	$\|E_{1,9,12}(x_i)\|$	$\|E_{1,9,15}(x_i)\|$
0.0	7.1734e-17	7.1734e-17	2.5849e-13	2.5849e-13
0.2	1.6581e-12	1.4319e-16	2.6713e-13	2.6718e-13
0.4	8.4701e-12	3.9159e-16	2.6106e-13	2.6135e-13
0.6	1.9584e-11	8.3624e-16	2.4192e-13	2.4262e-13
0.8	3.7650e-11	1.5149e-15	2.1405e-13	2.1542e-13
1.0	1.1300e-09	3.9365e-14	2.2807e-13	1.8852e-13

Table 4. Numerical results of the corrected error functions of $y_2(x_i)$ of Equation (35)

x_i	$\|E_{2,6,9}(x_i)\|$	$\|E_{2,6,12}(x_i)\|$	$\|E_{2,9,12}(x_i)\|$	$\|E_{2,9,15}(x_i)\|$
0.0	9.0937e-18	9.0937e-18	1.0165e-13	1.0165e-13
0.2	3.3935e-12	9.0210e-17	1.9282e-13	1.9294e-13
0.4	1.8602e-11	5.4901e-16	5.6857e-13	5.6917e-13
0.6	4.8260e-11	1.4033e-15	1.0416e-12	1.0431e-12
0.8	1.0590e-10	2.7071e-15	1.6337e-12	1.6366e-12
1.0	2.6006e-09	9.4434e-14	2.4733e-12	2.3793e-12

Table 5. Numerical results of the absolute error functions $e_{1,10}(x)$ of $y_1(x_i)$ of Equation (36)

	MHPM [8]	Bessel Collocation Method [22]	Present method		
x_i	$\|e_{1,10}(x_i)\|$	$\|e_{1,10}(x_i)\|$	$\|e_{1,10}(x_i)\|$	$\|E_{1,10,13}(x_i)\|$	$\|E_{1,10,15}(x_i)\|$
0.1	8.7765e-07	9.5559e-09	9.4270e-09	5.8922e-009	1.5115e-09
0.4	3.5106e-06	3.8223e-08	4.7241e-08	8.1562e-010	3.4479e-09
0.7	6.1436e-06	6.6891e-08	1.0233e-08	1.4400e-010	5.3863e-09
1.0	8.7765e-06	9.5559e-08	8.8797e-08	3.0570e-009	7.8197e-09

Table 6. Numerical results of the absolute error functions $e_{2,10}(x)$ of $y_2(x_i)$ of Equation (36)

	MHPM [8]	Bessel Collocation Method [22]	Present method		
x_i	$\|e_{2,10}(x_i)\|$	$\|e_{2,10}(x_i)\|$	$\|e_{2,10}(x_i)\|$	$\|E_{2,10,13}(x_i)\|$	$\|E_{2,10,15}(x_i)\|$
0.1	6.3461e-06	1.7995e-08	1.8764e-08	2.1278e-09	1.4651e-09
0.4	2.8625e-05	8.1196e-08	9.4569e-08	5.4951e-09	8.4035e-09
0.7	5.5765e-05	1.5822e-07	1.6867e-07	1.1447e-08	1.6892e-08
1.0	8.7765e-05	2.4907e-07	2.7485e-07	1.8293e-08	2.6850e-08