https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 70, Number 2, Pages 837–848 (2021)
DOI: 10.31801/cfsuasmas.869893
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: January 28, 2021; Accepted: April 25, 2021

EMBEDDINGS BETWEEN WEIGHTED TANDORI AND CESÀRO FUNCTION SPACES

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Abstract. We characterize the weights for which the two-operator inequality

$$\left\| \left(\int_0^x f(t)^p v(t)^p dt \right)^{\frac{1}{p}} \right\|_{q,u,(0,\infty)} \le c \left\| \operatorname*{ess\,sup}_{t\in(x,\infty)} f(t) \right\|_{r,w,(0,\infty)}$$

holds for all non-negative measurable functions on $(0,\infty)$, where $0 < p < q \le \infty$ and $0 < r < \infty$, namely, we find the least constants in the embeddings between weighted Tandori and Cesàro function spaces. We use the combination of duality arguments for weighted Lebesgue spaces and weighted Tandori spaces with weighted estimates for the iterated integral operators.

1. INTRODUCTION

Given two function spaces X, Y and an operator T, a standard problem is characterizing the conditions for which T maps X into Y. If X and Y are (quasi) Banach spaces of measurable functions, a bounded operator $\mathrm{T} : X \to Y$ satisfies the inequality $\| \mathrm{T} f\|_Y \le c\|f\|_X$ for all $f \in X$ where $c \in (0,\infty)$. When T is the identity operator I, we say that X is embedded into Y and write $X \hookrightarrow Y$. The least constant c in the embedding $X \hookrightarrow Y$ is $\| \mathrm{I} \|_{X\to Y}$.

In this paper, we find the optimal constants in the embedding between weighted Tandori and Cesàro function spaces. We shall begin with the definitions of the function spaces considered in this paper.

Given a measurable function f on E, we set

$$\|f\|_{p,E} := \left(\int_E |f(x)|^p dx \right), \quad 1 \le p < \infty$$

2020 *Mathematics Subject Classification.* Primary 46E30; Secondary 26D10.

Keywords and phrases. Cesàro function spaces, Copson function spaces, Tandori function spaces, embeddings, weighted inequalities, Hardy operator, Copson operator, iterated operators.

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and

$$\|f\|_{\infty,E} := \operatorname*{ess\,sup}_{x\in E} |f(x)|, \quad p=\infty.$$

If w is a weight on E, that is, measurable, positive and finite a.e. on E, then we denote by $L_{p,w}(E)$[1], the weighted Lebesgue space, the set of measurable functions satisfying $\|f\|_{p,w,E} := \|fw\|_{p,E} < \infty$.

Let $0 < p, q \le \infty$, u be a non-negative measurable function and v be a weight, the weighted Cesàro space $\mathrm{Ces}_{p,q}(u,v)$ is the set of all measurable functions such that $\|f\|_{\mathrm{Ces}_{p,q}(u,v)} < \infty$, where

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} := \big\|\|f\|_{p,v,(0,x)}\big\|_{q,u,(0,\infty)};$$

and the weighted Copson space $\mathrm{Cop}_{p,q}(u,v)$ is the set of all measurable functions such that $\|f\|_{\mathrm{Cop}_{p,q}(u,v)} < \infty$, where

$$\|f\|_{\mathrm{Cop}_{p,q}(u,v)} := \big\|\|f\|_{p,v,(x,\infty)}\big\|_{q,u,(0,\infty)}.$$

The classical Cesàro spaces $\mathrm{Ces}_{1,p}(x^{-1},1)$, $1 \le p < \infty$ were defined by Shiue [20] in 1970. When $1 < p < \infty$ Hassard and Hussein [12] proved that $\mathrm{Ces}_{1,p}(x^{-1},1)$ are separable Banach spaces and Bennett [4] showed that the spaces $\mathrm{Ces}_{1,p}(x^{-1},1)$ and $\mathrm{Cop}_{1,p}(1,x^{-1})$ coincide. Dual spaces of the classical Cesàro function spaces were considered in [4, 21]. In [1], factorization theorems for classical Cesàro function spaces were given and based on these results the dual spaces of classical Cesàro function spaces were presented. One weighted Cesàro function spaces $\mathrm{Ces}_{1,p}(w^{\frac{1}{p}},1)$ and their duals were considered in [13]. Recently, in [3], factorization of the spaces $\mathrm{Ces}_{1,p}(x^{-1}w^{\frac{1}{p}},1)$ and $\mathrm{Cop}_{1,p}(w^{\frac{1}{p}},x^{-1})$ are given.

We do not aim to give a thorough set of references on the history of these spaces. Instead, we refer the interested reader to survey paper [2], where the comprehensive history on the structure of Cesàro and Copson function spaces are given.

In this paper our primary focus is the following inequality

$$\|f\|_{\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \le c\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)} \tag{1}$$

for all measurable functions where $0 < p_i, q_i \le \infty$, $i = 1,2$.

There is more than one motivation to study inclusion between Cesàro and Copson spaces. First of all when $p_1 = q_1$ or $p_2 = q_2$, weighted Cesàro and Copson function spaces coincide with some weighted Lebesgue spaces (see [9, Lemmas 3.4-3.5]), thus inequality (1) is a generalization of the well-known weighted direct and reverse Hardy-type inequalities (e.g. [15, 7, 19]). Another justification is to give the characterization of pointwise multipliers between two spaces of Cesàro and Copson type, because it reduces to the characterization the embeddings between these spaces. In [11, Section 7] Grosse-Erdmann considered the multipliers between the spaces of p-summable sequences and Cesàro and Copson sequence spaces. He also introduced corresponding function spaces but the characterization of the multipliers

[1]When $E = (0,\infty)$, we simply write $L_{p,w}$ instead of $L_{p,w}(0,\infty)$.

between two spaces of Cesàro and Copson type remained open for both sequence and function spaces for a long time.

The characterization of the inequality (1) is given in one parameter case when $p_1 = p_2 = 1$, $q_1 = q_2 = p > 1$, $v_1(t) = t^{-\beta-1}$, $v_2(t) = t^{\alpha-1}$, $u_1(t) = t^{\beta-1/p}$ and $u_2(t) = t^{-\alpha-1/p}$, $t > 0$, $\alpha, \beta > 0$ in [5]. Moreover, it was shown that the inequality is reversed when $0 < p < 1$. In [6], inequality (1) is considered for two different parameters in the special case $p_1 = p_2 = 1$, $q_1 = p$, $q_2 = q$, $v_1(t) = t^{-1}$, $v_2(t) = 1$, $u_1(t)^p = v(t)$, $u_2(t)^q = w(t)t^{-q}$, $t > 0$, under the restriction $q \geq 1$ in order to characterize the embeddings between some Lorentz-type spaces. Recently, in [9] the two sided estimates for the best constant in (1) is given for four weights and four parameters $0 < p_1, p_2, q_1, q_2 < \infty$ under the restriction $p_2 \leq q_2$. Moreover, using these results, in [9, Theorems 3.11-3.12], the associate spaces of weighted Copson and Cesàro function spaces were characterized and in [10] pointwise multipliers between Cesàro and Copson function spaces are given for some ranges of parameters.

Furthermore, in 2015, Lesnik and Maligranda [16,17] began studying these spaces within an abstract framework, where they used a more general function space X instead of the weighted Lebesgue spaces. When X is a Banach space, they defined Cesàro space CX, Copson space C^*X and Tandori space $\widetilde{X}$ as the set of all measurable functions, respectively, with the following norms:

$$\|f\|_{CX} = \left\| \frac{1}{x} \int_0^x |f(t)| dt \right\|_X < \infty,$$
$$\|f\|_{C^*X} = \left\| \int_x^\infty \frac{|f(t)|}{t} dt \right\|_X < \infty,$$
$$\|f\|_{\widetilde{X}} = \left\| \operatorname*{ess\,sup}_{t \in (x,\infty)} |f(t)| \right\|_X < \infty.$$

In [18], they named $\widetilde{X}$ as the generalized Tandori spaces in honour of Tandori who provided dual spaces to the spaces $CL_\infty[0,1]$ in [22]. Their definition is related to our definition in the following way:

$$CL_{p,w} = \mathrm{Ces}_{1,p}(x^{-1}w(x), 1), \quad C^*L_{p,w} = \mathrm{Cop}_{1,p}(w, x^{-1}), \quad \widetilde{L}_{p,w} = \mathrm{Cop}_{\infty,p}(w, 1).$$

We should note that recently in [14] multipliers between CL_p and CL_q are given when $1 < q \leq p \leq \infty$.

We want to continue this research. In this paper, we will handle the inequality (1) when $p_1 = \infty$. In other words, we will consider the embeddings $\widetilde{L}_{r,w} \hookrightarrow \mathrm{Ces}_{p,q}(u, v)$, namely, we will give the characterization of the following inequality,

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} \leq C \|f\|_{\widetilde{L}_{r,w}} \tag{2}$$

for all measurable functions where $p, q, r \in (0, \infty)$ with $p < q$. The restriction on the parameters arises from the duality argument. The key ingredient of the proof is combining characterizations of the associate spaces of Tandori spaces, namely, the reverse Hardy-type inequality for supremal operators which was given in [19] with the characterizations of some iterated Hardy-type inequalities.

Throughout the paper, we put $0 \cdot \infty = \frac{0}{0} = 0$. We write $A \approx B$ if there exist positive constants α, β independent of relevant quantities appearing in expressions A and B such that

$$\alpha \leq \frac{A}{B} \leq \beta$$

holds.

The symbol $\mathfrak{M}$ will stand for the set of all measurable functions on $(0, \infty)$, and we denote the class of non-negative elements of $\mathfrak{M}$ by $\mathfrak{M}^+$.

We sometimes omit the differential element dx to make the formulas simpler when the expressions are too long.

The paper is structured as follows. In Section 2, we formulate the main results of this paper. In Section 3, we collect some properties and necessary background material. Finally, in the last section, we give the proofs of our main results.

2. MAIN RESULTS

It is convenient to start this section by recalling some properties of the weighted Cesàro and Copson spaces. Let $0 < p, q \leq \infty$. Assume that u is a non-negative measurable function and v is a weight. We will always assume that $\|u\|_{q,(t,\infty)} < \infty$ for all $t > 0$ and $\|u\|_{q,(0,t)} < \infty$ for all $t > 0$, when considering weighted Cesàro and Copson function spaces, respectively. Otherwise, these spaces consist only of functions equivalent to zero (see, [9, Lemmas 3.1-3.2]).

In this section, we will formulate the least constant in the embedding

$$\widetilde{L}_{r,w} \hookrightarrow \operatorname{Ces}_{p,q}(u, v). \tag{3}$$

Remark 1. *Observe that,*

$$\| \mathrm{I} \|_{\operatorname{Cop}_{\infty,r}(w,v_1) \to \operatorname{Ces}_{p,q}(u,v_2)} = \| \mathrm{I} \|_{\widetilde{L}_{r,w} \to \operatorname{Ces}_{p,q}(u, \frac{v_2}{v_1})}$$

holds. Therefore, it is enough to consider the three weighted case (3)*.*

Remark 2. *Note that, when $p = q$ or $r = \infty$, this problem is not interesting since it reduces to the characterizations of Hardy-type inequalities and can be found in [9], therefore we will consider the cases when $r < \infty$. On the other hand, we have the restriction $p < q$, which arises from the duality argument.*

Now we are in position to formulate the results of this paper. We begin with the cases where $q = \infty$.

Theorem 3. *Let $0 < p, r < \infty$. Assume that v is a weight, $w \in \mathfrak{M}^+$ such that $\|w\|_{r,(0,t)} < \infty$ for all $t \in (0, \infty)$ and $w \neq 0$ a.e. on $(0, \infty)$, and $u \in \mathfrak{M}^+$ such that $\|u\|_{\infty,(t,\infty)} < \infty$ for all $t \in (0, \infty)$.*

(i) If $r \leq p$, then

$$\| \mathrm{I} \|_{\widetilde{L}_{r,w} \to \operatorname{Ces}_{p,\infty}(u,v)} \approx I_1,$$

where

$$I_1 := \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x) \sup_{t\in(0,x)} \left(\int_0^t v^p\right)^{\frac{1}{p}} \left(\int_0^t w^r\right)^{-\frac{1}{r}} < \infty.$$

(ii) If $p < r$, then

$$\| \mathrm{I} \|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,\infty}(u,v)} \approx I_2 + I_3 + I_4$$

where

$$I_2 := \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\bigg(\int_0^x \left(\int_0^t v^p\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\bigg)^{\frac{r-p}{rp}} < \infty,$$

$$I_3 := \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\bigg(\int_x^\infty \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\bigg)^{\frac{r-p}{rp}} \left(\int_0^x v^p\right)^{\frac{1}{p}} < \infty,$$

and

$$I_4 := \left(\int_0^\infty w^r\right)^{-\frac{1}{r}} \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\bigg(\int_0^x v^p\bigg)^{\frac{1}{p}} < \infty.$$

When $q < \infty$, we consider the cases $r \le p$ and $p < r$ separately.

Theorem 4. *Let $0 < r \le p < q < \infty$. Assume that $v \in \mathfrak{M}^+$, $w \in \mathfrak{M}^+$ such that $\|w\|_{r,(0,t)} < \infty$ for all $t \in (0,\infty)$ and $w \ne 0$ a.e. on $(0,\infty)$, and $u \in \mathfrak{M}^+$ such that $\|u\|_{q,(t,\infty)} < \infty$ for all $t \in (0,\infty)$. Then*

$$\| \mathrm{I} \|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,q}(u,v)} \approx I_5 + I_6,$$

where

$$I_5 := \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \left(\int_0^t \left(\int_0^s v(y)^p dy\right)^{\frac{q}{p}} u(s)^q ds\right)^{\frac{1}{q}} < \infty,$$

and

$$I_6 := \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \left(\int_0^t v(s)^p ds\right)^{\frac{1}{p}} \left(\int_t^\infty u(s)^q ds\right)^{\frac{1}{q}} < \infty.$$

Theorem 5. *Let $0 < p < r < \infty$ and $0 < p < q < \infty$. Assume that $v \in \mathfrak{M}^+$, such that $v > 0$, $\|v\|_{p,(0,t)} < \infty$ for all $t \in (0,\infty)$ and $\|v\|_{p,(0,\infty)} = \infty$. Suppose that $w \in \mathfrak{M}^+$ such that $\|w\|_{r,(0,t)} < \infty$ for all $t \in (0,\infty)$ and $w \ne 0$ a.e. on $(0,\infty)$, and $u \in \mathfrak{M}^+$ such that $\|u\|_{q,(t,\infty)} < \infty$ for all $t \in (0,\infty)$. Let*

- $\displaystyle\int_0^t \left(\int_0^s v^p\right)^{\frac{r}{r-p}} \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds < \infty$ *for all $t \in (0,\infty)$,*
- $\displaystyle\int_0^1 \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds = \infty,$

- $\displaystyle\int_t^\infty \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds < \infty$ *for all* $t \in (0,\infty)$,
- $\displaystyle\int_1^\infty \left(\int_0^s v^p\right)^{\frac{r}{r-p}} \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds = \infty$

hold.

(i) If $r \le q$, *then*

$$\|\,\mathrm{I}\,\|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,q}(u,v)} \approx I_7 + I_8 + I_9,$$

where

$$I_7 := \left(\int_0^\infty w^r\right)^{-\frac{1}{r}} \left(\int_0^\infty \left(\int_0^y v(s)^p ds\right)^{\frac{q}{p}} u(y)^q dy\right)^{\frac{1}{q}} < \infty, \tag{4}$$

$$I_8 := \sup_{x\in(0,\infty)} \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} \left(\int_x^\infty u^q\right)^{\frac{1}{q}} < \infty,$$

and

$$I_9 := \sup_{x\in(0,\infty)} \left(\int_x^\infty \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{q}{p}} u(t)^q dt\right)^{\frac{1}{q}} < \infty.$$

(ii) If $q < r$, *then*

$$\|\,\mathrm{I}\,\|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,q}(u,v)} \approx I_7 + I_{10} + I_{11},$$

where I_7 *is defined in* (4),

$$\begin{aligned} I_{10} := \Bigg(\int_0^\infty \left(\int_x^\infty u^q\right)^{\frac{r}{r-q}} \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r(q-p)}{p(r-q)}} \\ \times \left(\int_0^x v^p\right)^{\frac{r}{r-p}} \left(\int_0^x w^r\right)^{-\frac{r}{r-p}} w(x)^r dx\Bigg)^{\frac{r-q}{rq}} < \infty, \end{aligned}$$

and

$$\begin{aligned} I_{11} := \Bigg(\int_0^\infty \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{q}{p}} u(t)^q dt\right)^{\frac{r}{r-q}} \left(\int_x^\infty \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r(q-p)}{p(r-q)}} \\ \times \left(\int_0^x w^r\right)^{-\frac{r}{r-p}} w(x)^r dx\Bigg)^{\frac{r-q}{rq}} < \infty. \end{aligned}$$

3. BACKGROUND MATERIAL

In this section we quote some known results. Let us start with the characterization of the reverse Hardy-type inequality for supremal operator, that is,

$$\left(\int_0^\infty f(t)^p \mathfrak{u}(t)^p dt\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \mathfrak{w}(t)^q \left(\operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right)^q dt\right)^{\frac{1}{q}} \tag{5}$$

for all non-negative measurable functions f on $(0,\infty)$ where $0<p,q<\infty$.

Theorem 6. *[19, Theorem 3.4] Let $0<p,q<\infty$. Assume that $\mathfrak{u}\in\mathfrak{M}^+$ and $\mathfrak{w}\in\mathfrak{M}^+$ such that $\int_0^t \mathfrak{w}^q<\infty$ for all $t\in(0,\infty)$ and $\mathfrak{w}\neq 0$ a.e. on $(0,\infty)$.*

(i) If $q\le p$, then inequality (5) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if $A_1<\infty$, where

$$A_1 := \sup_{t\in(0,\infty)} \left(\int_0^t \mathfrak{u}^p\right)^{\frac{1}{p}} \left(\int_0^t \mathfrak{w}^q\right)^{-\frac{1}{q}}. \tag{6}$$

Moreover, the least possible constant C in (5) satifies $C\approx A_1$.

(ii) If $p<q$, then inequality (5) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if $A_2<\infty$ and $A_3<\infty$, where

$$A_2 := \left(\int_0^\infty \left(\int_0^t \mathfrak{u}^p\right)^{\frac{q}{q-p}} \left(\int_0^t \mathfrak{w}^q\right)^{-\frac{q}{q-p}} \mathfrak{w}(t)^q dt\right)^{\frac{q-p}{pq}}, \tag{7}$$

and

$$A_3 := \left(\int_0^\infty \mathfrak{u}^p\right)^{\frac{1}{p}} \left(\int_0^\infty \mathfrak{w}^q\right)^{-\frac{1}{q}}. \tag{8}$$

Moreover, the least possible constant C in (5) satifies $C\approx A_2+A_3$.

We next recall the characterization of the weighted iterated inequality involving Hardy and Copson operators, that is,

$$\left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty g\right)\mathfrak{v}(s)ds\right)^q \mathfrak{w}(t)^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty g(t)^p\mathfrak{u}(t)^p dt\right)^{\frac{1}{p}}. \tag{9}$$

Note that the characterization of inequality (9) is given in [8]. In the next theorem, we provide a modified version of [8, Theorem 3.1], using the gluing lemmas presented in the recent paper [10]. Denote by

$$\mathcal{V}(t) := \int_0^t \mathfrak{v}(s)ds,\ t>0.$$

Theorem 7. *Let $1<p<\infty$ and $0<q<\infty$. Assume that $\mathfrak{u}\in\mathfrak{M}^+$ and $\mathfrak{v},\mathfrak{w}\in\mathfrak{M}^+$ such that $\mathfrak{v}(t)>0$, $\mathcal{V}(t)<\infty$ for all $t\in(0,\infty)$ and $\mathcal{V}(\infty)=\infty$,*

- $\int_0^t \mathcal{V}(s)^q\mathfrak{w}(s)^q ds<\infty$ *for all $t\in(0,\infty)$ and* $\int_1^\infty \mathcal{V}(s)^q\mathfrak{w}(s)^q ds=\infty$,
- $\int_t^\infty \mathfrak{w}(s)^q ds<\infty$ *for all $t\in(0,\infty)$ and* $\int_0^1 \mathfrak{w}(s)^q ds=\infty$.

(i) If $p\le q$, then (9) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if $B_1<\infty$ and $B_2<\infty$, where

$$B_1 := \sup_{x\in(0,\infty)} \left(\int_0^x \mathcal{V}(t)^q\mathfrak{w}(t)^q dt\right)^{\frac{1}{q}} \left(\int_x^\infty \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}},$$

and

$$B_2 := \sup_{x\in(0,\infty)} \left(\int_x^\infty \mathfrak{w}(t)^q dt\right)^{\frac{1}{q}} \left(\int_0^x \mathcal{V}(t)^{\frac{p}{p-1}} \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}.$$

Moreover, the least possible constant C in (9) *satifies $C \approx B_1 + B_2$.*

(ii) If $q < p$, then (9) *holds for all non-negative measurable functions f on* $(0,\infty)$ *if and only if $B_3 < \infty$ and $B_4 < \infty$, where*

$$B_3 := \left(\int_0^\infty \left(\int_x^\infty \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{q(p-1)}{p-q}} \left(\int_0^x \mathcal{V}(t)^q \mathfrak{w}(t)^q dt\right)^{\frac{q}{p-q}} \mathcal{V}(x)^q \mathfrak{w}(x)^q dx\right)^{\frac{p-q}{pq}},$$

and

$$B_4 := \left(\int_0^\infty \left(\int_x^\infty \mathfrak{w}(t)^q dt\right)^{\frac{q}{p-q}} \left(\int_0^x \mathcal{V}(t)^{\frac{p}{p-1}} \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{q(p-1)}{p-q}} \mathfrak{w}(x)^q dx\right)^{\frac{p-q}{pq}}.$$

Moreover, the least possible constant C in (9) *satifies $C \approx B_3 + B_4$.*

Proof. The proof is the combination of [8, Theorem 3.1, (iii)] and [10, Lemma 2.7] for the first case and [8, Theorem 3.1, (iv)] and [10, Lemma 2.8] for the second case. □

4. PROOFS

Denote by

$$\mathcal{R}(p,r;v,w) := \sup_{f\in\mathfrak{M}^+} \frac{\|f\|_{p,v,(0,\infty)}}{\left\| \operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right\|_{r,w,(0,\infty)}}.$$

Proof of Theorem 3 Let $0 < p, r < \infty$. We have

$$C = \sup_{f\in\mathfrak{M}} \frac{\|f\|_{\mathrm{Ces}_{p,\infty}(u,v)}}{\|f\|_{\widetilde{L}_{r,w}}} = \sup_{f\in\mathfrak{M}^+} \frac{\operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\|f\|_{p,v,(0,x)}}{\left\| \operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right\|_{r,w,(0,\infty)}}.$$

Fix $x \in (0,\infty)$, then

$$C = \sup_{f\in\mathfrak{M}^+} \frac{\operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\|f\chi_{(0,x)}\|_{p,v,(0,\infty)}}{\left\| \operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right\|_{r,w,(0,\infty)}}.$$

Observe that, interchanging supremum gives

$$C = \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\mathcal{R}(p,r;\tilde{v},w),$$

where $\tilde{v}(t) = \chi_{(0,x)}(t)v(t)$, $t \in (0,\infty)$. Thus, the problem is reduced to the characterization of reverse Hardy-type inequalities for supremal operator. It remains to apply [Theorem 6, (i)] when $r \le p$ and [Theorem 6, (ii)] when $p < r$.

$\square$

Proof of Theorem 4 Let $0 < r \le p < q < \infty$. We have

$$C = \sup_{f \in \mathfrak{M}} \frac{\|f\|_{\mathrm{Ces}_{p,q}(u,v)}}{\|f\|_{\widetilde{L}_{r,w}}}$$

Since $q/p \in (1,\infty)$, by the duality in weighted Lebesgue spaces, we have

$$\|f\|^p_{\mathrm{Ces}_{p,q}(u,v)} = \sup_{g \in \mathfrak{M}^+} \frac{\int_0^\infty \left(\int_0^t f(s)^p v(s)^p ds \right) g(t) dt}{\left(\int_0^\infty g(t)^{\frac{q}{q-p}} u(t)^{-\frac{qp}{q-p}} dt \right)^{\frac{q-p}{q}}}.$$

Interchanging supremum and Fubini's Theorem gives that

$$\begin{aligned} C &= \sup_{g \in \mathfrak{M}^+} \frac{1}{\left(\int_0^\infty g(t)^{\frac{q}{q-p}} u(t)^{-\frac{qp}{q-p}} dt \right)^{\frac{q-p}{qp}}} \sup_{f \in \mathfrak{M}^+} \frac{\left(\int_0^\infty f(s)^p v(s)^p \int_s^\infty g(t) dt\, ds \right)^{\frac{1}{p}}}{\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{s \in (t,\infty)} f(s) \right)^r w(t)^r dt \right)^{\frac{1}{r}}} \\ &=: \sup_{g \in \mathfrak{M}^+} \frac{\mathcal{R}(p,r;\tilde{v},w)}{\|g\|^{\frac{1}{p}}} \end{aligned} \tag{10}$$

where, $\tilde{v}(s) = v(s) \big(\int_s^\infty g(t) dt \big)^{\frac{1}{p}}$, $s \in (0,\infty)$, and

$$\|g\| := \left(\int_0^\infty g(t)^{\frac{q}{q-p}} u(t)^{-\frac{qp}{q-p}} dt \right)^{\frac{q-p}{q}}.$$

Note that $\mathcal{R}(p,r;\tilde{v},w)$ is the best constant in the inequality

$$\left(\int_0^\infty h(s)^p v(s)^p \int_s^\infty g(t) dt\, ds \right)^{\frac{1}{p}} \le c \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{s \in (t,\infty)} h(s) \right)^r w(t)^r dt \right)^{\frac{1}{r}}, \ h \in \mathfrak{M}^+$$

for every fixed $g \in \mathfrak{M}^+$. Now, we can apply Theorem 6 by taking the parameters p, r, and weights

$$\mathfrak{w}(s) = w(s) \quad \mathfrak{u}(s) = v(s) \left(\int_s^\infty g \right)^{\frac{1}{p}}, \quad s > 0.$$

Since $r \le p$, according to the first case in Theorem 6,

$$\mathcal{R}(p,r;\tilde{v},w) \approx \sup_{t \in (0,\infty)} \left(\int_0^t v(s)^p \left(\int_s^\infty g \right) ds \right)^{\frac{1}{p}} \left(\int_0^t w(s)^r ds \right)^{-\frac{1}{r}}$$

holds. Thus,

$$C \approx \sup_{g\in\mathfrak{M}^+} \frac{\sup\limits_{t\in(0,\infty)} \left(\int_0^t v(s)^p \left(\int_s^\infty g\right) ds\right)^{\frac{1}{p}} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}}}{\|g\|^{\frac{1}{p}}}.$$

Interchanging suprema yields that

$$C \approx \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \sup_{g\in\mathfrak{M}^+} \frac{\left(\int_0^\infty v(s)^p \left(\int_s^\infty g\right) \chi_{(0,t)}(s) ds\right)^{\frac{1}{p}}}{\|g\|^{\frac{1}{p}}}.$$

From Fubini's Theorem and duality in weighted Lebesgue spaces with $q/p \in (1,\infty)$ again, it follows that

$$\begin{aligned} C &= \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \sup_{g\in\mathfrak{M}^+} \frac{\left(\int_0^\infty g(y) \left(\int_0^y v(s)^p \chi_{(0,t)}(s) ds\right) dy\right)^{\frac{1}{p}}}{\left(\int_0^\infty g(y)^{\frac{q}{q-p}} u(y)^{-\frac{qp}{q-p}} dy\right)^{\frac{q-p}{qp}}} \\ &\approx \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \left(\int_0^\infty \left(\int_0^y v(s)^p \chi_{(0,t)}(s) ds\right)^{\frac{q}{p}} u(y)^q dy\right)^{\frac{1}{q}}. \end{aligned}$$

Observe that,

$$\begin{aligned} &\int_0^\infty \left(\int_0^y v(s)^p \chi_{(0,t)}(s) ds\right)^{\frac{q}{p}} u(y)^q dy \\ &\quad = \int_0^t \left(\int_0^y v(s)^p ds\right)^{\frac{q}{p}} u(y)^q dy + \left(\int_0^t v(s)^p ds\right)^{\frac{q}{p}} \left(\int_t^\infty u(y)^q dy\right). \end{aligned}$$

Thus we arrive at $C \approx I_5 + I_6$.

$\square$

Proof of Theorem 5 Let $0 < p < r < \infty$ and $0 < p < q < \infty$. Using the steps identical to the preceding proof, which relies on $q/p \in (1,\infty)$, duality in weighted Lebesgue spaces, and Fubini's Theorem one can see that (10) holds. Since $p < r$, applying the second case of Theorem 6, we obtain that

$$\begin{aligned} \mathcal{R}(p,r;\tilde{v},w) \approx& \left(\int_0^\infty \left(\int_0^t v(s)^p \left(\int_s^\infty g\right) ds\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} \\ &+ \left(\int_0^\infty v(s)^p \left(\int_s^\infty g\right) ds\right)^{\frac{1}{p}} \left(\int_0^\infty w^r\right)^{-\frac{1}{r}}. \end{aligned}$$

Then, $C \approx C_1 + C_2$, where

$$C_1 := \sup_{g\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_0^t v(s)^p \left(\int_s^\infty g\right) ds\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}}}{\|g\|^{\frac{1}{p}}}$$

and

$$C_2 := \left(\int_0^\infty w^r\right)^{-\frac{1}{r}} \sup_{g\in\mathfrak{M}^+} \frac{\left(\int_0^\infty v(s)^p \int_s^\infty g(y)dy\, ds\right)^{\frac{1}{p}}}{\|g\|^{\frac{1}{p}}}.$$

First observe that, using Fubini's Theorem and duality principle one more time, we have

$$C_2 = \left(\int_0^\infty w^r\right)^{-\frac{1}{r}} \left(\int_0^\infty \left(\int_0^y v(s)^p ds\right)^{\frac{q}{p}} u(y)^q dy\right)^{\frac{1}{q}},$$

and C_1^p is the best constant in the inequality (9) with parameters $p = \frac{q}{q-p}$ and $q = \frac{r}{r-p}$, and weights

$$\mathfrak{u}(s) = u(s)^{-p}, \quad \mathfrak{v}(s) = v(s)^p, \quad \mathfrak{w}(s) = \left(\int_0^s w^r\right)^{-1} w(s)^{r-p}, \quad s > 0.$$

It remains to apply Theorem 7. To this end we should again split this case into two parts.

(i) If $r \le q$, then applying the first case in Theorem 7, we obtain that $C_1 \approx I_8 + I_9$ and the result follows.

(ii) If $q < r$, then applying the second case in Theorem 7, we obtain that $C_1 \approx I_{10} + I_{11}$ and the result follows.

Declaration of Competing Interests The author declares no competing interests.

Acknowledgement The author thanks the anonymous referees for their helpful remarks, which have improved the final version of this paper.

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