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The Essentials of Clifford Algebras with Maple Programming

Mutlu AKAR*[1]

Abstract

Clifford algebra (geometric algebra) which has many applications in physics, robotics, Computer-Aided Manufacture, computer graphics, image processing, Computer-Aided Design etc. is one of the important subjects in mathematics. In this paper, after we give the definition of Clifford algebras, introduce their subspaces. Firstly, we develop an algorithm which obtains some concepts of Clifford algebras using Maple programming. Secondly, another algorithm calculates the norm of the multivector obtained by finding the Clifford product of any two vectors of the same finite dimension.

Keywords: Clifford algebras, Clifford product, multivector, Maple

1. INTRODUCTION

Quaternions, a generalization of complex numbers, were discovered by William Hamilton in 1843. After Hamilton's discovery of quaternions, J. T. Graves invented octonion algebra in 1843. However, Graves did not publish the article. Octonions were rediscovered by Cayley in 1845 and octonions are also known as Cayley numbers. Clifford algebras were first found in 1878 by the British mathematician W. K. Clifford and published in his article titled "Application of Grasmann's Extensive Algebra" in the journal "American Journal of Mathematics Pure and Applied". Clifford's geometric algebras; Grasmann defined it as a generalization of algebras, complex numbers and quaternions. W. K. Clifford used hyperbolic numbers to represent the sum of spins in 1882. Corrado Segre dealt with algebras called bicomplex numbers in his work with these algebras in 1892 [1].

Clifford algebra is increasingly used in almost every field. In recent years, Clifford algebras have applications in many fields such as robot vision, neural computing, computer vision, image and signal processing, control problems, electromagnetism, physics [2].

Clifford support vector machines were characterized as a speculation of real and complex valued support vector machines using Clifford algebras. By including the Clifford product in the core function in nonlinear support vector machines, more precise results have been obtained in classifications by improving multiple data input and data output. Thus, satellite control,

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neural computing, pattern recognition, etc. areas have been contributed [3].

In this study, after giving the basic concepts in Clifford algebras, for a number n inputting with the help of Maple programming, we give how to make the Clifford product in $C\ell_{n,0}$, which prints the basis of $C\ell_{n,0}$ (or $C\ell_{0,n}$) Clifford algebras by grouping their dimensions, the bases of their subspaces and the dimensions of their subspaces.

2. CLIFFORD ALGEBRAS

In this section, some basic concepts of Clifford algebras are given.

Definition 1. Let us consider bilinear form for $x, y \in \mathbb{R}^n$ and $r+s=n$

$$\begin{aligned}\langle x, y\rangle_{r,s} &= x_1 y_1 + \cdots + x_r y_r \\ &\quad - x_{r+1} y_{r+1} - \cdots - x_{r+s} y_{r+s}\,.\end{aligned} \tag{1}$$

If $\mathbb{R}^{r,s} = \mathbb{R}^n$ has the following quadratic form for $x \in \mathbb{R}^n$,

$$\langle x, x\rangle_{r,s} = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2 \tag{2}$$

$\left(\mathbb{R}^{r,s}, \langle . \rangle_{r,s}\right)$ is called a real quadratic space.

$\langle x, y\rangle_{n,0}$ an dot product for $x, y \in \mathbb{R}^{n,0}$ and $\langle x, x\rangle_{n,0} = |x|^2$. That is, Euclidean dot product is a special case of quadratic form [4].

Definition 2: For $r+s=n$

$$e_i^2 = 1,\ i = 1,\ldots,r,\ \ e_i^2 = -1,\ i = r+1,\ldots,n \tag{3}$$

and for

$$e_i e_j = -e_j e_i, \quad i, j = 1,\ldots,n, \quad i \neq j\ , \tag{4}$$

The algebra produced by the standard orthonormal basis of the $\{e_1, e_2, \ldots, e_n\}$ in $\mathbb{R}^n$ is called the real Clifford algebra $C\ell_{r,s}$ [4].

Example 1: Some known sets of numbers are actually examples of Clifford algebra. The algebra $C\ell_{0,1}$ is the usual algebra of complex numbers $\mathbb{C}$. The algebra $C\ell_{0,2}$ is the algebra of quaternions $\mathbb{H}$. Whereas the algebra $C\ell_{1,0}$ is often called the split complex numbers (or hyperbolic numbers).

An element $\zeta \in C\ell_{0,1}$ has the form $x_0 + x_1 e_1$ where $x_0, x_1 \in \mathbb{R}$ and $e_1^2 = -1$. We identify ζ with the point (x_0, x_1) in $\mathbb{R}^2$. In this way we view $\mathbb{C}$ as the plane $\mathbb{R}^2$ with an additional algebraic structure. Its conjugate by definition is $\overline{\zeta} = x_0 - x_1 e_1$. Notice that $\zeta\overline{\zeta} = x_0^2 + x_1^2 = \langle x, x\rangle_{2,0}$. Therefore, the complex numbers are associated with the quadratic space $\mathbb{R}^{2,0}$.

Similarly, an element $\zeta \in C\ell_{1,0}$ has the form $x_0 + x_1 e_1$ where $x_0, x_1 \in \mathbb{R}$ and $e_1^2 = 1$. The conjugate again by definition is $x_0 - x_1 e_1$. In this case $\zeta\overline{\zeta} = x_0^2 - x_1^2 = \langle x, x\rangle_{1,1}$. So the split complex numbers are associated with the quadratic space $\mathbb{R}^{1,1}$.

A quaternion $\zeta \in \mathbb{H}$ has the form $\zeta = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$. The conjugate of ζ is now $\overline{\zeta} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3$ so that $\zeta\overline{\zeta} = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Hence the quaternions are associated with the quadratic space $\mathbb{R}^{4,0}$ [4].

The Clifford algebra $C\ell_{r,s}$ is a $\sum_{p=0}^{n} \binom{n}{p} = 2^n$ dimensional vector space with a basis

$$\{1, e_1, e_2, \dots, e_n, e_1e_2, \dots, e_{n-1}e_n, \dots, e_1e_2\dots e_n\}. \qquad (5)$$

The vector space $C\ell_{r,s}$ can be written as the direct sum of subspaces

$$C\ell_{r,s} = \Lambda^0\mathbb{R}^n \oplus \Lambda^1\mathbb{R}^n \oplus \cdots \oplus \Lambda^n\mathbb{R}^n. \qquad (6)$$

The dimension of each subspace is $\binom{n}{p}$ for $p = 0, \dots, n$. The element $\mathcal{A}$ of Clifford algebra $C\ell_{r,s}$ is called a multivector. The element of subspaces $\Lambda^p\mathbb{R}^n$, $(p = 0, \dots, n)$ is a $p-$vector. For instance, the subspace $\Lambda^0\mathbb{R}^n = \mathbb{R}$ of dimension is 1 and its element is $0-$vector, that is a real number. The subspace $\Lambda^1\mathbb{R}^n = \mathbb{R}^n$, has the basis $\{e_1, e_2, \dots, e_n\}$, its element is $1-$vector, that is a vector and the dimension of subspace $\Lambda^1\mathbb{R}^n$ is n. $\{e_1e_2, e_1e_3, \dots, e_{n-1}e_n\}$ is a basis of subspace $\Lambda^2\mathbb{R}^n$ whose element is bivector ($2-$vector). $\{e_1e_2\dots e_n\}$ is a basis of subspace $\Lambda^n\mathbb{R}^n$ whose element is called pseudoscalar ($n-$vector). The multivector $\mathcal{A}$, which is the element of the vector space $C\ell_{r,s}$ given as direct sums of subspaces $\Lambda^p\mathbb{R}^p$, $(0 \le p \le n)$ given by Eq. (6) equation, can be written as in [5,6]

$$\mathcal{A} = \langle\mathcal{A}\rangle_0 + \langle\mathcal{A}\rangle_1 + \cdots + \langle\mathcal{A}\rangle_n. \qquad (7)$$

In Eq. (7) $\langle\mathcal{A}\rangle_p$ is the $p-$vector of the multivector $\mathcal{A}$ and denotes the projection of multivector $\mathcal{A} \in C\ell_{r,s}$ onto the subspace $\Lambda^p\mathbb{R}^p$. The notation $\langle\ \rangle$ represents the elements of each subspace with respect to the subscript [6,7].

Definition 3: $\mathbf{fg}$ is called the Clifford product or geometric product of two vectors $\mathbf{f}$ and $\mathbf{g}$ if

$$\mathbf{fg} = \mathbf{f}\bullet\mathbf{g} + \mathbf{f}\wedge\mathbf{g} \qquad (8)$$

where $\mathbf{f}\bullet\mathbf{g}$ is the inner or dot product while $\mathbf{f}\wedge\mathbf{g}$ is the wedge or outer (or exterior) product. Hence [3,5]

$$\mathbf{f}\bullet\mathbf{g} = \frac{1}{2}(\mathbf{fg} + \mathbf{gf}) \qquad (9)$$

$$\mathbf{f}\wedge\mathbf{g} = \frac{1}{2}(\mathbf{fg} - \mathbf{gf}). \qquad (10)$$

$\mathbf{f}\wedge\mathbf{g}$ is a bivector. The generalized form $\mathbf{f}\wedge\mathbf{g}\wedge\mathbf{h}$ of the wedge product is the trivector ($3-$vector). The most generalized form is pseudoscalar ($n-$vector). Clifford algebras are non-commutative, but is associative and distributive over addition [3].

For $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathbb{R}^n$,
$\mathbf{g} = (g_1, g_2, \dots, g_n) \in \mathbb{R}^n$ [5,8]

$$\begin{aligned}\mathbf{fg} &= (f_1e_1 + f_2e_2 + \cdots + f_ne_n)(g_1e_1 + g_2e_2 + \cdots + g_ne_n) \\ &= \sum_{i=1}^{n} f_ig_i + \sum_{j=1}^{n}\sum_{i=j+1}^{n}(f_jg_i - f_ig_j)e_je_i \\ &= \mathbf{f}\bullet\mathbf{g} + \mathbf{f}\wedge\mathbf{g}.\end{aligned} \qquad (11)$$

For example, we get the following equation for $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$, $\mathbf{g} = (g_1, g_2, g_3) \in \mathbb{R}^3$

$$\begin{aligned}\mathbf{f}\wedge\mathbf{g} = \mathbf{f}\times\mathbf{g} &= \begin{vmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} \\ &= (f_2g_3 - f_3g_2, f_3g_1 - f_1g_3, f_1g_2 - f_2g_1) \in \mathbb{R}^3.\end{aligned}$$

That is, in $\mathbb{R}^3$ wedge product is vector product.

Definition 4. Positive definite norm of a multivector $\mathcal{A}$ is defined as below [3,4]:

$$\|\mathcal{A}\|^2 = \mathcal{A}.\mathcal{A} \qquad (12)$$

where $\|\mathcal{A}\| = 0$ if and only if $\mathcal{A} = \mathbf{0}$.

We will use the notation $e_ie_je_ke_t = e_{ijkt}$ in the rest of our work [4].

2.1. Clifford Algebra $C\ell_2$

The Clifford algebra $C\ell_{2,0} = C\ell_2$ generated by the orthonormal basis $\{e_1, e_2\}$ of the real vector space $\mathbb{R}^2$ is a $2^{2+0} = 4$ dimensional space. For every $u \in C\ell_2$ since $\{1, e_1, e_2, e_{12}\}$ is the basis of $C\ell_2$

$$u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12}$$

can be written where u_0 is a scalar, $u_1 e_1$, $u_2 e_2$ are vectors, and $u_{12} e_{12}$ is a bivector.

For every $\mathbf{f}, \mathbf{g} \in \mathbb{R}^2$,

$$\mathbf{f} = f_1 e_1 + f_2 e_2, \qquad \mathbf{g} = g_1 e_1 + g_2 e_2$$

can be written, where $\{e_1, e_2\}$ of $\mathbb{R}^2$ is the standard orthonormal basis.

The Clifford products of the vectors $\mathbf{f}$ and $\mathbf{g}$ are as follows:

$$\begin{aligned} \mathbf{fg} &= (f_1 e_1 + f_2 e_2)(g_1 e_1 + g_2 e_2) \\ &= f_1 g_1 e_1 e_1 + f_1 g_2 e_1 e_2 + f_2 g_1 e_2 e_1 + f_2 g_2 e_2 e_2 . \end{aligned} \tag{13}$$

Since $e_1 e_1 = e_2 e_2 = 1$ and $e_2 e_1 = -e_1 e_2$, Eq. (13) is obtained as the sum of dot product and wedge product as below [9]:

$$\mathbf{fg} = (f_1 g_1 + f_2 g_2) + (f_1 g_2 - f_2 g_1) e_1 e_2 \tag{14}$$

where the wedge product $\mathbf{f} \wedge \mathbf{g}$ is a bivector and it means the area of the parallelogram built on the vectors $\mathbf{f}$ and $\mathbf{g}$. The value of this area is $|\mathbf{f} \wedge \mathbf{g}| = |f_1 g_2 - f_2 g_1|$ [8].

2.2. Clifford Algebra $C\ell_3$

The Clifford algebra $C\ell_{3,0} = C\ell_3$ generated by the orthonormal basis $\{e_1, e_2, e_3\}$ of the real vector space $\mathbb{R}^3$ is a $2^{3+0} = 8$ dimensional space.

For every $u \in C\ell_3$ since $\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$ is the basis of $C\ell_3$

$$\begin{aligned} u &= u_0 + u_1 e_1 + u_2 e_2 + u_3 e_3 + u_{12} e_{12} \\ &\quad + u_{13} e_{13} + u_{23} e_{23} + u_{123} e_{123} \end{aligned}$$

can be written where u_0 is a scalar, $u_1 e_1$, $u_2 e_2$, $u_3 e_3$ are vectors, $u_{12} e_{12}$, $u_{13} e_{13}$, $u_{23} e_{23}$ are bivectors, and $u_{123} e_{123}$ is a trivector.

For every $\mathbf{f}, \mathbf{g} \in \mathbb{R}^3$,

$$\mathbf{f} = f_1 e_1 + f_2 e_2 + f_3 e_3, \qquad \mathbf{g} = g_1 e_1 + g_2 e_2 + g_3 e_3$$

can be written, where $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is the standard orthonormal basis.

The Clifford products of the vectors $\mathbf{f}$ and $\mathbf{g}$ are as follows:

$$\begin{aligned} \mathbf{fg} &= (f_1 e_1 + f_2 e_2 + f_3 e_3)(g_1 e_1 + g_2 e_2 + g_3 e_3) \\ &= f_1 g_1 e_1 e_1 + f_1 g_2 e_1 e_2 + f_1 g_3 e_1 e_3 + f_2 g_1 e_2 e_1 \\ &\quad + f_2 g_2 e_2 e_2 + f_2 g_3 e_2 e_3 + f_3 g_1 e_3 e_1 + f_3 g_2 e_3 e_2 \\ &\quad + f_3 g_3 e_3 e_3 . \end{aligned} \tag{15}$$

Since $e_1 e_1 = e_2 e_2 = e_3 e_3 = 1$, $e_2 e_1 = -e_1 e_2$, $e_3 e_1 = -e_1 e_3$, $e_3 e_2 = -e_2 e_3$, Eq. (15) is obtained as the sum of dot product and wedge product as below [9]:

$$\begin{aligned} \mathbf{fg} &= (f_1 g_1 + f_2 g_2 + f_3 g_3) + (f_1 g_2 - f_2 g_1) e_1 e_2 \\ &\quad + (f_1 g_3 - f_3 g_1) e_1 e_3 + (f_2 g_3 - f_3 g_2) e_2 e_3 . \end{aligned} \tag{16}$$

The terms $e_1 e_2$, $e_1 e_3$, and $e_2 e_3$ in Eq. (16) are bivectors and are interpreted as the oriented area element lying on the planes defined by the vectors (e_1, e_2), (e_1, e_3), and (e_2, e_3), respectively. $e_1 e_2 e_3$ represents the directed volume element in $\mathbb{R}^3$ and is the trivector [9].

Since $\mathbf{f} \wedge \mathbf{g} = -\mathbf{g} \wedge \mathbf{f}$ for every $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ we get $\mathbf{gf} = \mathbf{f} \bullet \mathbf{g} - \mathbf{f} \wedge \mathbf{g}$. Hence [4]

$$\mathbf{fg} = \mathbf{gf} \Leftrightarrow \mathbf{f} \parallel \mathbf{g} \Leftrightarrow \mathbf{f} \wedge \mathbf{g} = \mathbf{0} \Leftrightarrow \mathbf{fg} = \mathbf{f} \bullet \mathbf{g},$$

$$\mathbf{fg} = -\mathbf{gf} \Leftrightarrow \mathbf{f} \perp \mathbf{g} \Leftrightarrow \mathbf{f} \bullet \mathbf{g} = 0 \Leftrightarrow \mathbf{fg} = \mathbf{f} \wedge \mathbf{g}.$$

Let's summarize the Clifford algebras in the following tables:

Table 1. Clifford algebras and their subspaces and dimensions

Clifford Algebras	Subspaces	Dimensions
$C\ell_0$	$\mathbb{R}$	$2^0 = 1$
$C\ell_1$	$\Lambda^0\mathbb{R}^1 \oplus \Lambda^1\mathbb{R}^1$	$2^1 = 2$
$C\ell_2$	$\Lambda^0\mathbb{R}^2 \oplus \Lambda^1\mathbb{R}^2 \oplus \Lambda^2\mathbb{R}^2$	$2^2 = 4$
$C\ell_3$	$\Lambda^0\mathbb{R}^3 \oplus \Lambda^1\mathbb{R}^3 \oplus \Lambda^2\mathbb{R}^3 \oplus \Lambda^3\mathbb{R}^3$	$2^3 = 8$
$C\ell_4$	$\Lambda^0\mathbb{R}^4 \oplus \Lambda^1\mathbb{R}^4 \oplus \Lambda^2\mathbb{R}^4 \oplus \Lambda^3\mathbb{R}^4 \oplus \Lambda^4\mathbb{R}^4$	$2^4 = 16$
$C\ell_5$	$\Lambda^0\mathbb{R}^5 \oplus \Lambda^1\mathbb{R}^5 \oplus \Lambda^2\mathbb{R}^5 \oplus \Lambda^3\mathbb{R}^5 \oplus \Lambda^4\mathbb{R}^5 \oplus \Lambda^5\mathbb{R}^5$	$2^5 = 32$
$\vdots$	$\vdots$	$\vdots$
$C\ell_n$	$\Lambda^0\mathbb{R}^n \oplus \Lambda^1\mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n \oplus \Lambda^3\mathbb{R}^n \oplus \Lambda^4\mathbb{R}^n \cdots \oplus \Lambda^n\mathbb{R}^n$	2^n

Table 2. The bases of Clifford algebras and the dimensions of their subspaces

Clifford Algebras	Bases	The numbers of elements of Bases (Pascal Triangle)
$C\ell_0$	$\{1\}$	1
$C\ell_1$	$\{1, e_1\}$	1 1
$C\ell_2$	$\{1, e_1, e_2, e_{12}\}$	1 2 1
$C\ell_3$	$\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$	1 3 3 1
$C\ell_4$	$\{1,$ $e_1, e_2, e_3, e_4,$ $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34},$ $e_{123}, e_{124}, e_{134}, e_{234},$ $e_{1234}\}$	1 4 6 4 1

$C\ell_5$	$\{1,$ $e_1,e_2,e_3,e_4,e_5,$ $e_{12},e_{13},e_{14},e_{15},e_{23},e_{24},e_{25},e_{34},e_{35},e_{45},$ $e_{123},e_{124},e_{125},e_{134},e_{135},e_{145},e_{234},e_{235},e_{245},e_{345},$ $e_{1234},e_{1235},e_{1245},e_{1345},e_{2345},$ $e_{12345}\}$	1 5 10 10 5 1
$\vdots$	$\vdots$	$\vdots$
$C\ell_n$	$\{1,$ $e_1,e_2,\ldots,e_n,$ $e_1e_2,e_1e_3,\ldots,e_{n-1}e_n,\ldots,e_1e_2\ldots e_n\}$	$\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{2}\cdots\binom{n}{n}$

3. MAPLE APPLICATIONS

In this section, how to calculate some basic concepts of Clifford algebras with the help of Maple programming is given. Then, a program that calculates the norm of the multivector obtained by finding the Clifford product of any two vectors of the same finite dimension in $\mathbb{R}^n$ $\left(n\in\mathbb{Z}^+\right)$ is mentioned.

3.1. Clifford Algebras in Maple

In Maple programming, an algorithm has been developed that prints $C\ell_{n,0}$ (or $C\ell_{0,n}$) the basis and size of Clifford algebras by grouping the bases of the subspaces (multivector types) and subspaces according to the pascal triangle for any number n $(0\leq n\leq 12)$ inputted.

```
> restart:
> c:=proc(n) local b,d;
> b[1]:=seq(e[i1],i1=1..n):
>
b[2]:=seq(seq(e[i1]*e[i2],i2=i1+1..n),i1=1..n):
>
b[3]:=seq(seq(seq(e[i1]*e[i2]*e[i3],i3=i2+1..n),i2=i1+1..n),i1=1..n):
>
b[4]:=seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4],i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
>
b[5]:=seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5],i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
>
b[6]:=seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6],i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
>
b[7]:=seq(seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6]*e[i7],i7=i6+1..n),i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
>
b[8]:=seq(seq(seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6]*e[i7]*e[i8],i8=i7+1..n),i7=i6+1..n),i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
>
b[9]:=seq(seq(seq(seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6]*e[i7]*e[i8]*e[i9],i9=i8+1..n),i8=i7+1..n),i7=i6+1..n),i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
```

```
> b[10]:=seq(seq(seq(seq(seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6]*e[i7]*e[i8]*e[i9]*e[i10],i10=i9+1..n),i9=i8+1..n),i8=i7+1..n),i7=i6+1..n),i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
> b[11]:=seq(seq(seq(seq(seq(seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6]*e[i7]*e[i8]*e[i9]*e[i10]*e[i11],i11=i10+1..n),i10=i9+1..n),i9=i8+1..n),i8=i7+1..n),i7=i6+1..n),i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
> b[12]:=seq(seq(seq(seq(seq(seq(seq(seq(seq(seq(seq(seq(e[i1]*e[i2]*e[i3]*e[i4]*e[i5]*e[i6]*e[i7]*e[i8]*e[i9]*e[i10]*e[i11]*e[i12],i12=i11+1..n),i11=i10+1..n),i10=i9+1..n),i9=i8+1..n),i8=i7+1..n),i7=i6+1..n),i6=i5+1..n),i5=i4+1..n),i4=i3+1..n),i3=i2+1..n),i2=i1+1..n),i1=1..n):
> print(The_basis_of_Clifford_algebra,[1,b[1],b[2],b[3],b[4],b[5],b[6],b[7],b[8],b[9],b[10],b[11],b[12]]);
> print(The_dimension_of_Clifford_algebra,2**n);
> print(This_Clifford_algebra_has,n+1,subspaces_and_their_dimension_respectively,seq(binomial(n,d),d=0..n));
> for d from 1 to n do
>print(d-vectors_and_their_numbers,b[d],binomial(n,d));
> od;
> end:
```

Let's get the information we mentioned above for Clifford algebra $C\ell_{4,0}$ by inputting 4:

```
> c(4);
```

$$The\_basis\_of\_Clifford\_algebra, [1, e_1, e_2, e_3, e_4, e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3, e_2 e_4, e_3 e_4, e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_3 e_4, e_2 e_3 e_4, e_1 e_2 e_3 e_4]$$

$$The\_dimension\_of\_Clifford\_algebra, 16$$

$$This\_Clifford\_algebra\_has, 5, subspaces\_and\_their\_dimension\_respectively, 1, 4, 6, 4, 1$$

$$1 - vectors\_and\_their\_numbers, e_1, e_2, e_3, e_4, 4$$

$$2 - vectors\_and\_their\_numbers, e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3, e_2 e_4, e_3 e_4, 6$$

$$3 - vectors\_and\_their\_numbers, e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_3 e_4, e_2 e_3 e_4, 4$$

$$4 - vectors\_and\_their\_numbers, e_1 e_2 e_3 e_4, 1$$

3.2. Clifford Product and Norm in Maple

In this subsection, we develop the following algorithm that finds the Clifford product in $C\ell_{n,0}$

of any two vectors of the same finite dimension inputted and the norm of the multivector we have obtained.

```
> restart:
> P:=proc(x,y) local P,l,N,A,R;
> if nops(x)=nops(y) then
> l:=sum(x[i]*y[i],i=1..nops(x)):
>
N:=l+sum(sum((x[j]*y[i]-x[i]*y[j])*e[j]*e[i],i=j+1..nops(x)),j=1..nops(x)):
>
A:=seq(seq((x[j]*y[i]-x[i]*y[j]),i=j+1..nops(x)),j=1..nops(x)):
>
R:=(sqrt(l**2+sum(A[k]**2,k=1..binomial(nops(x),2))));
> else print("The dimensions of the vectors inputted must be the same.")
> fi;
> end:
```

Let's calculate the norm of the resulting multivector by finding the Clifford product of the vectors $\mathbf{x}=(-1,1,2,4,-0.4)\in\mathbb{R}^5$ and $\mathbf{y}=(1,-3,-0.7,5,2)\in\mathbb{R}^5$ in $C\ell_{5,0}$:

```
> x:=[-1,1,2,4,-0.4]:
> y:=[1,-3,-0.7,5,2]:
>
> P(x,y);
> print(N);print([l,A]);
```

$$13.8+2\,e_1\,e_2-1.3\,e_1\,e_3-9\,e_1\,e_4-1.6\,e_1\,e_5+5.3\,e_2\,e_3+17\,e_2\,e_4+0.8\,e_2\,e_5+12.80000000\,e_3\,e_4+3.720000000\,e_3\,e_5+10.0\,e_4\,e_5$$

$$[13.8, 2, -1.3, -9, -1.6, 5.3, 17, 0.8, 12.8, 3.72, 10.0]$$

$$29.58206213$$

4. CONCLUSION

Clifford algebras, a topic that is getting popular day by day, are more involved especially in technological developments. It is often difficult to operate with multidimensional vectors. For this reason, these algorithms written in Maple will contribute to faster progress by facilitating the operations in the applications of Clifford algebras such as computer graphics, physics, robotics, computer-aided manufacturing, image processing, machine learning, computer-aided design, etc.

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