Universal Journal of Mathematics and Applications, 4 (2) (2021) 59-69
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.870050

Hermite-Hadamard type Inequalities via p–Harmonic Exponential type Convexity and Applications

Muhammad Tariq[1]

[1]*Mehran University of Engineering and Technology, Jamshoro, Pakistan*

Article Info

Keywords: *Hermite–Hadamard inequality; Hölder's inequality; Convex function; Harmonic convex function; p–Harmonic exponential convex function.*
2010 AMS: *26A51, 26A33, 26D07, 26D10, 26D15*
Received: *28 January 2021*
Accepted: *10 May 2021*
Available online: *30 June 2021*

Abstract

In this work, we introduce the idea and concept of p–harmonic exponential type convex functions. We elaborate on the newly introduced idea by examples and some interesting algebraic properties. In addition, we attain the novel version of Hermite–Hadamard type inequality in the mode of the newly introduced definition and on the basis of lemmas, some refinements of the Hermite–Hadamard type inequalities in the support of the newly introduced idea are established. Finally, we investigate and explore some integral inequalities in the form of applications for the arithmetic, geometric, harmonic and logarithmic means. The amazing tools and interesting ideas of this work may inspire and motivate further research in this direction furthermore.

1. Introduction

Theory of convexity present an active and attractive field of research. Many researchers endeavor, attempt and maintain his work on the concept of convexity, extend and generalize its variant forms in different ways using innovative ideas and fruitful techniques. This theory provides us with unified and unique framework to develop and organize highly efficient numerical tools to tackle and solve a wide class of problems that arise in pure and applied mathematics. In recent years, the concept of convexity has been improved, generalized, and extended in many directions. A number of studies have shown that the theory of convex functions has a close relationship with the theory of inequalities.

The integral inequalities are useful and have remarkable importance in optimization theory, functional analysis, physics and statistical theory. In the research area, inequalities have a lot of applications in probability, statistical problems and numerical quadrature formulas [10, 19, 20]. Due to many generalizations and extensions convex analysis and inequalities have become an attractive, interesting and absorbing field for the researchers and for attention reader can refer to [7, 17, 18, 21, 29].

It is well known that the harmonic mean is the special case of power mean. This mean has a lot of applications in different field of sciences which are computer science, geometry, probability, finance, trigonometry, statistics and electric circuit theory. Harmonic mean is the most appropriate measure for rates and ratios because it equalizes the weights of each data point. Harmonic mean is used to define the harmonic convex set. In 2003, first time harmonic convex set was introduced by Shi [27]. Harmonic and p–harmonic convex function was first time introduced and discussed by Anderson et al. [2] and Noor et al. [22] respectively. Nowadays a lot of people are working on exponential type convexity [5, 6]. Dragomir [9] introduced the class of exponential type convexity. After Dragomir, Awan [3] studied and investigated a new class of exponentially convex functions. Kadakal introduced a new definition of exponential type convexity in [16]. The amazing importance and applications of exponential type convexity is used to manipulate for statistical learning, information sciences, data mining, stochastic optimization and sequential prediction [1, 26, 28] and the references therein.
The principal focus and main aim of this note is to explore and define the idea of p–harmonic exponential type convex functions and in the support of these newly introduced functions, we attain its algebraic properties. Some interesting examples with logic are given as well. In addition, we attain the novel version of Hermite–Hadamard inequality in the mode of the newly discussed idea. Furthermore, we explore a new lemma and in order to this lemma, we attain some refinements of Hermite–Hadamard-type inequality in the manner of this newly explored definition. Finally, as applications, some new inequalities for the arithmetic, geometric and harmonic means are established. The

***Email address and ORCID number*: captaintariq2187@gmail.com, 0000-0001-8372-2532, (M. Tariq)**

awe-inspiring concepts and formidable tools of this paper may invigorate and revitalize for additional research in this worthy and absorbing field. Before we start, we need the following necessary known definitions and literature.

2. Preliminaries

In this section we recall some known concepts.

Definition 2.1. *[21] Let $\psi : I \to \mathbb{R}$ be a real valued function. A function ψ is said to be convex, if*

$$\psi(\kappa\sigma_1 + (1-\kappa)\sigma_2) \leq \kappa\psi(\sigma_1) + (1-\kappa)\psi(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\kappa \in [0,1]$.

Definition 2.2. *[15] A function $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be harmonic convex, if*

$$\psi\left(\frac{\sigma_1\sigma_2}{\kappa\sigma_2 + (1-\kappa)\sigma_1}\right) \leq \kappa\psi(\sigma_1) + (1-\kappa)\psi(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\kappa \in [0,1]$.

For the harmonic convex function, İşcan [15] provided the Hermite–Hadamard type inequality.

Definition 2.3. *[23] A function $\psi : I \to \mathbb{R}$ is said to be p–harmonic convex, if*

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p + (1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \leq \kappa\psi(\sigma_1) + (1-\kappa)\psi(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\kappa \in [0,1]$.

Note that $\kappa = \frac{1}{2}$ in the above Definition 2.3, we get the following inequality

$$\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p + \sigma_2^p}\right]^{\frac{1}{p}}\right) \leq \frac{\psi(\sigma_1) + \psi(\sigma_2)}{2},$$

holds for all $\sigma_1, \sigma_2 \in I$.
The function ψ is called Jensen p–harmonic convex function.
If we put $p = -1$ and $p = 1$, then p–harmonic convex sets and p–harmonic convex functions collapses to classical convex sets, harmonic convex sets and harmonic convex functions respectively.
We organise the paper in following way. Firstly, we will give the idea and its algebraic properties of p–harmonic exponential type convex functions. Secondly, we will derive the new sort of Hermite–Hadamard type and refinements of Hermite–Hadamard type inequalities by using the newly introduced idea. Finally, we will give some applications for means and conclusion.

3. p–harmonic Exponential Type Convex Functions and its Properties

We are going to introduce a new definition called p–harmonic exponential type convex function and will study some of their algebraic properties. Throughout the paper, one thing get in mind p–harmonic exp convex function represents p–harmonic exponential type convex function.

Definition 3.1. *A function $\psi : I \subseteq (0,+\infty) \to [0,+\infty)$ is called p–harmonic exp convex, if*

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p + (1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \leq \left(e^{\kappa} - 1\right)\psi(\sigma_1) + \left(e^{1-\kappa} - 1\right)\psi(\sigma_2),$$

holds for every $\sigma_1, \sigma_2 \in I$, and $\kappa \in [0,1]$.

Remark 3.2. (i) *Taking $p = 1$ in Definition 3.1, we obtain the following new definition about harmonically exp type convex function:*

$$\psi\left(\frac{\sigma_1\sigma_2}{\kappa\sigma_2 + (1-\kappa)\sigma_1}\right) \leq \left(e^{\kappa} - 1\right)\psi(\sigma_1) + \left(e^{1-\kappa} - 1\right)\psi(\sigma_2).$$

(ii) *Taking $p = -1$ in Definition 3.1, then we get a definition namely exponential type convex function which is defined by Kadakal et al. [16].*

That is the beauty of this newly introduce definition if we put the different values of p, then we obtain new inequalities and also found some results which connect with previous results.

Lemma 3.3. *The following inequalities $e^{\kappa} - 1 \geq \kappa$ and $e^{1-\kappa} - 1 \geq 1 - \kappa$ are hold. If for all $\kappa \in [0,1]$.*

Proof. The rest of the proof is clearly seen. □

Proposition 3.4. *Every p–harmonic convex function is p–harmonic exp convex function.*

Proof. Using the definition of p–harmonic convex function and from the lemma 3.3 , since $\kappa \le e^{\kappa}-1$ and $1-\kappa \le e^{1-\kappa}-1$ for all $\kappa \in [0,1]$, we have

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \le \kappa\psi(\sigma_1)+(1-\kappa)\psi(\sigma_2) \le \left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2).$$

□

Proposition 3.5. *Every p–harmonic exp convex function is p–harmonic h–convex function with $h(\kappa)=(e^{\kappa}-1)$.*

Proof.

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \le \left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2) \le h(\kappa)\psi(\sigma_1)+h(1-\kappa)\psi(\sigma_2).$$

□

Remark 3.6. (i) *If $p=1$ in Proposition 3.5, then as a result we get harmonically convex function, which is introduced by Noor et al. in [25].*
(ii) *If $p=-1$ in Proposition 3.5, then as a result we get h–convex function, which is defined by Varošanec et al. [29].*

Now we make and investigate some examples by way of newly introduced definition.

Example 3.7. *If $\psi(\sigma)=\sigma^{p+1}$, $\forall\sigma\in(0,\infty)$ is p–harmonic convex function, then according to Proposition 3.4, it is a p–harmonic exp convex function.*

Example 3.8. *If $\psi(\sigma)=\frac{1}{\sigma^{2p}}$, $\forall\sigma\in\mathbb{R}\setminus\{0\}$ is p–harmonic convex function, then according to Proposition 3.4, it is a p–harmonic exp convex function.*

Now, we will discuss and investigate some of its algebraic properties.

Theorem 3.9. *Let $\psi,\varphi:[\sigma_1,\sigma_2]\to\mathbb{R}$. If ψ and φ are two p–harmonic exp convex functions, then*
(i) *$\psi+\varphi$ is a p–harmonic exp convex function.*
(ii) *For $c\in\mathbb{R}(c\ge 0)$, $c\psi$ is a p–harmonic exp convex function.*

Proof. (i) Let ψ and φ be a p–harmonic exp convex, then

$$\begin{aligned}(\psi+\varphi)\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) &= \psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)+\varphi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\\ &\le \left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)+\left(e^{\kappa}-1\right)\varphi(\sigma_1)+\left(e^{1-\kappa}-1\right)\varphi(\sigma_2)\\ &= \left(e^{\kappa}-1\right)[\psi(\sigma_1)+\varphi(\sigma_1)]+\left(e^{1-\kappa}-1\right)[\psi(\sigma_2)+\varphi(\sigma_2)]\\ &= \left(e^{\kappa}-1\right)(\psi+\varphi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(\psi+\varphi)(\sigma_2).\end{aligned}$$

(ii) Let ψ be a p–harmonic exp convex, then

$$\begin{aligned}(c\psi)\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) &\le c\left[\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)\right]\\ &= \left(e^{\kappa}-1\right)c\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)c\psi(\sigma_2)\\ &= \left(e^{\kappa}-1\right)(c\psi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(c\psi)(\sigma_2),\end{aligned}$$

which completes the proof. □

Remark 3.10. (i) *If $p=1$ in Theorem 3.9, then as a result we get the $\psi+\varphi$ and $c\psi$ are harmonic exp convex functions.*
(ii) *If $p=-1$ in Theorem 3.9, then as a result we get Theorem 2.1 in [16].*

Theorem 3.11. *Let $\psi:I=[\sigma_1,\sigma_2]\to J$ be p–harmonic convex function and $\varphi:J\to\mathbb{R}$ is non-decreasing and exp convex function. Then the function $\varphi\circ\psi:I=[\sigma_1,\sigma_2]\to\mathbb{R}$ is a p–harmonic exp convex function.*

Proof. $\forall\,\sigma_1,\sigma_2\in I$, and $\kappa\in[0,1]$, we have

$$\begin{aligned}(\varphi\circ\psi)\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) &= \varphi\left(\psi\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\\ &\le \varphi\left(\kappa\psi(\sigma_1)+(1-\kappa)\psi(\sigma_2)\right)\\ &\le \left(e^{\kappa}-1\right)\varphi(\psi(\sigma_1))+\left(e^{1-\kappa}-1\right)\varphi(\psi(\sigma_2))\\ &= \left(e^{\kappa}-1\right)(\varphi\circ\psi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(\varphi\circ\psi)(\sigma_2).\end{aligned}$$

□

Remark 3.12. *(i) In case of being $p=1$, as a result we attain the following new inequality*

$$(\varphi\circ\psi)\left[\frac{\sigma_1\sigma_2}{\kappa\sigma_2+(1-\kappa)\sigma_1}\right]\leq\left(e^{\kappa}-1\right)(\varphi\circ\psi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(\varphi\circ\psi)(\sigma_2).$$

(ii) In case of being $p=-1$, then as a result the above Theorem collapses to the Theorem 2.2 in [16].

Theorem 3.13. *Let $0<\sigma_1<\sigma_2$, $\psi_j:[\sigma_1,\sigma_2]\to[0,+\infty)$ be a class of p–harmonic exp convex functions and $\psi(u)=\sup_j\psi_j(u)$. Then ψ is a p–harmonic exp convex function and $U=\{u\in[\sigma_1,\sigma_2]:\psi(u)<+\infty\}$ is an interval.*

Proof. Let $\sigma_1,\sigma_2\in U$ and $\kappa\in[0,1]$, then

$$\begin{aligned}\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)&=\sup_j\psi_j\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\\&\leq\left(e^{\kappa}-1\right)\sup_j\psi_j(\sigma_1)+\left(e^{1-\kappa}-1\right)\sup_j\psi_j(\sigma_2)\\&=\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)<+\infty,\end{aligned}$$

which completes the proof. □

Remark 3.14. *In case of being $p=-1$ in Theorem 3.13, as a result we get Theorem 2.3 in [16].*

Theorem 3.15. *If $\psi:[\sigma_1,\sigma_2]\to\mathbb{R}$ is a p–harmonic exp convex then ψ is bounded on $[\sigma_1,\sigma_2]$.*

Proof. Let $x\in[\sigma_1,\sigma_2]$ and $L=\max\left\{\psi(\sigma_1),\psi(\sigma_2)\right\}$, then there $\exists\ \kappa\in[0,1]$ such that $x=\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}$. Thus, since $e^{\kappa}\leq e$ and $e^{1-\kappa}\leq e$, we have

$$\begin{aligned}\psi(x)=\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)&\leq\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)\\&\leq\left(e^{\kappa}+e^{1-\kappa}-2\right)\cdot L\\&\leq 2L[e-1]=M,\end{aligned}$$

The above proof clearly shows that ψ is bounded above from M. For bounded below, the readers using the identical concept as in Theorem (2.4) in [16]. □

Remark 3.16. *In case of being $p=-1$, we obtain Theorem 2.4 in [16].*

4. Hermite–Hadamard type inequality via p–harmonic exponential type convexity

The main object of this section is to investigate and prove a new version of Hermite–Hadamard type inequality using p–harmonic exp convexity.

Theorem 4.1. *Let $\psi:[\sigma_1,\sigma_2]\to[0,+\infty)$ be a p–harmonic exp convex function. If $\psi\in L[\sigma_1,\sigma_2]$, then*

$$\frac{1}{2\left(\sqrt{e}-1\right)}\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)\leq\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu\leq\left[\psi(\sigma_1)+\psi(\sigma_2)\right](e-2).$$

Proof. Since ψ is a p–harmonic exp convex function, we have

$$\psi\left(\left[\frac{x^py^p}{\kappa y^p+(1-\kappa)x^p}\right]^{\frac{1}{p}}\right)\leq\left(e^{\kappa}-1\right)\psi(x)+\left(e^{1-\kappa}-1\right)\psi(y),$$

which lead to

$$\psi\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{\frac{1}{p}}\right)\leq\left(\sqrt{e}-1\right)\psi(x)+\left(\sqrt{e}-1\right)\psi(y).$$

Using the change of variables, we get

$$\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)\leq\left(\sqrt{e}-1\right)\times\left\{\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\left(\kappa\sigma_2^p+(1-\kappa)\sigma_1^p\right)}\right]^{\frac{1}{p}}\right)+\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\left(\kappa\sigma_1^p+(1-\kappa)\sigma_2^p\right)}\right]^{\frac{1}{p}}\right)\right\}.$$

Integrating the above inequality with respect to κ on $[0,1]$, we obtain

$$\frac{1}{2\left(\sqrt{e}-1\right)}\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)\leq\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu,$$

which completes the left side inequality.

For the right side inequality, first of all we change the variable of integration by $\nu=\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}$ and using Definition 3.1 for the function ψ, we have

$$\begin{aligned}\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu &= \int_0^1\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)d\kappa\\ &\le \int_0^1\left[\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)\right]d\kappa\\ &= \psi(\sigma_1)\int_0^1\left(e^{t}-1\right)d\kappa+\psi(\sigma_2)\int_0^1\left(e^{1-\kappa}-1\right)d\kappa\\ &= \Big[\psi(\sigma_1)+\psi(\sigma_2)\Big](e-2),\end{aligned}$$

which completes the proof. □

Remark 4.2. (i) *In case of being $p=-1$, then as a result we obtain Theorem 3.1 in [16].*
(ii) *In case of being $p=1$, then as a result we obtain Corollary 1 in [11].*

5. Refinements of Hermite–Hadamard type inequality via p–harmonic exponential type convexity

In this section, in order to prove our main results regarding on some Hermite–Hadamard type inequalities for p–harmonic exp convex function, we need the following lemmas:

Lemma 5.1. . *Let $\psi: I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$, then*

$$\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\int_0^1\frac{\mu(\kappa)}{A_\kappa^{p+1}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)d\kappa,$$

where $A_\kappa=\left[\kappa\sigma_2^p+(1-\kappa)\sigma_1^p\right]^{\frac{1}{p}}$ and $\mu(\kappa)=(1-2\kappa)$.

Proof. Let

$$I=\frac{\sigma_2^p-\sigma_1^p}{2p\sigma_1^p\sigma_2^p}\int_0^1(1-2\kappa)\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{1+\frac{1}{p}}\psi'\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)$$

Using integration by parts

$$\begin{aligned}I&=\frac{\sigma_2^p-\sigma_1^p}{2p\sigma_1^p\sigma_2^p}\left\{\left|\frac{-p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}(1-2\kappa)\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\right|_0^1-\frac{2p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_0^1\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)d\kappa\right\}\\ &=\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx.\end{aligned}$$

□

Lemma 5.2. *[24]. Let $\psi: I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$, then*

$$\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\int_0^1\frac{\mu(\kappa)}{A_\kappa^{p+1}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)d\kappa,$$

where $A_\kappa=\left[\kappa\sigma_2^p+(1-\kappa)\sigma_1^p\right]^{\frac{1}{p}}$ and

$$\mu(\kappa)=\begin{cases}\kappa-\frac{1}{8}, & \text{if } \kappa\in[0,\frac{1}{3})\\ \kappa-\frac{1}{2}, & \text{if } \kappa\in[\frac{1}{3},\frac{2}{3})\\ \kappa-\frac{7}{8}, & \text{if } \kappa\in[\frac{2}{3},1].\end{cases}$$

Theorem 5.3. *Let $\psi : I = [\sigma_1, \sigma_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable function on the I° of I. If $\psi' \in L[\sigma_1, \sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on I, $q \geq 1$, then*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{G_1^{1-\frac{1}{q}}\left[G_2|\psi'(\sigma_1)|^q + G_3|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_1 = \int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa, \quad G_2 = \int_0^1 \frac{|1-2\kappa|(e^\kappa - 1)}{A_\kappa^{1+p}}d\kappa,$$

$$G_3 = \int_0^1 \frac{|1-2\kappa|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}d\kappa.$$

Proof. Using Lemma 5.1, properties of modulus, power mean inequality and p–harmonic exp convexity of the $|\psi'|^q$, we have

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa$$

$$\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{|1-2\kappa|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q + (e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}$$

$$\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{|1-2\kappa|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa + \int_0^1 \frac{|1-2\kappa|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{G_1^{1-\frac{1}{q}}\left[G_2|\psi'(\sigma_1)|^q + G_3|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

which completes the proof. □

Corollary 5.4. *Under the assumptions of Theorem 5.3 with $p = -1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right|$$

$$\leq \frac{(\sigma_2-\sigma_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left\{\left[|\psi'(\sigma_1)|^q + |\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 5.5. *Under the assumptions of Theorem 5.3 with $p = 1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^2}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2-\sigma_1)}{2}\left\{G_1'^{1-\frac{1}{q}}\left[G_2'|\psi'(\sigma_1)|^q + G_3'|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_1' = \int_0^1 \frac{|1-2t|}{A_\kappa^2}d\kappa, \quad G_2' = \int_0^1 \frac{|1-2\kappa|(e^\kappa-1)}{A_\kappa^2}d\kappa,$$

$$G_3' = \int_0^1 \frac{|1-2\kappa|(e^{1-\kappa}-1)}{A_\kappa^2}d\kappa.$$

Theorem 5.6. *Let $\psi : I = [\sigma_1, \sigma_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable function on the I° of I. If $\psi' \in L[\sigma_1, \sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on I, $r, q \geq 1$, $\frac{1}{r} + \frac{1}{q} \geq 1$ then*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_a^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p} \times \left\{G_4^{\frac{1}{r}}\left[G_5|\psi'(\sigma_1)|^q + G_6|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_4 = \int_0^1 |1-2\kappa|^r d\kappa, \quad G_5 = \int_0^1 \frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,$$

$$G_6 = \int_0^1 \frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa.$$

Proof. Using Lemma 5.1, properties of modulus, Hölder's inequality and p–harmonic exp convexity of the $|\psi'|^q$, we have

$$\begin{aligned}\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| &\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa\\ &\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{\frac{1}{r}}\left(\int_0^1\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\\ &\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{\frac{1}{r}}\right.\\ &\quad\times\left.\left(\int_0^1\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}\right\}\\ &= \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{G_4^{\frac{1}{r}}\left[G_5|\psi'(\sigma_1)|^q+G_6|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

which completes the proof. □

Corollary 5.7. *Under the assumptions of Theorem 5.6 with $p=-1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right|\leq\frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{\frac{1}{r}}(e-2)\left(|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\right)^{\frac{1}{q}}.$$

Corollary 5.8. *Under the assumptions of Theorem 5.6 with $p=1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^2}dx\right|\leq\frac{\sigma_1\sigma_2(\sigma_2-\sigma_1)}{2}\left\{G_4'^{\frac{1}{r}}\left[G_5'|\psi'(\sigma_1)|^q+G_6'|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_4'=\int_0^1|1-2\kappa|^r d\kappa,\quad G_5'=\int_0^1\frac{(e^\kappa-1)}{A_\kappa^{2q}}d\kappa,$$

$$G_6'=\int_0^1\frac{(e^{1-\kappa}-1)}{A_\kappa^{2q}}d\kappa.$$

Theorem 5.9. *Let $\psi: I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on I, $q\geq 1$ then*

$$\begin{aligned}&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\ &\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\left\{B_1^{1-\frac{1}{q}}[B_4|\psi'(\sigma_1)|^q+B_5|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right.\\ &\left.+B_2^{1-\frac{1}{q}}[B_6|\psi'(\sigma_1)|^q+B_7|\psi'(\sigma_2)|^q]^{\frac{1}{q}}+B_3^{1-\frac{1}{q}}[B_8|\psi'(\sigma_1)|^q+B_9|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right\},\end{aligned}$$

where

$$B_1=\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{p+1}}d\kappa,\quad B_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{p+1}}d\kappa,\quad B_3=\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{p+1}}d\kappa,$$

$$B_4=\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^\kappa-1)}{A_\kappa^{p+1}}d\kappa,\quad B_5=\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^{1-\kappa}-1)}{A_\kappa^{p+1}}d\kappa,$$

$$B_6=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^\kappa-1)}{A_\kappa^{p+1}}d\kappa,\quad B_7=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^{1-\kappa}-1)}{A_\kappa^{p+1}}d\kappa,$$

$$B_8=\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|(e^\kappa-1)}{A_\kappa^{p+1}}d\kappa,\quad B_9=\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|(e^{1-\kappa}-1)}{A_\kappa^{p+1}}d\kappa.$$

Proof. Using Lemma 5.2, properties of modulus, power mean inequality and p-harmonic exp convexity of the $|\psi'|^q$, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\ &\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left[\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa\right]\end{aligned}$$

$$\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left[\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right]\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}$$
$$\times\left[\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left[\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa+\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_t^{1+p}}dt\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa+\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}\right]$$
$$=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\left\{B_1^{1-\frac{1}{q}}[B_4|\psi'(\sigma_1)|^q+B_5|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right.$$
$$\left.+B_2^{1-\frac{1}{q}}[B_6|\psi'(\sigma_1)|^q+B_7|\psi'(\sigma_2)|^q]^{\frac{1}{q}}+B_3^{1-\frac{1}{q}}[B_8|\psi'(\sigma_1)|^q+B_9|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 5.10. *Under the assumptions of Theorem 5.9 with $p=-1$, we have the following new result*

$$\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\psi\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\psi(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right|$$
$$\leq(\sigma_2-\sigma_1)\left\{\left(\frac{17}{576}\right)\left[0.0069|\psi'(\sigma_1)|^q+0.036|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{0.183}{360}\right)\left[|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}+\left(\frac{17}{576}\right)\left[0.036|\psi'(\sigma_1)|^q+0.0069|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 5.11. *Let $\psi: I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on $I, r, q\geq 1$ and $\frac{1}{r}+\frac{1}{q}\geq 1$ then*

$$\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right|$$

$$\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{10}|\psi'(\sigma_1)|^q+B_{11}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{12}|\psi'(\sigma_1)|^q+B_{13}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{14}|f'(a)|^q dt+B_{15}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right\},$$

where

$$B_{10}=\int_0^{\frac{1}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,\quad B_{11}=\int_0^{\frac{1}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa,$$

$$B_{12}=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,\quad B_{13}=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa,$$

$$B_{14}=\int_{\frac{2}{3}}^{1}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,\quad B_{15}=\int_{\frac{2}{3}}^{1}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa.$$

Proof. Using Lemma 5.2, properties of modulus, Hölder's inequality and p–harmonic exponential convexity of the $|\psi'|^q$, we have

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p} \\
&\times\left[\int_0^{\frac{1}{3}}|\kappa-\frac{1}{8}|\left|\frac{1}{A_\kappa^{1+p}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}|\kappa-\frac{1}{2}|\left|\frac{1}{A_\kappa^{1+p}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{2}{3}}^{1}|\kappa-\frac{7}{8}|\left|\frac{1}{A_\kappa^{1+p}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa\right] \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\left\{\left(\int_0^{\frac{1}{3}}|\kappa-\frac{1}{8}|^r d\kappa\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right. \\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\kappa-\frac{1}{2}|^r d\kappa\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{2}{3}}^{1}|\kappa-\frac{7}{8}|^r d\kappa\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right\}\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\int_0^{\frac{1}{3}}|\kappa-\frac{1}{8}|^r d\kappa\right)^{\frac{1}{r}}\right. \\
&\times\left(\int_0^{\frac{1}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\kappa-\frac{1}{2}|^r d\kappa\right)^{\frac{1}{r}} \\
&\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}|\kappa-\frac{7}{8}|^r d\kappa\right)^{\frac{1}{r}} \\
&\left.\times\left(\int_{\frac{2}{3}}^{1}\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}\right\} \\
&=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_1)|^q d\kappa+\int_0^{\frac{1}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\right. \\
&\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}|\varphi'(\sigma_1)|^q d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}} \\
&\left.\times\left(\int_{\frac{2}{3}}^{1}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_1)|^q d\kappa+\int_{\frac{2}{3}}^{1}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}\right\} \\
&=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{10}|\psi'(\sigma_1)|^q+B_{11}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{12}|\psi'(\sigma_1)|^q+B_{13}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{14}|\psi'(\sigma_1)|^q d\kappa+B_{15}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right\},
\end{aligned}
$$

which completes the proof. □

Corollary 5.12. *Under the assumptions of Theorem 5.11 with $p=-1$, we have the following new result*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\psi\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\psi(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right| \\
&\leq(\sigma_2-\sigma_1)\left[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(0.0623|\psi'(\sigma_1)|^q+0.4372|\psi'(\sigma_2)|^q)^{\frac{1}{q}}+\left(\frac{1}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}0.2188(|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(0.4372|\psi'(\sigma_1)|^q+0.0623|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right].
\end{aligned}
$$

6. Applications

In this section, we recall the following special means of two positive numbers σ_1,σ_2 with $\sigma_1<\sigma_2$:

(1) The arithmetic mean

$$A=A(\sigma_1,\sigma_2)=\frac{\sigma_1+\sigma_2}{2}.$$

(2) The geometric mean

$$G=G(\sigma_1,\sigma_2)=\sqrt{\sigma_1\sigma_2}.$$

(3) The harmonic mean

$$H=H(\sigma_1,\sigma_2)=\frac{2\sigma_1\sigma_2}{\sigma_1+\sigma_2}.$$

(4) The logarithmic mean

$$L = L(\sigma_1,\sigma_2) = \frac{\sigma_2-\sigma_1}{\ln\sigma_2-\ln\sigma_1}.$$

These means have a lot of applications in areas and different type of numerical approximations. However, the following simple relationship is known in the literature.

$$H(\sigma_1,\sigma_2) \le G(\sigma_1,\sigma_2) \le L(\sigma_1,\sigma_2) \le A(\sigma_1,\sigma_2).$$

Proposition 6.1. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H_p(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2^{1-p}-\sigma_1^{1-p}}{1-p}\right) \le A(\sigma_1,\sigma_2)[2e-4]. \tag{6.1}$$

Proof. Taking $\psi(\sigma)=\sigma$ for $\nu>0$ in Theorem 4.1, then inequality (6.1) is easily captured. □

Proposition 6.2. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H_{2p}^{-1}(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2^{\frac{1}{2}-p}-\sigma_1^{\frac{1}{2}-p}}{\frac{1}{2}-p}\right)^{-1} \le A^{-1}(\sqrt{\sigma_1},\sqrt{\sigma_2})[2e-4]. \tag{6.2}$$

Proof. Taking $\psi(\sigma)=\frac{1}{\sqrt{\sigma}}$ for $\sigma>0$ in Theorem 4.1, then inequality (6.2) is easily captured. □

Proposition 6.3. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2-\sigma_1}{L(\sigma_1,\sigma_2)}\right) \le A(\sigma_1^p,\sigma_2^p)[2e-4]. \tag{6.3}$$

Proof. Taking $\psi(\sigma)=\sigma^p$ for $\sigma>0$ in Theorem 4.1, then inequality (6.3) is easily captured. □

Proposition 6.4. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H_p^2(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2^{2-p}-\sigma_1^{2-p}}{2-p}\right) \le A(\sigma_1^2,\sigma_2^2)[2e-4]. \tag{6.4}$$

Proof. Taking $\psi(\sigma)=\sigma^2$ for $\sigma>0$ in Theorem 4.1, then inequality (6.4) is easily captured. □

Proposition 6.5. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}\ln G(\sigma_1,\sigma_2) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{-\ln x}{x^{p+1}}dx \le \ln H_p(\sigma_1^p,\sigma_2^p)[2e-4]. \tag{6.5}$$

Proof. Taking $\psi(\sigma)=-\ln\sigma$ for $\sigma>0$ in Theorem 4.1, then inequality (6.5) is easily captured. □

Proposition 6.6. *Let $0<\sigma_1<\sigma_2$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}e^{H(\sigma_1,\sigma_2)} \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{e^x}{x^{p+1}}dx \le A(e^{\sigma_1},e^{\sigma_2})[2e-4]. \tag{6.6}$$

Proof. Taking $\psi(\sigma)=e^{\sigma}$ for $\sigma>0$ in Theorem 4.1, then inequality (6.6) is easily captured. □

Proposition 6.7. *Let $0<\sigma_1<\sigma_2$. Then we get the following inequality*

$$A(\sin\sigma_1,\sin\sigma_2)[2e-4] \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\sin x}{x^{p+1}}dx \le \frac{1}{2(\sqrt{e}-1)}\sin H_p(\sigma_1,\sigma_2). \tag{6.7}$$

Proof. Taking $\psi(\nu)=\sin(-\nu)$ for $\nu\in(0,\frac{\pi}{2})$ in Theorem 4.1, then inequality (6.7) is easily captured. □

Remark 6.8. *The above discussed means are well–known in literature because these means have fruitful importance and magnificent applications in machine learning, probability, statistics and numerical approximation [4, 8]. But we believe that in the future we will try to find the applications of He Chengtian mean (also called as He Chengtian average), which was introduced by the first time a famous ancient Chinese mathematician He Chengtian [12]. This mean was extended to solve nonlinear oscillators and it is called as He's max–min approach (also called as He's max–min method), which was further developed into a frequency–amplitude formulation for nonlinear oscillators [13, 14].*

7. Conclusion

We have introduced and investigated some algebraic properties of a new class of functions namely p–harmonic exp convex. We showed that our new introduced class of function have some nice properties. New version of Hermite–Hadamard type inequality and an integral identity for the differentiable function are obtained. It is the time to find the applications and importance of these inequalities along with efficient numerical tools and methods. The interesting tools and fruitful ideas of this paper can be extended and generalized on the co-ordinates along with fractional calculus. Further, this new concept will be opening new door of investigations toward fractal integration and differentiations in convexity, preinvexity and fractal image processing. We hope the consequences and techniques of this article will energize and inspire the researcher to explore a more interesting sequel in this area.

Acknowledgements

The author are pleased and wish to express their heartfelt and thanks to the referees for their helpful suggestions to improve the final version of this paper.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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