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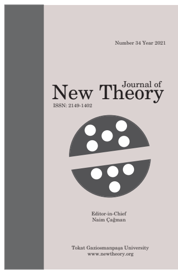

Rough Approximations of Complex Quadripartitioned Single Valued Neutrosophic Sets

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Abstract — The quadripartitioned single valued neutrosophic set consisting of the real-valued amplitude terms: truth-membership grade, contradiction-membership grade, ignorance-membership grade and falsity-membership grade, cannot handle complex-valued information. In this paper, the ranges of grades of truth-membership, contradiction-membership, ignorance-membership and falsity-membership are extended from the interval [0,1] to unit circle in the complex plane, and thus the notion of complex quadripartitioned single valued neutrosophic set is proposed. Further, some fundamental operations and relations on the complex quadripartitioned single valued neutrosophic sets are studied. Secondly, the rough approximations of complex quadripartitioned single valued neutrosophic sets are derived, and then their related remarkable properties are discussed. Finally, a formulation is proposed to measure rough degree of complex quadripartitioned single valued neutrosophic sets in the approximate space.

Keywords — *Single valued neutrosophic set, quadripartitioned single valued neutrosophic set, complex quadripartitioned single valued neutrosophic set, rough approximations, rough degree*

Mathematics Subject Classification (2020) — 03E72, 94D05

1. Introduction

To tackle real world issues, the techniques generally employed in classical mathematics are not always beneficial due to uncertainties and ambiguities. In 1965, Zadeh [1] proposed the fuzzy set (FS) as an effective mathematical tool to deal with such issues. In the following years, Atanassov [2] created an intuitionistic fuzzy set (IFS) that offers both the truth-membership degree and the falsity-membership degree of an object into the set. Many authors established several fuzzy models in the different aspects, i.e., relations, aggregation operators, matrix representations [3–11]. Smarandache [12] developed the neutrosophic logic sprouted from branch of philosophy *neutrosophy* which means the study of neutralities, and then initiated the theory of neutrosophic sets (NSs), a generalization of the IFSs, in which each element is characterized by a truth-membership function, indeterminate-membership function and the falsity-membership function, each of which belongs to the the non-standard unit interval $]0^-, 1^+[$. In 2010, Wang et al. [13] said that the NS is difficult to truly apply to practical problems in real world scenarios, and therefore enlivened the idea of single valued neutrosophic set (SVNS), in which each element is characterized by a truth-membership function, indeterminate-membership function and the falsity-membership function, each of which belongs to the the unit interval $[0, 1]$. For more details, refer to [14–16]. Many authors studied the generalized types of NSs and SVNSs such as interval-valued [17–19], bipolar-valued [20–22], cubic [23–26], and their practical applications [27–29].

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In 2017, Ali and Smarandache [30] introduced the framework of complex neutrosophic set (CNS) characterized by complex-valued truth-membership function, complex-valued indeterminate-membership function and complex-valued falsity-membership function. They put forward that the CNS is the mainstream over all because it is not only the extension of all the current frameworks but also represents the information in a complete and comprehensive way. Al-Quran and Alkhazaleh [31] studied the relations between the (single valued) CNSs with their applications in decision making.

In 1982, Pawlak [32] developed the notion of rough set which expresses vagueness in the concepts of the lower and upper approximations of a set and it employs the boundary region of a set. In [33,34], the authors established the models of rough FSs and rough IFSs. In 2014, Broumi et al. [35] introduced a hybrid structure of rough NSs and discussed its basic operations in the approximation space. In [36], the multi-attribute decision making method based on the rough accuracy score function with rough neutrosophic attribute values was constructed. Samuel and Narmadhagnanam [37] studied the tangent logarithmic distance measure and cosecant similarity measure between rough NSs. In 2018, Abdel-Basset and Mohamed [38] proposed the combination of SVNS and rough set will deal with all aspects of vagueness, incompleteness and inconsistency of data and information. Nowadays, many authors have concerned with the rough approximations of NSs and SVNSs in crisp and neutrosophic spaces, in which both constructive and axiomatic approaches are employed.

By splitting the indeterminacy in the structure of SVNSs into two parts as Unknown (or ignorance) and Contradiction, Chatterjee et al. [39] proposed the notion of quadripartitioned single valued neutrosophic set (QSVNS) based on Belnap's [40] four-valued logic. Mohan and Krishnaswamy [41] presented the axiomatic characterizations of the combined structure of QSVNS and rough set. In [42, 43], the researchers discussed the bipolarity hybridizations of QSVNSs, and some basic set-theoretic terminologies of the emerging QSVNSs. Currently, QSVNS theory has become a very successful and flourishing area of research in different aspects of both theory and practice.

In this study, we introduce the complex quadripartitioned single valued neutrosophic sets (CQSVNSs) by extending the QSVNSs, whose complex-valued truth-membership function, complex-valued contradiction-membership function, complex-valued ignorance-membership function and complex-valued falsity-membership function are the combination of real-valued truth amplitude term in association phase term, real-valued contradiction amplitude term in association phase term, real-valued ignorance amplitude term in association phase term and real-valued falsity amplitude term in association phase term, respectively. Moreover, their set-theoretic operations such as intersection, union, complement, cartesian product, algebraic products are derived. We develop the rough approximations of CQSVNSs and discuss their axiomatic characterizations. Further, we investigate the approximate precision degree and the rough degree in the novel model.

The structure of the paper is organized as follows. In Section 2, some concepts required in our work are briefly recalled. Section 3 is devoted to the construction, operations and relations of CQSVNSs. Section 4 introduces the model of rough CQSVNS in the approximation space. In Section 5, the level cut sets of lower and upper approximations and the rough degree of CQSVNS in the approximation space are studied. Section 6 gives brief conclusion and future research directions.

2. Preliminaries

In this section, we give some preliminary information that will be useful in the following sections.

Definition 2.1. [13] Let $\mathcal{A}$ be a universe of discourse. A single valued nuetrosophic set (SVNS) N in $\mathcal{A}$ is characterized in the following form

$$N = \{(a, \langle t_N(a), \iota_N(a), f_N(a)\rangle) : a \in \mathcal{A}\} \tag{1}$$

where $t_N, \iota_N, f_N : \mathcal{A} \to [0, 1]$ are termed the functions of truth-membership, indeterminacy-membership and falsity-membership, respectively. Also, $t_N(a)$, $\iota_N(a)$ and $f_N(a)$ denote the grades of truth-membership, indeterminacy-membership and falsity-membership of $a \in \mathcal{A}$ to the set N respectively with the condition $0 \leq t_N(a_) + \iota_N(a) + f_N(a) \leq 3$ for each $a \in \mathcal{A}$.

Definition 2.2. [30] Let $\mathcal{A}$ be a universe of discourse. A complex single valued nuetrosophic set (CSVNS) C in $\mathcal{A}$ is characterized in the following form

$$C = \{(a, \langle t_C(a), \iota_C(a), f_C(a)\rangle) : a \in \mathcal{A}\} \tag{2}$$

where $t_C(a) = \Gamma_C(a).e^{i\gamma_C(a)}$, $\iota_C(a) = \Delta_C(a).e^{i\delta_C(a)}$ and $f_C(a) = \Omega_C(a).e^{i\omega_C(a)}$ (for $i = \sqrt{-1}$) denote the complex truth-membership grade, complex indeterminacy-membership grade and complex falsity-membership grade of $a \in \mathcal{A}$ to the set C, respectively. In addition, the amplitude terms $\Gamma_C(a)$, $\Delta_C(a)$, $\Omega_C(a)$ and the phase terms $\gamma_C(a)$, $\delta_C(a)$, $\omega_C(a)$ satisfy the following conditions: $0 \leq \Gamma_C(a) + \Delta_C(a) + \Omega_C(a) \leq 3$ for $\Gamma_C(a), \Delta_C(a), \Omega_C(a) \in [0,1]$ and $0 \leq \gamma_C(a) + \delta_C(a) + \omega_C(a) \leq 6\pi$ for $\gamma_C(a), \delta_C(a), \omega_C(a) \in [0, 2\pi]$.

Chatterjee et al. [39] split the indeterminacy in structure of SVNS into two parts signifying contradiction and unknown (ignorance), and thereby initiated the theory of quadripartitioned single valued neutrosophic set. The term quadripartitioned means something that is divided into the four characteristic features.

Definition 2.3. [39] Let $\mathcal{A}$ be a universe of discourse. A quadripartitioned single valued nuetrosophic set (QSVNS) Q in $\mathcal{A}$ is an object having the following form

$$Q = \{(a, \langle t_Q(a), c_Q(a), u_Q(a), f_Q(a)\rangle) : a \in \mathcal{A}\} \tag{3}$$

where $t_Q, c_Q, u_Q, f_Q : \mathcal{A} \to [0,1]$ are termed the functions of truth-membership, contradiction-membership, ignorance-membership and falsity-membership, respectively. Also, $t_Q(a)$, $c_Q(a)$, $u_Q(a)$ and $f_Q(a)$ denote the grades of truth-membership, contradiction-membership, ignorance-membership and falsity-membership of $a \in \mathcal{A}$ to the set Q respectively with the condition $0 \leq t_Q(a) + c_Q(a) + u_Q(a) + f_Q(a) \leq 4$ for each $a \in \mathcal{A}$.

Remark 2.4. A QSVNS Q can be decomposed to yield two SVNS, say Q_T and Q_F, where the respective membership functions of both these are described as $t_{Q_T}(a) = t_Q(a) = t_{Q_F}(a)$; $\iota_{Q_T}(a) = c_Q(a)$; $\iota_{Q_F}(a) = u_Q(a)$; $f_{Q_T}(a) = f_Q(a) = f_{Q_F}(a)$ for all $a \in \mathcal{A}$.

In this respect, it needs to be specified that while performing set-theoretic operations on these SVNSs, the behavior of ι_{Q_T} is treated similar to that of t_{Q_T} while the behavior of ι_{Q_F} is modelled in a way similar to that of f_{Q_F}.

Assume $\mathcal{A}$ and $\mathcal{B}$ is any non-empty crisp sets. The subset of cartesian product of $\mathcal{A}$ and $\mathcal{B}$ is called a relation from $\mathcal{A}$ to $\mathcal{B}$. Especially, the subset of cartesian product of $\mathcal{A} \times \mathcal{A}$ is a relation on $\mathcal{A}$. The relation $\Re$ on $\mathcal{A}$ is said to be

1. reflexive when $(a_j, a_j) \in \Re$ for all $a_j \in \mathcal{A}$.
2. symmetric when $(a_j, a_k) \in \Re \Rightarrow (a_k, a_j) \in \Re$ for all $a_j, a_k \in \mathcal{A}$.
3. transitive when $(a_j, a_k) \in \Re$ and $(a_k, a_l) \in \Re \Rightarrow (a_j, a_l) \in \Re$ for all $a_j, a_k, a_l \in \mathcal{A}$.

If $\Re$ is reflexive, symmetric and transitive then it is called an equivalence relation on $\mathcal{A}$.

Definition 2.5. [32] Let $\mathcal{A}$ be any non-empty crisp set and $\Re$ an equivalence relation on $\mathcal{A}$. Then, $(\mathcal{A}, \Re)$ is said to be (Pawlak) approximation space. If B is a subset of $\mathcal{A}$, then the sets

$$\underline{appr}_{\Re}(B) = \{b : [b]_{\Re} \subseteq B\} \tag{4}$$

and

$$\overline{appr}_{\Re}(B) = \{b : [b]_{\Re} \cap B \neq \emptyset\} \tag{5}$$

are called the lower and upper approximations of B , respectively, where $[b]_{\Re}$ stands for the equivalence class of $\Re$ containing the object $b \in B \subseteq \mathcal{A}$. The pair $appr_{\Re}(B) = (\underline{appr}_{\Re}(B), \overline{appr}_{\Re}(B))$ is said to be rough set of B in the (Pawlak) approximation space $(\mathcal{A}, \Re)$. Especially, if $\underline{appr}_{\Re}(B) = \overline{appr}_{\Re}(B)$ then B is called a definable. The positive region, negative region and boundary region of B are defined as $\mathfrak{P}_{\Re}(B) = \underline{appr}_{\Re}(B)$, $\mathfrak{N}_{\Re}(B) = \mathcal{A} - \overline{appr}_{\Re}(B)$ and $\mathfrak{B}_{\Re}(B) = \overline{appr}_{\Re}(B) - \underline{appr}_{\Re}(B)$, respectively.

3. Complex Quadripartitioned Single Valued Neutrosophic Set Theory

In this section, we initiate the theory of complex quadripartitioned single valued neutrosophic set and discuss some basic complex quadripartitioned single valued neutrosophic operations and relations.

3.1. Construction of Complex Quadripartitioned Single Valued Neutrosophic Set

It can be observed that QSVNSs are insufficient to describe the complex information based on the four-valued logic. To eliminate this drawback, the framework of complex quadripartitioned single valued neutrosophic set is constructed as follows.

Definition 3.1. Let $\mathcal{A}$ be a universe of discourse. A complex quadripartitioned single valued neutrosophic set (CQSVNS) $\mathfrak{C}$ in $\mathcal{A}$ is characterized by a truth-membership function $t_{\mathfrak{C}}$, a contradiction-membership function $c_{\mathfrak{C}}$, an ignorance-membership function $u_{\mathfrak{C}}$ and a falsity-membership function $f_{\mathfrak{C}}$ that assign an element $a \in \mathcal{A}$ a complex-valued degree of $t_{\mathfrak{C}}(a)$, $c_{\mathfrak{C}}(a)$, $u_{\mathfrak{C}}(a)$ and $f_{\mathfrak{C}}(a)$ in $\mathfrak{C}$. The values $t_{\mathfrak{C}}(a)$, $c_{\mathfrak{C}}(a)$, $u_{\mathfrak{C}}(a)$, $f_{\mathfrak{C}}(a)$ and their sum may all within the unit circle in the complex plane, and are of the form: $t_{\mathfrak{C}}(a) = \Gamma_{\mathfrak{C}}(a).e^{i\gamma_{\mathfrak{C}}}(a)$, $c_{\mathfrak{C}}(a) = \Lambda_{\mathfrak{C}}(a).e^{i\lambda_{\mathfrak{C}}}(a)$, $u_{\mathfrak{C}}(a) = \Psi_{\mathfrak{C}}(a).e^{i\psi_{\mathfrak{C}}}(a)$ and $f_{\mathfrak{C}}(a) = \Omega_{\mathfrak{C}}(a).e^{i\omega_{\mathfrak{C}}}(a)$ (where $i = \sqrt{-1}$). In addition, the amplitude terms $\Gamma_{\mathfrak{C}}(a)$, $\Lambda_{\mathfrak{C}}(a)$, $\Psi_{\mathfrak{C}}(a)$, $\Omega_{\mathfrak{C}}(a)$ and the phase terms $\gamma_{\mathfrak{C}}(a)$, $\lambda_{\mathfrak{C}}(a)$, $\psi_{\mathfrak{C}}(a)$, $\omega_{\mathfrak{C}}(a)$ satisfy the following conditions:

$$0 \leq \Gamma_{\mathfrak{C}}(a) + \Lambda_{\mathfrak{C}}(a) + \Psi_{\mathfrak{C}}(a) + \Omega_{\mathfrak{C}}(a) \leq 4 \quad \text{for} \quad \Gamma_{\mathfrak{C}}(a), \Lambda_{\mathfrak{C}}(a), \Psi_{\mathfrak{C}}(a), \Omega_{\mathfrak{C}}(a) \in [0,1] \tag{6}$$

and

$$0 \leq \gamma_{\mathfrak{C}}(a) + \lambda_{\mathfrak{C}}(a) + \psi_{\mathfrak{C}}(a) + \omega_{\mathfrak{C}}(a) \leq 8\pi \quad \text{for} \quad \gamma_{\mathfrak{C}}(a), \lambda_{\mathfrak{C}}(a), \psi_{\mathfrak{C}}(a), \omega_{\mathfrak{C}}(a) \in [0,2\pi] \tag{7}$$

Simply, a CQSVNS can be given in the following form:

$$\begin{aligned} \mathfrak{C} &= \{(a, \langle t_{\mathfrak{C}}(a), c_{\mathfrak{C}}(a), u_{\mathfrak{C}}(a), f_{\mathfrak{C}}(a)\rangle) : a \in \mathcal{A}\} \\ &= \{(a, \langle \Gamma_{\mathfrak{C}}(a).e^{i\gamma_{\mathfrak{C}}}(a), \Lambda_{\mathfrak{C}}(a).e^{i\lambda_{\mathfrak{C}}}(a), \Psi_{\mathfrak{C}}(a).e^{i\psi_{\mathfrak{C}}}(a), \Omega_{\mathfrak{C}}(a).e^{i\omega_{\mathfrak{C}}}(a)\rangle) : a \in \mathcal{A}\} \end{aligned} \tag{8}$$

The complex membership value $\langle \Gamma_{\mathfrak{C}}(a).e^{i\gamma_{\mathfrak{C}}}(a), \Lambda_{\mathfrak{C}}(a).e^{i\lambda_{\mathfrak{C}}}(a), \Psi_{\mathfrak{C}}(a).e^{i\psi_{\mathfrak{C}}}(a), \Omega_{\mathfrak{C}}(a).e^{i\omega_{\mathfrak{C}}}(a)\rangle$ for a of $\mathfrak{C}$ is simply denoted $((\Gamma_{\mathfrak{C}}, \gamma_{\mathfrak{C}}), (\Lambda_{\mathfrak{C}}, \lambda_{\mathfrak{C}}), (\Psi_{\mathfrak{C}}, \psi_{\mathfrak{C}}), (\Omega_{\mathfrak{C}}, \omega_{\mathfrak{C}}))$ and named as complex quadripartitioned single valued neutrosophic number (CQSVNN).

Example 3.2. Bronchitis is an inflammation of the lining of the bronchial tubes that carry air to the lungs. The bronchitis can be acute or chronic. The symptoms of acute bronchitis are usually a mild headache, cough and production of mucus. The sets of symptoms of acute bronchitis is $\mathcal{A} = \{a_1 \ (\textit{a mild headache}),\ a_2 \ (\textit{cough}),\ a_3 \ (\textit{production of mucus})\}$. While these symptoms usually improve in about a week, they may take a few weeks. By using the data of many patients who survived the disease, a doctor (expert) can create the following CQSVNS in $\mathcal{A}$ depends on the membership "recovery time of symptoms".

$$\mathfrak{C} = \left\{ \begin{array}{l} (a_1, \langle 0.4e^{i2\pi(1)}, 0.7e^{i2\pi(\frac{3}{4})}, 0.6e^{i2\pi(\frac{2}{3})}, 0.6e^{i2\pi(\frac{3}{5})}\rangle) \\ (a_2, \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.7e^{i2\pi(\frac{2}{3})}, 0e^{i2\pi(0)}, 0.6e^{i2\pi(\frac{1}{3})}\rangle), \\ (a_3, \langle 0.4e^{i2\pi(\frac{1}{3})}, 0.6e^{i2\pi(\frac{1}{4})}, 0.2e^{i2\pi(\frac{2}{5})}, 1e^{i2\pi(0)}\rangle) \end{array} \right\}$$

Definition 3.3. Let $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$. For $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$ and $\beta_1, \beta_2, \beta_3, \beta_4 \in [0, 2\pi]$, the $((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4))$-level cut set of $\mathfrak{C}$, denoted by $\mathfrak{C}^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$, is defined as follows:

$$\mathfrak{C}^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \left\{ a \in \mathcal{A} : \left(\begin{array}{l} \Gamma_{\mathfrak{C}}(a) \geq \alpha_1, \ \Lambda_{\mathfrak{C}}(a) \geq \alpha_2, \ \Psi_{\mathfrak{C}}(a) \leq \alpha_3, \ \Omega_{\mathfrak{C}}(a) \leq \alpha_4, \\ \gamma_{\mathfrak{C}}(a) \geq \beta_1, \ \lambda_{\mathfrak{C}}(a) \geq \beta_2, \ \psi_{\mathfrak{C}}(a) \leq \beta_3, \ \omega_{\mathfrak{C}}(a) \leq \beta_4 \end{array} \right) \right\} \tag{9}$$

Example 3.4. Consider the CQSVNS $\mathfrak{C}$ in Example 3.2. Then, $((0.3, \frac{\pi}{5}), (0.5, \frac{\pi}{2}), (0.7, \frac{4\pi}{3}), (1, \pi))$-level cut set of $\mathfrak{C}$ is $\mathfrak{C}^{(\frac{\pi}{5},\frac{\pi}{2},\frac{4\pi}{3},\pi)}_{(0.3,0.5,0.7,1)} = \{a_3\}$.

Proposition 3.5. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$. If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ then $(\mathfrak{C}_1)^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} \subseteq (\mathfrak{C}_2)^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$.

PROOF. It can be proved easily according to the Definition 3.3, therefore omitted. □

Definition 3.6. A CQSVNS $\mathfrak{C}$ in $\mathcal{A}$ is said to be a null CQSVNS, denoted by Φ, if its complex membership degrees are respectively $t_\Phi(a) = \Gamma_\Phi(a).e^{i\gamma_\Phi}(a) = 0$, $c_\Phi(a) = \Lambda_\Phi(a).e^{i\lambda_\Phi}(a) = 0$, $u_\Phi(a) = \Psi_\Phi(a).e^{i\psi_\Phi}(a) = e^{i2\pi}$ and $f_\Phi(a) = \Omega_\Phi(a).e^{i\omega_\Phi}(a) = e^{i2\pi}$ for all $a \in \mathcal{A}$.

Definition 3.7. A CQSVNS $\mathfrak{C}$ in $\mathcal{A}$ is said to be a absolute CQSVNS, denoted by $\widehat{\mathcal{A}}$, if its complex membership degrees are respectively $t_{\widehat{\mathcal{A}}}(a) = \Gamma_{\widehat{\mathcal{A}}}(a).e^{i\gamma_{\widehat{\mathcal{A}}}}(a) = e^{i2\pi}$, $c_{\widehat{\mathcal{A}}}(a) = \Lambda_{\widehat{\mathcal{A}}}(a).e^{i\lambda_{\widehat{\mathcal{A}}}}(a) = e^{i2\pi}$, $u_{\widehat{\mathcal{A}}}(a) = \Psi_{\widehat{\mathcal{A}}}(a).e^{i\psi_{\widehat{\mathcal{A}}}}(a) = 0$ and $f_{\widehat{\mathcal{A}}}(a) = \Omega_{\widehat{\mathcal{A}}}(a).e^{i\omega_{\widehat{\mathcal{A}}}}(a) = 0$ for all $a \in \mathcal{A}$.

3.2. Operations of Complex Quadripartitioned Single Valued Neutrosophic Set

In this part, we study the set-theoretic operations on the CQSVNSs and the properties related to them.

Definition 3.8. Let $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be three CQSVNSs in $\mathcal{A}$. Then,

(a) $\mathfrak{C}_1$ is said to be a CQSVN subset of $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$, if the following conditions are satisfied:

$$\begin{pmatrix} t_{\mathfrak{C}_1}(a) \leq t_{\mathfrak{C}_2}(a), \text{ i.e., } \Gamma_{\mathfrak{C}_1}(a) \leq \Gamma_{\mathfrak{C}_2}(a) \text{ and } \gamma_{\mathfrak{C}_1}(a) \leq \gamma_{\mathfrak{C}_2}(a) \\ c_{\mathfrak{C}_1}(a) \leq c_{\mathfrak{C}_2}(a), \text{ i.e., } \Lambda_{\mathfrak{C}_1}(a) \leq \Lambda_{\mathfrak{C}_2}(a) \text{ and } \lambda_{\mathfrak{C}_1}(a) \leq \lambda_{\mathfrak{C}_2}(a) \\ u_{\mathfrak{C}_1}(a) \geq u_{\mathfrak{C}_2}(a), \text{ i.e., } \Psi_{\mathfrak{C}_1}(a) \geq \Psi_{\mathfrak{C}_2}(a) \text{ and } \psi_{\mathfrak{C}_1}(a) \geq \psi_{\mathfrak{C}_2}(a) \\ f_{\mathfrak{C}_1}(a) \geq f_{\mathfrak{C}_2}(a), \text{ i.e., } \Omega_{\mathfrak{C}_1}(a) \geq \Omega_{\mathfrak{C}_2}(a) \text{ and } \omega_{\mathfrak{C}_1}(a) \geq \omega_{\mathfrak{C}_2}(a) \end{pmatrix} \tag{10}$$

(b) $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are said to be a CQSVN equal, denoted by $\mathfrak{C}_1 = \mathfrak{C}_2$, if the following conditions are satisfied:

$$\begin{pmatrix} t_{\mathfrak{C}_1}(a) = t_{\mathfrak{C}_2}(a), \text{ i.e., } \Gamma_{\mathfrak{C}_1}(a) = \Gamma_{\mathfrak{C}_2}(a) \text{ and } \gamma_{\mathfrak{C}_1}(a) = \gamma_{\mathfrak{C}_2}(a) \\ c_{\mathfrak{C}_1}(a) = c_{\mathfrak{C}_2}(a), \text{ i.e., } \Lambda_{\mathfrak{C}_1}(a) = \Lambda_{\mathfrak{C}_2}(a) \text{ and } \lambda_{\mathfrak{C}_1}(a) = \lambda_{\mathfrak{C}_2}(a) \\ u_{\mathfrak{C}_1}(a) = u_{\mathfrak{C}_2}(a), \text{ i.e., } \Psi_{\mathfrak{C}_1}(a) = \Psi_{\mathfrak{C}_2}(a) \text{ and } \psi_{\mathfrak{C}_1}(a) = \psi_{\mathfrak{C}_2}(a) \\ f_{\mathfrak{C}_1}(a) = f_{\mathfrak{C}_2}(a), \text{ i.e., } \Omega_{\mathfrak{C}_1}(a) = \Omega_{\mathfrak{C}_2}(a) \text{ and } \omega_{\mathfrak{C}_1}(a) = \omega_{\mathfrak{C}_2}(a) \end{pmatrix} \tag{11}$$

(c) the complement of $\mathfrak{C}$, denoted by $\sim \mathfrak{C}$, is defined as

$$\sim \mathfrak{C} = \{(a, \langle t_{\sim\mathfrak{C}}(a), c_{\sim\mathfrak{C}}(a), u_{\sim\mathfrak{C}}(a), f_{\sim\mathfrak{C}}(a)\rangle) : a \in \mathcal{A}\}, \tag{12}$$

where $t_{\sim\mathfrak{C}}(a) = f_{\mathfrak{C}}(a)$, $c_{\sim\mathfrak{C}}(a) = u_{\mathfrak{C}}(a)$, $u_{\sim\mathfrak{C}}(a) = c_{\mathfrak{C}}(a)$, and $f_{\sim\mathfrak{C}}(a) = t_{\mathfrak{C}}(a)$ for all $a \in \mathcal{A}$.

(d) the intersection of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \cap \mathfrak{C}_2$, is defined as

$$\begin{aligned} \mathfrak{C}_1 \cap \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a), c_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a), u_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a), f_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \tag{13}$$

where

$$\Gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a) \wedge \Gamma_{\mathfrak{C}_2}(a), \quad \Lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a) \wedge \Lambda_{\mathfrak{C}_2}(a),$$

$$\Psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a) \vee \Psi_{\mathfrak{C}_2}(a), \quad \Omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a) \vee \Omega_{\mathfrak{C}_2}(a),$$

$$\gamma_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a) \wedge \Gamma_{\mathfrak{C}_2}(a), \quad \lambda_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a) \wedge \lambda_{\mathfrak{C}_2}(a),$$

$$\psi_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a) \vee \psi_{\mathfrak{C}_2}(a), \quad \omega_{\mathfrak{C}_1\cap\mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a) \vee \omega_{\mathfrak{C}_2}(a).$$

(e) the union of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \cup \mathfrak{C}_2$, is defined as

$$\begin{aligned}\mathfrak{C}_1 \cup \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a), c_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a), u_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a), f_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (14)$$

where

$$\Gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a) \vee \Gamma_{\mathfrak{C}_2}(a), \quad \Lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a) \vee \Lambda_{\mathfrak{C}_2}(a),$$

$$\Psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a) \wedge \Psi_{\mathfrak{C}_2}(a), \quad \Omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a) \wedge \Omega_{\mathfrak{C}_2}(a),$$

$$\gamma_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a) \vee \Gamma_{\mathfrak{C}_2}(a), \quad \lambda_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a) \vee \lambda_{\mathfrak{C}_2}(a),$$

$$\psi_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a) \wedge \psi_{\mathfrak{C}_2}(a), \quad \omega_{\mathfrak{C}_1\cup\mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a) \wedge \omega_{\mathfrak{C}_2}(a).$$

Example 3.9. Let $\mathcal{A} = \{a_1, a_2\}$ be a universal set. Assume that two CQSVNS are

$$\mathfrak{C}_1 = \{(a_1, \langle 0.5e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(0)}, 0.1e^{i2\pi(1)}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.5e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{1}{10})}\rangle)\}$$

and

$$\mathfrak{C}_2 = \{(a_1, \langle 0.6e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(\frac{5}{7})}, 0.9e^{i2\pi(\frac{9}{10})}, 0.1e^{i2\pi(\frac{5}{6})}\rangle), (a_2, \langle 0.7e^{i2\pi(\frac{2}{3})}, 0.4e^{i2\pi(1)}, 0.2e^{i2\pi(\frac{1}{5})}, 0.8e^{i2\pi(\frac{1}{2})}\rangle)\}$$

The complement of $\mathfrak{C}_1$ is

$$\sim \mathfrak{C}_1 = \{(a_1, \langle 0.1e^{i2\pi(1)}, 1e^{i2\pi(0)}, 0.7e^{i2\pi(\frac{7}{10})}, 0.5e^{i2\pi(\frac{1}{2})}\rangle), (a_2, \langle 0.7e^{i2\pi(\frac{1}{10})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.5e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{1}{2})}\rangle)\}$$

The intersection of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is

$$\mathfrak{C}_1 \cap \mathfrak{C}_2 = \{(a_1, \langle 0.5e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{9}{10})}, 0.1e^{i2\pi(1)}\rangle), (a_2, \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.4e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.8e^{i2\pi(\frac{1}{2})}\rangle)\}$$

The union of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is

$$\mathfrak{C}_1 \cup \mathfrak{C}_2 = \{(a_1, \langle 0.6e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{5}{7})}, 0.9e^{i2\pi(0)}, 0.1e^{i2\pi(\frac{5}{6})}\rangle), (a_2, \langle 0.7e^{i2\pi(\frac{2}{3})}, 0.5e^{i2\pi(1)}, 0.2e^{i2\pi(\frac{1}{5})}, 0.7e^{i2\pi(\frac{1}{10})}\rangle)\}$$

Proposition 3.10. For three CQSVNSs $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathcal{A}$, $\sim \mathfrak{C}$, $\mathfrak{C}_1 \cap \mathfrak{C}_2$ and $\mathfrak{C}_1 \cup \mathfrak{C}_2$ are also CQSVNSs in $\mathcal{A}$.

Proof. By considering the concepts in Definition 3.8, these results can be proved easily. □

Proposition 3.11. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be three CQSVNSs in $\mathcal{A}$. Then, the following are hold.

(i) $\mathfrak{C}_1 * \mathfrak{C}_2$ and $\mathfrak{C}_2 * \mathfrak{C}_3 \Rightarrow \mathfrak{C}_1 * \mathfrak{C}_3$ for each $* \in \{\subseteq, =\}$

(ii) $\mathfrak{C}_1 \Diamond \mathfrak{C}_2 = \mathfrak{C}_2 \Diamond \mathfrak{C}_1$ for each $\Diamond \in \{\cap, \cup\}$

(iii) $\mathfrak{C}_1 \Diamond (\mathfrak{C}_2 \Diamond \mathfrak{C}_3) = (\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \Diamond \mathfrak{C}_3$ for each $\Diamond \in \{\cap, \cup\}$

(iv) $\mathfrak{C}_1 \Diamond (\mathfrak{C}_2 \Box \mathfrak{C}_3) = (\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \Box (\mathfrak{C}_1 \Diamond \mathfrak{C}_3)$ for each $\Diamond, \Box \in \{\cap, \cup\}$

(v) $(\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \Box \mathfrak{C}_3 = (\mathfrak{C}_1 \Box \mathfrak{C}_3) \Diamond (\mathfrak{C}_2 \Box \mathfrak{C}_3)$ for each $\Diamond, \Box \in \{\cap, \cup\}$

(vi) $\sim (\mathfrak{C}_1 \Diamond \mathfrak{C}_2) = \sim \mathfrak{C}_1 \Box \sim \mathfrak{C}_2$ for each $\Diamond, \Box \in \{\cap, \cup\}$ and $\Diamond \neq \Box$

Proof. We will prove (vi), others can be demonstrated by similar techniques.
(vi): Assume that $\Diamond = \cap$ and $\Box = \cup$. According to the operations of complement and intersection in Definition 3.8, we can write

$$\sim (\mathfrak{C}_1 \cap \mathfrak{C}_2) = \{(a, \langle f_{\mathfrak{C}_1}(a) \vee f_{\mathfrak{C}_2}(a), u_{\mathfrak{C}_1}(a) \vee u_{\mathfrak{C}_2}(a), c_{\mathfrak{C}_1}(a) \wedge c_{\mathfrak{C}_2}(a), t_{\mathfrak{C}_1}(a) \wedge t_{\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\} \quad (15)$$

Likewise, we obtain for $d = 1, 2$,

$$\sim \mathfrak{C}_d = \{(a, \langle t_{\sim\mathfrak{C}_d}(a), c_{\sim\mathfrak{C}_d}(a), u_{\sim\mathfrak{C}_d}(a), f_{\sim\mathfrak{C}_d}(a)\rangle) : a \in \mathcal{A}\} = \{(a, \langle f_{\mathfrak{C}_d}(a), u_{\mathfrak{C}_d}(a), c_{\mathfrak{C}_d}(a), t_{\mathfrak{C}_d}(a)\rangle) : a \in \mathcal{A}\}$$

and so

$$\sim \mathfrak{C}_1 \cup \sim \mathfrak{C}_2 = \{(a, \langle f_{\mathfrak{C}_1}(a) \vee f_{\mathfrak{C}_2}(a), u_{\mathfrak{C}_1}(a) \vee u_{\mathfrak{C}_2}(a), c_{\mathfrak{C}_1}(a) \wedge c_{\mathfrak{C}_2}(a), t_{\mathfrak{C}_1}(a) \wedge t_{\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\} \quad (16)$$

From Eqs. (15) and (16), we have $\sim (\mathfrak{C}_1 \cap \mathfrak{C}_2) = \sim \mathfrak{C}_1 \cup \sim \mathfrak{C}_2$. It is shown in a similar way that $\sim (\mathfrak{C}_1 \cup \mathfrak{C}_2) = \sim \mathfrak{C}_1 \cap \sim \mathfrak{C}_2$. □

Definition 3.12. Let $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be three CQSVNSs in $\mathcal{A}$ and $n > 0$ be a real number. Then, the following operational laws are hold.

(a)

$$\begin{aligned} \mathfrak{C}_1 \oplus \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a), c_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a), u_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a), f_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\} \end{aligned} \quad (17)$$

where $\Gamma_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a) + \Gamma_{\mathfrak{C}_2}(a) - \Gamma_{\mathfrak{C}_1}(a)\Gamma_{\mathfrak{C}_2}(a)$, $\Lambda_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a) + \Lambda_{\mathfrak{C}_2}(a) - \Lambda_{\mathfrak{C}_1}(a)\Lambda_{\mathfrak{C}_2}(a)$, $\Psi_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a)\Psi_{\mathfrak{C}_2}(a)$, $\Omega_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a)\Omega_{\mathfrak{C}_2}(a)$, $\gamma_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a) + \gamma_{\mathfrak{C}_2}(a) - \gamma_{\mathfrak{C}_1}(a)\gamma_{\mathfrak{C}_2}(a)$, $\lambda_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a) + \lambda_{\mathfrak{C}_2}(a) - \lambda_{\mathfrak{C}_1}(a)\lambda_{\mathfrak{C}_2}(a)$, $\psi_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a)\psi_{\mathfrak{C}_2}(a)$, and $\omega_{\mathfrak{C}_1\oplus\mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a)\omega_{\mathfrak{C}_2}(a)$.

(b)

$$\begin{aligned} \mathfrak{C}_1 \otimes \mathfrak{C}_2 &= \{(a, \langle t_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a), c_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a), u_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a), f_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a).e^{i\gamma_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a)}, \Lambda_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a).e^{i\lambda_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a)}, \\ \Psi_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a).e^{i\psi_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a)}, \Omega_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a).e^{i\omega_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (18)$$

where $\Gamma_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \Gamma_{\mathfrak{C}_1}(a)\Gamma_{\mathfrak{C}_2}(a)$, $\Lambda_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \Lambda_{\mathfrak{C}_1}(a)\Lambda_{\mathfrak{C}_2}(a)$, $\Psi_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \Psi_{\mathfrak{C}_1}(a) + \Psi_{\mathfrak{C}_2}(a) - \Psi_{\mathfrak{C}_1}(a)\Psi_{\mathfrak{C}_2}(a)$, $\Omega_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \Omega_{\mathfrak{C}_1}(a) + \Omega_{\mathfrak{C}_2}(a) - \Omega_{\mathfrak{C}_1}(a)\Omega_{\mathfrak{C}_2}(a)$, $\gamma_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \gamma_{\mathfrak{C}_1}(a)\gamma_{\mathfrak{C}_2}(a)$, $\lambda_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \lambda_{\mathfrak{C}_1}(a)\lambda_{\mathfrak{C}_2}(a)$, $\psi_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \psi_{\mathfrak{C}_1}(a) + \psi_{\mathfrak{C}_2}(a) - \psi_{\mathfrak{C}_1}(a)\psi_{\mathfrak{C}_2}(a)$, and $\omega_{\mathfrak{C}_1\otimes\mathfrak{C}_2}(a) = \omega_{\mathfrak{C}_1}(a) + \omega_{\mathfrak{C}_2}(a) - \omega_{\mathfrak{C}_1}(a)\omega_{\mathfrak{C}_2}(a)$.

(c)

$$\begin{aligned} n\mathfrak{C} &= \{(a, \langle t_{n\mathfrak{C}}(a), c_{n\mathfrak{C}}(a), u_{n\mathfrak{C}}(a), f_{n\mathfrak{C}}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{n\mathfrak{C}}(a).e^{i\gamma_{n\mathfrak{C}}(a)}, \Lambda_{n\mathfrak{C}}(a).e^{i\lambda_{n\mathfrak{C}}(a)}, \\ \Psi_{n\mathfrak{C}}(a).e^{i\psi_{n\mathfrak{C}}(a)}, \Omega_{n\mathfrak{C}}(a).e^{i\omega_{n\mathfrak{C}}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (19)$$

where $\Gamma_{n\mathfrak{C}}(a) = 1 - (1 - \Gamma_{\mathfrak{C}}(a))^n$, $\Lambda_{n\mathfrak{C}}(a) = 1 - (1 - \Lambda_{\mathfrak{C}}(a))^n)$, $\Psi_{n\mathfrak{C}}(a) = (\Psi_{\mathfrak{C}}(a))^n$, $\Omega_{n\mathfrak{C}}(a) = (\Omega_{\mathfrak{C}}(a))^n$, $\gamma_{n\mathfrak{C}}(a) = 1 - (1 - \gamma_{\mathfrak{C}}(a))^n$, $\lambda_{n\mathfrak{C}}(a) = 1 - (1 - \lambda_{\mathfrak{C}}(a))^n$, $\psi_{n\mathfrak{C}}(a) = (\psi_{\mathfrak{C}}(a))^n$, and $\omega_{n\mathfrak{C}}(a) = (\omega_{\mathfrak{C}}(a))^n$.

(d)

$$\begin{aligned} \mathfrak{C}^n &= \{(a, \langle t_{\mathfrak{C}^n}(a), c_{\mathfrak{C}^n}(a), u_{\mathfrak{C}^n}(a), f_{\mathfrak{C}^n}(a)\rangle) : a \in \mathcal{A}\}, \\ &= \left\{ \left(a, \left\langle \begin{array}{c} \Gamma_{\mathfrak{C}^n}(a).e^{i\gamma_{\mathfrak{C}^n}(a)}, \Lambda_{\mathfrak{C}^n}(a).e^{i\lambda_{\mathfrak{C}^n}(a)}, \\ \Psi_{\mathfrak{C}^n}(a).e^{i\psi_{\mathfrak{C}^n}(a)}, \Omega_{\mathfrak{C}^n}(a).e^{i\omega_{\mathfrak{C}^n}(a)} \end{array} \right\rangle \right) : a \in \mathcal{A} \right\}, \end{aligned} \quad (20)$$

where $\Gamma_{\mathfrak{C}^n}(a) = (\Gamma_{\mathfrak{C}}(a))^n$, $\Lambda_{\mathfrak{C}^n}(a) = (\Lambda_{\mathfrak{C}}(a))^n$, $\Psi_{\mathfrak{C}^n}(a) = 1 - (1 - \Psi_{\mathfrak{C}}(a))^n$, $\Omega_{\mathfrak{C}^n}(a) = 1 - (1 - \Omega_{\mathfrak{C}}(a))^n$, $\gamma_{\mathfrak{C}^n}(a) = (\gamma_{\mathfrak{C}}(a))^n$, $\lambda_{\mathfrak{C}^n}(a) = (\lambda_{\mathfrak{C}}(a))^n$, $\psi_{\mathfrak{C}^n}(a) = 1 - (1 - \psi_{\mathfrak{C}}(a))^n$, and $\omega_{\mathfrak{C}^n}(a) = 1 - (1 - \omega_{\mathfrak{C}}(a))^n$.

Example 3.13. Consider the CQSVNSs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in Example 3.9 and $n = 2$. Then we have

$$\mathfrak{C}_1 \oplus \mathfrak{C}_1 = \left\{ \begin{array}{l} (a_1, \langle 0.8e^{i2\pi(\frac{2}{3})}, 0.76e^{i2\pi(\frac{32}{35})}, 0.9e^{i2\pi(0)}, 0.01e^{i2\pi(\frac{5}{6})}\rangle), \\ (a_2, \langle 0.82e^{i2\pi(\frac{5}{6})}, 0.7e^{i2\pi(1)}, 0.08e^{i2\pi(\frac{3}{25})}, 0.56e^{i2\pi(\frac{1}{20})}\rangle) \end{array} \right\},$$

$$\mathfrak{C}_1 \otimes \mathfrak{C}_1 = \left\{ \begin{array}{l} (a_1, \langle 0.3e^{i2\pi(\frac{1}{6})}, 0.14e^{i2\pi(\frac{1}{2})}, 1e^{i2\pi(\frac{9}{10})}, 0.19e^{i2\pi(1)}\rangle), \\ (a_2, \langle 0.28e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(1)}, 0.52e^{i2\pi(\frac{17}{25})}, 0.94e^{i2\pi(\frac{11}{20})}\rangle) \end{array} \right\},$$

$$2\mathfrak{C}_1 = \left\{ \begin{array}{l} (a_1, \langle 0.75e^{i2\pi(\frac{3}{4})}, 0.91e^{i2\pi(\frac{91}{100})}, 1e^{i2\pi(0)}, 0.01e^{i2\pi(1)}\rangle), \\ (a_2, \langle 0.64e^{i2\pi(\frac{3}{4})}, 0.75e^{i2\pi(1)}, 0.16e^{i2\pi(\frac{9}{25})}, 0.49e^{i2\pi(\frac{1}{100})}\rangle) \end{array} \right\},$$

and

$$\mathfrak{C}_1^2 = \left\{ \begin{array}{l} (a_1, \langle 0.25e^{i2\pi(\frac{1}{4})}, 0.49e^{i2\pi(\frac{49}{100})}, 1e^{i2\pi(0)}, 0.19e^{i2\pi(1)}\rangle), \\ (a_2, \langle 0.16e^{i2\pi(\frac{1}{4})}, 0.25e^{i2\pi(1)}, 0.64e^{i2\pi(\frac{21}{25})}, 0.91e^{i2\pi(\frac{19}{100})}\rangle) \end{array} \right\}$$

Proposition 3.14. If $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are three CQSVNSs in $\mathcal{A}$ and $n > 0$ is a real number then $\mathfrak{C}_1 \oplus \mathfrak{C}_2$, $\mathfrak{C}_1 \otimes \mathfrak{C}_2$, $n\mathfrak{C}$ and $\mathfrak{C}^n$ are also CQSVNSs in $\mathcal{A}$.

PROOF. By considering Definition 3.12, these results can be proved easily. □

Proposition 3.15. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$ and $n, m > 0$ be two real numbers. Then,

(i) $\mathfrak{C}_1 \blacklozenge \mathfrak{C}_2 = \mathfrak{C}_2 \blacklozenge \mathfrak{C}_1$ for each $\blacklozenge \in \{\oplus, \otimes\}$

(ii) $n(\mathfrak{C}_1 \oplus \mathfrak{C}_2) = n\mathfrak{C}_1 \oplus n\mathfrak{C}_2$

(iii) $n\mathfrak{C}_1 \oplus m\mathfrak{C}_1 = (n + m)\mathfrak{C}_1$

(iv) $(\mathfrak{C}_1 \oplus \mathfrak{C}_2)^n = \mathfrak{C}_1^n \otimes \mathfrak{C}_2^n$

(v) $\mathfrak{C}_1^n \otimes \mathfrak{C}_1^m = \mathfrak{C}_1^{n+m}$

(vi) $\sim (\mathfrak{C}_1 \blacklozenge \mathfrak{C}_2) = \sim \mathfrak{C}_1 \blacksquare \sim \mathfrak{C}_1$ for each $\blacklozenge, \blacksquare \in \{\oplus, \otimes\}$ and $\blacklozenge \neq \blacksquare$

PROOF. It can be proved similar to calculations in the proof of Proposition 3.11. □

Proposition 3.16. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be three CQSVNSs in $\mathcal{A}$. Then,

(i) $(\mathfrak{C}_1 \lozenge \mathfrak{C}_2) \blacklozenge \mathfrak{C}_3 = (\mathfrak{C}_1 \blacklozenge \mathfrak{C}_3) \lozenge (\mathfrak{C}_2 \blacklozenge \mathfrak{C}_3)$ for each $\lozenge \in \{\cap, \cup\}$ and $\blacklozenge \in \{\oplus, \otimes\}$

(ii) $(\mathfrak{C}_1 \lozenge \mathfrak{C}_2) \blacklozenge (\mathfrak{C}_1 \square \mathfrak{C}_2) = (\mathfrak{C}_1 \blacklozenge \mathfrak{C}_2)$ for each $\blacklozenge \in \{\oplus, \otimes\}$, $\lozenge, \square \in \{\cap, \cup\}$ and $\lozenge \neq \square$

PROOF. From Definitions 3.8 and 3.12, they can be proved easily. □

Definition 3.17. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$. Then, the cartesian product of $\mathfrak{C}_1$ and $\mathfrak{C}_2$, denoted by $\mathfrak{C}_1 \times \mathfrak{C}_2$, is defined as

$$\begin{aligned} \mathfrak{C}_1 \times \mathfrak{C}_2 &= \left\{ \left((a_j, a_k), \left\langle \begin{array}{l} t_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k), c_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k), \\ u_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k), f_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) \end{array} \right\rangle \right) : (a_j, a_k) \in \mathcal{A} \times \mathcal{A} \right\}, \\ &= \left\{ \left((a_j, a_k), \left\langle \begin{array}{l} \Gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)}, \\ \Lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)}, \\ \Psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)}, \\ \Omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k).e^{i\omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)} \end{array} \right\rangle \right) : (a_j, a_k) \in \mathcal{A} \times \mathcal{A} \right\} \end{aligned} \tag{21}$$

where $\Gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Gamma_{\mathfrak{C}_1}(a_j) \wedge \Gamma_{\mathfrak{C}_2}(a_k)$, $\Lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Lambda_{\mathfrak{C}_1}(a_j) \wedge \Lambda_{\mathfrak{C}_2}(a_k)$, $\Psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Psi_{\mathfrak{C}_1}(a_j) \vee \Psi_{\mathfrak{C}_2}(a_k)$, $\Omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \Omega_{\mathfrak{C}_1}(a_j) \vee \Omega_{\mathfrak{C}_2}(a_k)$, $\gamma_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \gamma_{\mathfrak{C}_1}(a_j) \wedge \Gamma_{\mathfrak{C}_2}(a_k)$, $\lambda_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k) = \lambda_{\mathfrak{C}_1}(a_j)) \wedge \lambda_{\mathfrak{C}_2}(a_k)$, $\psi_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)) = \psi_{\mathfrak{C}_1}(a_j) \vee \psi_{\mathfrak{C}_2}(a_k)$, and $\omega_{\mathfrak{C}_1 \times \mathfrak{C}_2}(a_j, a_k)) = \omega_{\mathfrak{C}_1}(a_j) \vee \omega_{\mathfrak{C}_2}(a_k)$.

Example 3.18. Consider the CQSVNSs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in Example 3.9. Then, the cartesian product of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is

$$\mathfrak{C}_1 \times \mathfrak{C}_2 = \left\{ \begin{array}{l} ((a_1,a_1), \langle 0.5e^{i2\pi(\frac{1}{4})}, 0.2e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{9}{10})}, 0.1e^{i2\pi(1)}\rangle), \\ ((a_1,a_2), \langle 0.5e^{i2\pi(\frac{1}{4})}, 0.4e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{1}{3})}, 0.8e^{i2\pi(1)}\rangle), \\ ((a_2,a_1), \langle 0.4e^{i2\pi(\frac{1}{3})}, 0.2e^{i2\pi(\frac{5}{7})}, 0.9e^{i2\pi(\frac{9}{10})}, 0.7e^{i2\pi(\frac{5}{6})}\rangle), \\ ((a_2,a_2), \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.4e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.8e^{i2\pi(\frac{1}{2})}\rangle) \end{array} \right\}$$

Also, the cartesian product of $\mathfrak{C}_1 \times \mathfrak{C}_1$ is

$$\mathfrak{C}_1 \times \mathfrak{C}_1 = \left\{ \begin{array}{l} ((a_1,a_1), \langle 0.5e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(0)}, 0.1e^{i2\pi(1)}\rangle), \\ ((a_1,a_2), \langle 0.4e^{i2\pi(\frac{1}{4})}, 0.5e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(1)}\rangle), \\ ((a_2,a_1), \langle 0.4e^{i2\pi(\frac{1}{4})}, 0.5e^{i2\pi(\frac{7}{10})}, 1e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(1)}\rangle), \\ ((a_2,a_2), \langle 0.4e^{i2\pi(\frac{1}{2})}, 0.5e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{1}{10})}\rangle) \end{array} \right\}$$

Proposition 3.19. For two CQSVNSs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathcal{A}$, $\mathfrak{C}_1 \times \mathfrak{C}_2$ is a CQSVNS in $\mathcal{A} \times \mathcal{A}$.

Proof. By considering Definition 3.17, this result can be demonstrated easily. □

Proposition 3.20. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be three CQSVNSs in $\mathcal{A}$. Then,

- **(i)** $\mathfrak{C}_1 * \mathfrak{C}_2 \Rightarrow (\mathfrak{C}_1 \times \mathfrak{C}_3) * (\mathfrak{C}_2 \times \mathfrak{C}_3)$ for each $* \in \{\subseteq, =\}$
- **(ii)** $\mathfrak{C}_1 \times (\mathfrak{C}_2 \times \mathfrak{C}_3) = (\mathfrak{C}_1 \times \mathfrak{C}_2) \times \mathfrak{C}_3$
- **(iii)** $\mathfrak{C}_1 \times (\mathfrak{C}_2 \Diamond \mathfrak{C}_3) = (\mathfrak{C}_1 \times \mathfrak{C}_2) \Diamond (\mathfrak{C}_1 \times \mathfrak{C}_3)$ for each $\Diamond \in \{\cap, \cup\}$
- **(iv)** $(\mathfrak{C}_1 \Diamond \mathfrak{C}_2) \times \mathfrak{C}_3) = (\mathfrak{C}_1 \times \mathfrak{C}_3) \Diamond (\mathfrak{C}_2 \times \mathfrak{C}_3)$ for each $\Diamond \in \{\cap, \cup\}$

Proof. We will prove the assertion (i), the other can be demonstrated in a similar way.

(i): Assume that $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$. By considering Eq. (10), for truth-membership grades, we have $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_2}(a_j)$, i.e. $\Gamma_{\mathfrak{C}_1}(a_j) \leq \Gamma_{\mathfrak{C}_2}(a_j)$ and $\gamma_{\mathfrak{C}_1}(a_j) \leq \gamma_{\mathfrak{C}_2}(a_j)$. There are three cases.

Case 1: If $t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_2}(a_j)$ then $t_{\mathfrak{C}_3}(a_k) \wedge t_{\mathfrak{C}_1}(a_j) = t_{\mathfrak{C}_3}(a_k)$ and $t_{\mathfrak{C}_3}(a_j) \wedge t_{\mathfrak{C}_2}(a_k) = t_{\mathfrak{C}_3}(a_k)$. It follows $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$.

Case 2: If $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j)$ then $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_1}(a_j)$ and $t_{\mathfrak{C}_3}(a_j) \wedge t_{\mathfrak{C}_2}(a_k) = t_{\mathfrak{C}_3}(a_k)$. Since $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_3}(a_k)$, it is obtained that $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$.

Case 3: If $t_{\mathfrak{C}_1}(a_j) \leq t_{\mathfrak{C}_2}(a_j) \leq t_{\mathfrak{C}_3}(a_k)$ then $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_1}(a_j)$ and $t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) = t_{\mathfrak{C}_2}(a_j)$. It follows $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$.

As a result of these three cases, $t_{\mathfrak{C}_1}(a_j) \wedge t_{\mathfrak{C}_3}(a_k) \leq t_{\mathfrak{C}_2}(a_j) \wedge t_{\mathfrak{C}_3}(a_k)$ for every $a_j, a_k \in \mathcal{A}$. By making similar calculations, it can be shown that $c_{\mathfrak{C}_1}(a_j) \wedge c_{\mathfrak{C}_3}(a_k) \leq c_{\mathfrak{C}_2}(a_j) \wedge c_{\mathfrak{C}_3}(a_k)$, $u_{\mathfrak{C}_1}(a_j) \vee u_{\mathfrak{C}_3}(a_k) \geq u_{\mathfrak{C}_2}(a_j) \vee u_{\mathfrak{C}_3}(a_k)$ and $f_{\mathfrak{C}_1}(a_j) \vee f_{\mathfrak{C}_3}(a_k) \geq f_{\mathfrak{C}_2}(a_j) \vee f_{\mathfrak{C}_3}(a_k)$ for every $a_j, a_k \in \mathcal{A}$. So we have $\mathfrak{C}_1 \times \mathfrak{C}_3 \subseteq \mathfrak{C}_2 \times \mathfrak{C}_3$ if $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$. This is obvious for situation of equality.

□

3.3. Relations on Complex Quadripartitioned Single Valued Neutrosophic Set

In this part, we discuss the complex quadripartitioned single valued neutrosophic relation and equivalence complex quadripartitioned single valued neutrosophic relation with desired properties.

Definition 3.21. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$. Then, a complex quadripartitioned single valued neutrosophic relation (CQSVN relation) from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ is a (non-null) CQSVN subset of $\mathfrak{C}_1 \times \mathfrak{C}_2$. Thus, a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ is denoted by $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$, where $\Im(\mathfrak{C}_1, \mathfrak{C}_2) \subseteq \mathfrak{C}_1 \times \mathfrak{C}_2$. $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$ can be represented as the set

$$\begin{aligned} \Im(\mathfrak{C}_1, \mathfrak{C}_2) &= \left\{ \left((a_j, a_k), \left\langle \begin{array}{c} t_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k), c_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k), \\ u_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k), f_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \end{array} \right\rangle \right) : (a_j, a_k) \in \mathcal{A} \times \mathcal{A} \right\}, \\ &= \left\{ \left((a_j, a_k), \left\langle \begin{array}{c} \Gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k).e^{i\gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)}, \\ \Lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k).e^{i\lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)}, \\ \Psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k).e^{i\psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)}, \\ \Omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k).e^{i\omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)} \end{array} \right\rangle \right) : (a_j, a_k) \in \mathcal{A} \times \mathcal{A} \right\} \end{aligned} \quad (22)$$

Especially, a CQSVN subset of $\mathfrak{C}_1 \times \mathfrak{C}_1$ is called a CQSVN relation on $\mathfrak{C}_1$ and denoted by $\Im(\mathfrak{C}_1)$.

Example 3.22. We consider $\mathfrak{C}_1 \times \mathfrak{C}_2$ given in Example 3.18. If

$$\Im(\mathfrak{C}_1, \mathfrak{C}_2) = \left\{ \begin{array}{l} ((a_1, a_1), \langle 0.3e^{i2\pi(\frac{2}{9})}, 0.2e^{i2\pi(\frac{1}{2})}, 1e^{i2\pi(1)}, 0.9e^{i2\pi(1)} \rangle), \\ ((a_1, a_2), \langle 0.2e^{i2\pi(\frac{1}{4})}, 0.1e^{i2\pi(\frac{1}{4})}, 1e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(1)} \rangle), \\ ((a_2, a_1), \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.2e^{i2\pi(\frac{5}{9})}, 0.9e^{i2\pi(1)}, 0.8e^{i2\pi(\frac{5}{6})} \rangle), \\ ((a_2, a_2), \langle 0.3e^{i2\pi(\frac{4}{9})}, 0.1e^{i2\pi(\frac{2}{5})}, 0.7e^{i2\pi(\frac{4}{5})}, 0.9e^{i2\pi(\frac{2}{3})} \rangle) \end{array} \right\},$$

then $\Im(\mathfrak{C}_1, \mathfrak{C}_2) \subseteq \mathfrak{C}_1 \times \mathfrak{C}_2$ and so $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$.

Definition 3.23. If $\Im$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ then the inverse $\Im^{-1}$ is a CQSVN relation from $\mathfrak{C}_2$ to $\mathfrak{C}_1$ and is defined as follows:

$$\Im^{-1}(\mathfrak{C}_2, \mathfrak{C}_1) = \left\{ \left((a_k, a_j), \left\langle \begin{array}{c} t_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j), c_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j), \\ u_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j), f_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) \end{array} \right\rangle \right) : (a_k, a_j) \in \mathcal{A} \times \mathcal{A} \right\}, \quad (23)$$

where $t_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) = t_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k)$, $c_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) = c_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k)$, $u_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) = u_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k)$ and $f_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) = f_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k)$.

Example 3.24. We consider the CQSVN relation $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$ from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ in Example 3.22. Then,

$$\Im^{-1}(\mathfrak{C}_2, \mathfrak{C}_1) = \left\{ \begin{array}{l} ((a_1, a_1), \langle 0.3e^{i2\pi(\frac{2}{9})}, 0.2e^{i2\pi(\frac{1}{2})}, 1e^{i2\pi(1)}, 0.9e^{i2\pi(1)} \rangle), \\ ((a_1, a_2), \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.2e^{i2\pi(\frac{5}{9})}, 0.9e^{i2\pi(1)}, 0.8e^{i2\pi(\frac{5}{6})} \rangle), \\ ((a_2, a_1), \langle 0.2e^{i2\pi(\frac{1}{4})}, 0.1e^{i2\pi(\frac{1}{4})}, 1e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(1)} \rangle), \\ ((a_2, a_2), \langle 0.3e^{i2\pi(\frac{4}{9})}, 0.1e^{i2\pi(\frac{2}{5})}, 0.7e^{i2\pi(\frac{4}{5})}, 0.9e^{i2\pi(\frac{2}{3})} \rangle) \end{array} \right\},$$

is the inverse of $\Im(\mathfrak{C}_1, \mathfrak{C}_2)$, and further is a CQSVN relation from $\mathfrak{C}_2$ to $\mathfrak{C}_1$.

Definition 3.25. If $\Im$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ and $\widetilde{\Im}$ is a CQSVN relation from $\mathfrak{C}_2$ to $\mathfrak{C}_3$ then the composition $\Im \circ \widetilde{\Im}$, is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_3$, is defined as follows:

$$(\Im \circ \widetilde{\Im})(\mathfrak{C}_1, \mathfrak{C}_3) = \left\{ \left((a_j, a_l), \left\langle \begin{array}{c} t_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l), c_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l), \\ u_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l), f_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l) \end{array} \right\rangle \right) : (a_j, a_l) \in \mathcal{A} \times \mathcal{A} \right\}, \quad (24)$$

where $t_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l) = \big(\bigvee_{a_k}\{\Gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \wedge \Gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k, a_l)\}\big).e^{i(\bigvee_{a_k}\{\gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \wedge \gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$,

$c_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l) = \big(\bigvee_{a_k}\{\Lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \wedge \Lambda_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k, a_l)\}\big).e^{i(\bigvee_{a_k}\{\lambda_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \wedge \lambda_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$,

$u_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l) = \big(\bigwedge_{a_k}\{\Psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \vee \Psi_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k, a_l)\}\big).e^{i(\bigwedge_{a_k}\{\psi_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \vee \psi_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$,

$f_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l) = \big(\bigwedge_{a_k}\{\Omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \vee \Omega_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k, a_l)\}\big).e^{i(\bigwedge_{a_k}\{\omega_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k) \vee \omega_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})}$.

Proposition 3.26. Let $\Im$ and $\widetilde{\Im}$ be two CQSVN relations from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ and from $\mathfrak{C}_2$ to $\mathfrak{C}_3$, respectively. Then, the following assertions are true.

(i) $(\Im^{-1})^{-1} = \Im$

(ii) $(\Im \circ \widetilde{\Im})^{-1} = \widetilde{\Im}^{-1} \circ \Im^{-1}$

PROOF. **(i):** The proof is straightforward.
(ii): If the composition $\Im \circ \widetilde{\Im}$ is a CQSVN relation from $\mathfrak{C}_1$ to $\mathfrak{C}_3$ then the inverse $(\Im \circ \widetilde{\Im})^{-1}$ is a CQSVN relation from $\mathfrak{C}_3$ to $\mathfrak{C}_1$. By the definitions of inverse and composition of CQSVN relations, we can write

$$\begin{aligned}
&t_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = t_{\Im\circ\widetilde{\Im}(\mathfrak{C}_1,\mathfrak{C}_3)}(a_j, a_l) \\
&= \Big(\bigvee_{a_k}\{\Gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j, a_k) \wedge \Gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k, a_l)\}\Big).e^{i(\bigvee_{a_k}\{\gamma_{\Im(\mathfrak{C}_1,\mathfrak{C}_2)}(a_j,a_k)\wedge\gamma_{\widetilde{\Im}(\mathfrak{C}_2,\mathfrak{C}_3)}(a_k,a_l)\})} \\
&= \Big(\bigvee_{a_k}\{\Gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j) \wedge \Gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l, a_k)\}\Big).e^{i(\bigvee_{a_k}\{\gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j)\wedge\gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l,a_k)\})} \\
&= \Big(\bigvee_{a_k}\{\Gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l, a_k) \wedge \Gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k, a_j)\}\Big).e^{i(\bigvee_{a_k}\{\gamma_{\widetilde{\Im}^{-1}(\mathfrak{C}_3,\mathfrak{C}_2)}(a_l,a_k)\wedge\gamma_{\Im^{-1}(\mathfrak{C}_2,\mathfrak{C}_1)}(a_k,a_j)\})} \\
&= t_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)
\end{aligned} \tag{25}$$

By using the similar techniques, we can demonstrate the equalities:
$c_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = c_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)$, $u_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = u_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)$ and $f_{(\Im\circ\widetilde{\Im})^{-1}(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j) = f_{(\widetilde{\Im}^{-1}\circ\Im^{-1})(\mathfrak{C}_3,\mathfrak{C}_1)}(a_l, a_j)$. So, we have $(\Im \circ \widetilde{\Im})^{-1} = \widetilde{\Im}^{-1} \circ \Im^{-1}$. □

Definition 3.27. A CQSVN relation $\Im$ on $\mathfrak{C}$ is said to be

(a) reflexive if $t_{\Im(\mathfrak{C})}(a_j, a_j) = e^{i2\pi}$, $c_{\Im(\mathfrak{C})}(a_j, a_j) = e^{i2\pi}$, $u_{\Im(\mathfrak{C})}(a_j, a_j) = 0$ and $f_{\Im(\mathfrak{C})}(a_j, a_j) = 0$ for all $a_j \in \mathcal{A}$.

(b) symmetric if $t_{\Im(\mathfrak{C})}(a_j, a_k) = t_{\Im(\mathfrak{C})}(a_k, a_j)$, $c_{\Im(\mathfrak{C})}(a_j, a_k) = c_{\Im(\mathfrak{C})}(a_k, a_j)$, $u_{\Im(\mathfrak{C})}(a_j, a_k) = u_{\Im(\mathfrak{C})}(a_k, a_j)$ and $f_{\Im(\mathfrak{C})}(a_j, a_k) = f_{\Im(\mathfrak{C})}(a_k, a_j)$ for all $a_j, a_k \in \mathcal{A}$.

(c) transitive if $\Im \circ \Im \subseteq \Im$.
(e.g., for amplitude term and phase term of truth-membership, it is characterized as follows: $\Gamma_{\Im(\mathfrak{C})}(a_j, a_l) \geq \bigvee_{a_k}\{\Gamma_{\Im(\mathfrak{C})}(a_j, a_k) \wedge \Gamma_{\Im(\mathfrak{C})}(a_k, a_l)\}$, $\gamma_{\Im(\mathfrak{C})}(a_j, a_l) \geq \bigvee_{a_k}\{\gamma_{\Im(\mathfrak{C})}(a_j, a_k) \wedge \gamma_{\Im(\mathfrak{C})}(a_k, a_l)\}$ for all $a_j, a_k, a_l \in \mathcal{A}$. Likewise, it can be interpreted in accordance with the concept of composition of CQSVN relations for contradiction-membership, ignorance-membership and falsity-membership.)

Example 3.28. For the CQSVNS $\mathfrak{C}_1\times\mathfrak{C}_1$ in Example 3.18, $\Im(\mathfrak{C}_1) = \mathfrak{C}_1\times\mathfrak{C}_1$ is a CQSVN relation on $\mathfrak{C}_1$. $\Im(\mathfrak{C}_1)$ is not reflexive (e.g., $t_{\Im(\mathfrak{C}_1)}(a_j, a_j) \neq e^{i2\pi}$). Since $t_{\Im(\mathfrak{C}_1)}(a_1, a_2) = t_{\Im(\mathfrak{C}_1)}(a_2, a_1)$, $c_{\Im(\mathfrak{C}_1)}(a_1, a_2) = c_{\Im(\mathfrak{C}_1)}(a_2, a_1)$, $u_{\Im(\mathfrak{C}_1)}(a_1, a_2) = u_{\Im(\mathfrak{C}_1)}(a_2, a_1)$ and $f_{\Im(\mathfrak{C}_1)}(a_1, a_2) = f_{\Im(\mathfrak{C}_1)}(a_2, a_1)$, $\Im(\mathfrak{C}_1)$ is symmetric. Since $\Im(\mathfrak{C}_1) \circ \Im(\mathfrak{C}_1) \subseteq \Im(\mathfrak{C}_1)$, $\Im(\mathfrak{C}_1)$ is transitive.

Proposition 3.29. Let $\Im$ be a CQSVN relations on $\mathfrak{C}$. Then,

(i) if $\Im$ is a reflexive CQSVN relation, then $\Im^{-1}$ is also reflexive.

(ii) if $\Im$ is a symmetric CQSVN relation, then $\Im^{-1}$ is also symmetric.

(iii) if $\Im$ is a transitive CQSVN relation, then $\Im^{-1}$ is also transitive.

PROOF. The proofs are straightforward. □

Definition 3.30. A CQSVN relation $\Im$ on $\mathfrak{C}$ is said to be equivalence CQSVN relation if $\Im$ is reflexive, symmetric and transitive.

Proposition 3.31. If $\Im$ is an equivalence CQSVN relation on $\mathfrak{C}$ then $\Im^{-1}$ is also an equivalence CQSVN relation on $\mathfrak{C}$.

PROOF. The proof is obvious from Definition 3.30 and Proposition 3.29. □

4. Rough Sets Combined Complex Quadripartitioned Single Valued Neutrosophic Sets

In this section, we introduce the concept of rough complex quadripartitioned single valued neutrosophic set by combining both rough set and CQSVNS. Also, we investigate the axiomatic characterizations.

Definition 4.1. Let $\mathcal{A}$ be a non-empty set and $\Re$ be a an equivalence relation on $\mathcal{A}$. Assume that $\mathfrak{C}$ is a CQSVNS in $\mathcal{A}$.
The lower approximation of $\mathfrak{C}$ in the approximation space $(\mathcal{A}, \Re)$, denoted by $\underline{appr}_{\Re}(\mathfrak{C})$, is defined as

$$\underline{appr}_{\Re}(\mathfrak{C}) = \left\{ \left(a_j, \left\langle \begin{array}{c} \Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)} \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\}, \tag{26}$$

where, for all $a_j \in \mathcal{A}$,

$$\Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k), \quad \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Lambda_{\mathfrak{C}}(a_k),$$

$$\Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Psi_{\mathfrak{C}}(a_k), \quad \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Omega_{\mathfrak{C}}(a_k),$$

$$\gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \gamma_{\mathfrak{C}}(a_k), \quad \lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \lambda_{\mathfrak{C}}(a_k),$$

$$\psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \psi_{\mathfrak{C}}(a_k), \quad \omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \omega_{\mathfrak{C}}(a_k)$$

The upper approximation of $\mathfrak{C}$ in the approximation space $(\mathcal{A}, \Re)$, denoted by $\overline{appr}_{\Re}(\mathfrak{C})$, is defined as

$$\overline{appr}_{\Re}(\mathfrak{C}) = \left\{ \left(a_j, \left\langle \begin{array}{c} \Gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)}, \\ \Omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).e^{i\omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j)} \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\}, \tag{27}$$

where, for all $a_j \in \mathcal{A}$,

$$\Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k), \quad \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \Lambda_{\mathfrak{C}}(a_k),$$

$$\Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Psi_{\mathfrak{C}}(a_k) \quad \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Omega_{\mathfrak{C}}(a_k),$$

$$\underline{appr}_{\Re}(\mathfrak{C})(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \gamma_{\mathfrak{C}}(a_k), \quad \lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\Re}} \lambda_{\mathfrak{C}}(a_k),$$

$$\psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \psi_{\mathfrak{C}}(a_k), \quad \omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \omega_{\mathfrak{C}}(a_k)$$

It is easy to see that $\underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ are two CQSVNSs in $\mathcal{A}$. $appr_{\mathfrak{R}}(\mathfrak{C}) = (\underline{appr}_{\mathfrak{R}}(\mathfrak{C}), \overline{appr}_{\mathfrak{R}}(\mathfrak{C}))$ is called the rough complex quadripartitioned single valued neutrosophic set (rough CQSVNS) in the approximation space $(\mathcal{A}, \mathfrak{R})$. Furthermore, the positive region, negative region and boundary region of CQSVNS $\mathfrak{C}$ are defined as $\mathfrak{P}_{\mathfrak{R}}(\mathfrak{C}) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$, $\mathfrak{N}_{\mathfrak{R}}(\mathfrak{C}) =\sim \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\mathfrak{B}_{\mathfrak{R}}(\mathfrak{C}) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C}) \cap \sim \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$, respectively. If $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ then the CQSVNS $\mathfrak{C}$ is called a definable CQSVNS in $(\mathcal{A}, \mathfrak{R})$, otherwise $\mathfrak{C}$ is a rough. It can be easily demonstrated that null CQSVNS and absolute CQSVNS are definable.

Example 4.2. Suppose that $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is a universal set and

$$\mathfrak{C} = \left\{ \begin{array}{l} (a_1, \langle 0.5e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{2}{5})}, 1e^{i2\pi(1)}, 1e^{i2\pi(\frac{3}{10})} \rangle), (a_2, \langle 0.4e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{1}{2})}, 0e^{i2\pi(0)} \rangle), \\ (a_3, \langle 0.7e^{i2\pi(\frac{1}{9})}, 0.2e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{2}{7})}, 0.5e^{i2\pi(1)} \rangle), (a_4, \langle 0.1e^{i2\pi(\frac{1}{5})}, 0.4e^{i2\pi(\frac{2}{3})}, 0.1e^{i2\pi(\frac{2}{5})}, 0.1e^{i2\pi(\frac{1}{10})} \rangle), \\ (a_5, \langle 0.2e^{i2\pi(\frac{1}{3})}, 0.5e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{2}{7})}, 0.7e^{i2\pi(\frac{3}{8})} \rangle), (a_6, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.4e^{i2\pi(\frac{3}{5})}, 0.2e^{i2\pi(\frac{1}{4})} \rangle) \end{array} \right\}$$

is a CQSVNS in $\mathcal{A}$. Also, let $\mathfrak{R}$ be an equivalence relation on $\mathcal{A}$ such that the equivalence classes are $[a_1]_{\mathfrak{R}} = \{a_1, a_3\}$, $[a_2]_{\mathfrak{R}} = \{a_2\}$, and $[a_4]_{\mathfrak{R}} = \{a_4, a_4, a_6\}$. Then, the lower and upper approximations of $\mathfrak{C}$, i.e. $\underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C})$, respectively, in the approximation space $(\mathcal{A}, \mathfrak{R})$ are as follows:

$$\left\{ \begin{array}{l} (a_1, \langle 0.5e^{i2\pi(\frac{1}{9})}, 0.2e^{i2\pi(\frac{1}{3})}, 1e^{i2\pi(1)}, 1e^{i2\pi(1)} \rangle), (a_2, \langle 0.4e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{1}{2})}, 0e^{i2\pi(0)} \rangle), \\ (a_3, \langle 0.5e^{i2\pi(\frac{1}{9})}, 0.2e^{i2\pi(\frac{1}{3})}, 1e^{i2\pi(1)}, 1e^{i2\pi(1)} \rangle), (a_4, \langle 0.1e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{3}{8})} \rangle), \\ (a_5, \langle 0.1e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{3}{8})} \rangle), (a_6, \langle 0.1e^{i2\pi(\frac{1}{3})}, 0.4e^{i2\pi(\frac{3}{5})}, 0.6e^{i2\pi(\frac{3}{5})}, 0.7e^{i2\pi(\frac{3}{8})} \rangle) \end{array} \right\},$$

and

$$\left\{ \begin{array}{l} (a_1, \langle 0.7e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{2}{5})}, 0.4e^{i2\pi(\frac{2}{7})}, 0.5e^{i2\pi(\frac{3}{10})} \rangle), (a_2, \langle 0.4e^{i2\pi(\frac{2}{3})}, 0.9e^{i2\pi(\frac{1}{4})}, 0.7e^{i2\pi(\frac{1}{2})}, 0e^{i2\pi(0)} \rangle), \\ (a_3, \langle 0.7e^{i2\pi(\frac{1}{2})}, 0.7e^{i2\pi(\frac{2}{5})}, 0.4e^{i2\pi(\frac{2}{7})}, 0.5e^{i2\pi(\frac{3}{10})} \rangle), (a_4, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.1e^{i2\pi(\frac{2}{7})}, 0.2e^{i2\pi(\frac{1}{10})} \rangle), \\ (a_5, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.1e^{i2\pi(\frac{2}{7})}, 0.2e^{i2\pi(\frac{1}{10})} \rangle), (a_6, \langle 0.2e^{i2\pi(1)}, 0.7e^{i2\pi(1)}, 0.1e^{i2\pi(\frac{2}{7})}, 0.2e^{i2\pi(\frac{1}{10})} \rangle) \end{array} \right\}$$

So, it is a rough CQSVNS.

Proposition 4.3. For the lower and upper approximations of CQSVNSs $\mathfrak{C}$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$, the following properties are hold.

- **(i)** $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}) \subseteq \mathfrak{C} \subseteq \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(ii)** $\mathfrak{C}_1 \subseteq \mathfrak{C}_2 \Rightarrow \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \subseteq \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \subseteq \overline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$
- **(iii)** $\underline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\overline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(iv)** $\underline{appr}_{\mathfrak{R}}(\overline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\underline{appr}_{\mathfrak{R}}(\mathfrak{C})) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(v)** $\underline{appr}_{\mathfrak{R}}(\sim \mathfrak{C}) =\sim \underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\sim \mathfrak{C}) =\sim \overline{appr}_{\mathfrak{R}}(\mathfrak{C})$
- **(vi)** $\underline{appr}_{\mathfrak{R}}(\mathfrak{C}_1 \cap \mathfrak{C}_2) = \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \cap \underline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C}_1 \cup \mathfrak{C}_2) = \overline{appr}_{\mathfrak{R}}(\mathfrak{C}_1) \cup \overline{appr}_{\mathfrak{R}}(\mathfrak{C}_2)$

PROOF.

- **(i):** Let $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$, and $\underline{appr}_{\mathfrak{R}}(\mathfrak{C})$ and $\overline{appr}_{\mathfrak{R}}(\mathfrak{C})$ be lower and upper approximations of $\mathfrak{C}$, respectively. For every $a_j \in \mathcal{A}$, we calculate (by considering Definitions 3.8 (a) and 4.1), for the amplitude term of complex truth-membership,

$$\Gamma_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k) \leq \Gamma_{\mathfrak{C}}(a_j) \leq \bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \Gamma_{\mathfrak{C}}(a_k) = \Gamma_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j)$$

and for the phase term of complex falsity-membership,

$$\omega_{\underline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j) = \bigvee_{a_k \in [a_j]_{\mathfrak{R}}} \omega_{\mathfrak{C}}(a_k) \geq \omega_{\mathfrak{C}}(a_j) \geq \bigwedge_{a_k \in [a_j]_{\mathfrak{R}}} \omega_{\mathfrak{C}}(a_k) = \omega_{\overline{appr}_{\mathfrak{R}}(\mathfrak{C})}(a_j).$$

Proceeding with similar calculations, we obtain that

$$\Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \Lambda_{\mathfrak{C}}(a_j) \leq \Lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j), \quad \Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \Psi_{\mathfrak{C}}(a_j) \geq \Psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j),$$

$$\Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \Omega_{\mathfrak{C}}(a_j) \geq \Omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j), \quad \gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \gamma_{\mathfrak{C}}(a_j) \leq \gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j),$$

$$\lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \lambda_{\mathfrak{C}}(a_j) \leq \lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j), \quad \psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \psi_{\mathfrak{C}}(a_j) \geq \psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j).$$

Therefore, we have $\underline{appr}_{\Re}(\mathfrak{C}) \subseteq \mathfrak{C} \subseteq \overline{appr}_{\Re}(\mathfrak{C})$

(ii): It is obvious from the definitions of lower and upper approximations of CQSVNS.

(iii): According to the definition of lower approximation of CQSVNS, we can write, for ever $a_j \in \mathcal{A}$,

$$\Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k) = \Gamma_{\mathfrak{C}}(a_{k_*})$$

where $a_{k_*} \in [a_j]_{\Re}$. It follows

$$\Gamma_{\underline{appr}_{\Re}(\underline{appr}_{\Re}(\mathfrak{C}))}(a_j) = \bigwedge_{a_k \in [a_j]_{\Re}} \Big(\bigwedge_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k)\Big) = \Gamma_{\mathfrak{C}}(a_{k_*})$$

So, $\Gamma_{\underline{appr}_{\Re}(\underline{appr}_{\Re}(\mathfrak{C}))}(a_j) = \Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j)$ for ever $a_j \in \mathcal{A}$. It can be shown similarly for other amplitude terms and phase terms. These demonstrate that $\underline{appr}_{\Re}(\underline{appr}_{\Re}(\mathfrak{C})) = \underline{appr}_{\Re}(\mathfrak{C})$. The property $\underline{appr}_{\Re}(\underline{appr}_{\Re}(\mathfrak{C})) = \underline{appr}_{\Re}(\mathfrak{C})$ can be proved similarly.

(iv): The proof is similar to the proof of (iii).

(v): According to the Definitions 3.8 (c) and 4.1, we can obtain

$$\begin{aligned}
\underline{appr}_{\Re}(\sim \mathfrak{C}) &= \left\{ \left(a_j, \left\langle \begin{array}{c} \Big(\bigwedge_{a_k \in [a_j]_{\Re}} \Gamma_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\Re}} \gamma_{\sim\mathfrak{C}}(a_k))}, \\ \Big(\bigwedge_{a_k \in [a_j]_{\Re}} \Lambda_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\Re}} \lambda_{\sim\mathfrak{C}}(a_k))}, \\ \Big(\bigvee_{a_k \in [a_j]_{\Re}} \Psi_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\Re}} \psi_{\sim\mathfrak{C}}(a_k))}, \\ \Big(\bigvee_{a_k \in [a_j]_{\Re}} \Omega_{\sim\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\Re}} \omega_{\sim\mathfrak{C}}(a_k))} \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\} \\
&= \left\{ \left(a_j, \left\langle \begin{array}{c} \Big(\bigvee_{a_k \in [a_j]_{\Re}} \Omega_{\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\Re}} \omega_{\mathfrak{C}}(a_k))}, \\ \Big(\bigvee_{a_k \in [a_j]_{\Re}} \Psi_{\mathfrak{C}}(a_k)\Big).e^{i(\bigvee_{a_k \in [a_j]_{\Re}} \psi_{\mathfrak{C}}(a_k))}, \\ \Big(\bigwedge_{a_k \in [a_j]_{\Re}} \Lambda_{\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\Re}} \lambda_{\mathfrak{C}}(a_k))}, \\ \Big(\bigwedge_{a_k \in [a_j]_{\Re}} \Gamma_{\mathfrak{C}}(a_k)\Big).e^{i(\bigwedge_{a_k \in [a_j]_{\Re}} \gamma_{\mathfrak{C}}(a_k))}, \end{array} \right\rangle \right) : a_j \in \mathcal{A} \right\} \\
&= \sim \underline{appr}_{\Re}(\mathfrak{C}).
\end{aligned} \tag{28}$$

The property of $\overline{appr}_{\Re}(\sim \mathfrak{C}) =\sim \overline{appr}_{\Re}(\mathfrak{C})$ can be demonstrated similarly.

(vi): Based on the Definition 3.8 (c) and (d) and Definition 4.1, it can be proved similar to the proof of (v).

□

5. Level Cut Set-based Rough Degree of Complex Quadripartitioned Single Valued Neutrosophic Set

In this section, we introduce the approximate precision and rough degree of CQSVNS and give some theoretical results.

For the CQSVNS $\mathfrak{C}$ in $\mathcal{A}$, we know that $\underline{appr}_{\Re}(\mathfrak{C})$ and $\overline{appr}_{\Re}(\mathfrak{C})$ are two CQSVNSs. Thus, the $((\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3),(\alpha_4,\beta_4))$-level cut sets of $\underline{appr}_{\Re}(\mathfrak{C})$ and $\overline{appr}_{\Re}(\mathfrak{C})$ can be described as follows.

Definition 5.1. The $((\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3),(\alpha_4,\beta_4))$-level cut sets of $\underline{appr}_{\Re}(\mathfrak{C})$ and $\overline{appr}_{\Re}(\mathfrak{C})$, denoted by $(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ and $(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$, are defined as follows, respectively:

$$(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \left\{ a_j \in \mathcal{A} : \begin{pmatrix} \Gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_1,\ \Lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_2, \\ \Psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_3,\ \Omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_4, \\ \gamma_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_1,\ \lambda_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_2, \\ \psi_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_3,\ \omega_{\underline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_4 \end{pmatrix} \right\} \tag{29}$$

and

$$(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1,\beta_2,\beta_3,\beta_4)}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \left\{ a_j \in \mathcal{A} : \begin{pmatrix} \Gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_1,\ \Lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \alpha_2, \\ \Psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_3,\ \Omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \alpha_4, \\ \gamma_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_1,\ \lambda_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \geq \beta_2, \\ \psi_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_3,\ \omega_{\overline{appr}_{\Re}(\mathfrak{C})}(a_j) \leq \beta_4 \end{pmatrix} \right\} \tag{30}$$

Definition 5.2. Let $(\mathcal{A},\Re)$ be an approximation space and $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$. Also, let the $((\alpha_1^2,\beta_1^2),(\alpha_2^2,\beta_2^2),(\alpha_3^2,\beta_3^2),(\alpha_4^2,\beta_4^2))$-level cut set of $\overline{appr}_{\Re}(\mathfrak{C})$ be not null. The level cut set-based approximate precision of CQSVNS $\mathfrak{C}$ can be defined as

$$\sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} = \frac{\left|(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}\right|}{\left|(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2)}_{(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)}\right|} \tag{31}$$

where the notation $|\cdot|$ denotes the cardinality of set and $0 < \alpha_1^2 \leq \alpha_1^1 \leq 1$, $0 < \alpha_2^2 \leq \alpha_2^1 \leq 1$, $0 < \alpha_3^1 \leq \alpha_3^2 \leq 1$, $0 < \alpha_4^1 \leq \alpha_4^2 \leq 1$, $0 < \beta_1^2 \leq \beta_1^1 \leq 1$, $0 < \beta_2^2 \leq \beta_2^1 \leq 1$, $0 < \beta_3^1 \leq \beta_3^2 \leq 1$, $0 < \beta_4^1 \leq \beta_4^2 \leq 1$.
The level cut set-based rough degree of CQSVNS $\mathfrak{C}$ is denoted and defined by

$$\rho(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} = 1 - \sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \tag{32}$$

Note 5.3. From now on, the $((\alpha_1^2,\beta_1^2),(\alpha_2^2,\beta_2^2),(\alpha_3^2,\beta_3^2),(\alpha_4^2,\beta_4^2))$-level cut set of $\overline{appr}_{\Re}(\mathfrak{C})$ is not null.

Theorem 5.4. Let $(\mathcal{A},\Re)$ be an approximation space and $\mathfrak{C}$ be a CQSVNS in $\mathcal{A}$. Then, the approximate precision $\sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$ and the rough degree $\rho(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$ of CQSVNS $\mathfrak{C}$ provide the following properties.

(i) $0 \leq \sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq 1$

(ii) $0 \leq \rho(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq 1$

PROOF.

(i): By Proposition 4.3 (i), we know that $\underline{appr}_{\Re}(\mathfrak{C}) \subseteq \overline{appr}_{\Re}(\mathfrak{C})$. Since $0 < \alpha_p^2 \leq \alpha_p^1 \leq 1$, $0 < \beta_p^2 \leq \beta_p^1 \leq 1$ for $p = 1,2$ and $0 < \alpha_q^1 \leq \alpha_q^2 \leq 1$, $0 < \beta_q^1 \leq \beta_q^2 \leq 1$ for $p = 3,4$, we can say that

$$\left|(\underline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}\right| \leq \left|(\overline{appr}_{\Re}(\mathfrak{C}))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}\right|$$

So, we have $0 \leq \sigma(\mathfrak{C})^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq 1$.

(ii): It is obvious from (i) and Eq. (32).

□

Example 5.5. Consider the lower approximation $\overline{appr}_{\Re}(\mathfrak{C})$ and upper approximation $\overline{appr}_{\Re}(\mathfrak{C})$ of $\mathfrak{C}$ in Example 4.2. We can find that the $((0.3, \frac{\pi}{3}), (0.7, \frac{\pi}{2}), (0.7, \frac{4\pi}{3}), (0.3, 0))$-level cut set of $\underline{appr}_{\Re}(\mathfrak{C})$ is

$$(\underline{appr}_{\Re}(\mathfrak{C}))^{(\frac{\pi}{3},\frac{\pi}{2},\frac{4\pi}{3},0)}_{(0.3,0.7,0.7,0.3)} = \{a_2\}$$

and $((0.2, \frac{\pi}{3}), (0.7, \frac{2\pi}{5}), (0.7, \frac{4\pi}{3}), (0.5, \frac{\pi}{5}))$-level cut set of $\overline{appr}_{\Re}(\mathfrak{C})$ is

$$(\overline{appr}_{\Re}(\mathfrak{C}))^{(\frac{\pi}{3},\frac{2\pi}{5},\frac{4\pi}{3},\frac{\pi}{5})}_{(0.2,0.7,0.7,0.5)} = \{a_2, a_4, a_5, a_6\}$$

Hence, we calculate the approximation precision and rough degree as

$$\sigma(\mathfrak{C})^{((\frac{\pi}{3},\frac{\pi}{3}),(\frac{\pi}{2},\frac{2\pi}{5}),(\frac{4\pi}{3},\frac{4\pi}{3}),(0,\frac{\pi}{5}))}_{((0.3,0.2),(0.7,0.2),(0.7,0.7),(0.3,0.5))} = \frac{1}{4}$$

and

$$\rho(\mathfrak{C})^{((\frac{\pi}{3},\frac{\pi}{3}),(\frac{\pi}{2},\frac{2\pi}{5}),(\frac{4\pi}{3},\frac{4\pi}{3}),(0,\frac{\pi}{5}))}_{((0.3,0.2),(0.7,0.2),(0.7,0.7),(0.3,0.5))} = \frac{3}{4}$$

Proposition 5.6. Let $(\mathcal{A}, \Re)$ be an approximation space, and $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two CQSVNSs in $\mathcal{A}$.

(i) If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ and $(\overline{appr}_{\Re}(\mathfrak{C}_1))^{(\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2)}_{(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)} = (\overline{appr}_{\Re}(\mathfrak{C}_2))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}$ then

$$\sigma(\mathfrak{C}_1)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq \sigma(\mathfrak{C}_2)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$$

and

$$\rho(\mathfrak{C}_1)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \geq \rho(\mathfrak{C}_2)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$$

(ii) If $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ and $(\underline{appr}_{\Re}(\mathfrak{C}_1))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)} = (\underline{appr}_{\Re}(\mathfrak{C}_2))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}$ then

$$\sigma(\mathfrak{C}_1)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \geq \sigma(\mathfrak{C}_2)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$$

and

$$\rho(\mathfrak{C}_1)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})} \leq \rho(\mathfrak{C}_2)^{(\beta_1^{(1,2)},\beta_2^{(1,2)},\beta_3^{(1,2)},\beta_4^{(1,2)})}_{(\alpha_1^{(1,2)},\alpha_2^{(1,2)},\alpha_3^{(1,2)},\alpha_4^{(1,2)})}$$

PROOF.

(i): Since $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$, we have $(\underline{appr}_{\Re}(\mathfrak{C}_1))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)} \subseteq (\underline{appr}_{\Re}(\mathfrak{C}_2))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}$ by Propositions 3.5 and 4.3 (i). From the assumption, we have $(\overline{appr}_{\Re}(\mathfrak{C}_1))^{(\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2)}_{(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)} = (\overline{appr}_{\Re}(\mathfrak{C}_2))^{(\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1)}_{(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)}$. Therefore, the proof is clear from Eqs. (31) and (32).

(ii): It can be proved similar to proof of (i).

□

6. Conclusion

The QSVNS based on the four-valued logic is an effective mathematical tool for managing ambiguity. In this study based on extension of these sets, we introduced the concept of CQSVNSs and carried out theoretical study of various set-theoretic operations on them. Then, we described the lower and upper approximations of CQSVNSs in the approximation space and discussed their properties. Meanwhile, we gave the definitions of rough CQSVN cut sets and then presented how to measure the rough degree of CQSVN in the approximation space. It is worth mentioning that the CQSVNs and rough CQSVNs can be used for dealing with many problems in real life. Future works may involve the different types of distance measures between two CQSVNs (or rough CQSVNSs) and their applications in the medical diagnosis, pattern recognition and clustering analysis.

Conflicts of Interest

The authors declare no conflict of interest.

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