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Construction of surfaces with constant mean curvature along a timelike curve

Verilen bir timelike eğri boyunca sabit ortalama eğrilikli yüzeyler

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Construction of Surfaces With Constant Mean Curvature Along A Timelike Curve

Highlights

- *Surfaces constructed using a given timelike curve*
- *Constraints for surfaces to have constant mean curvature along the given curve are obtained*
- *The method is illustrated with examples*
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Graphical Abstract

We construct surfaces with constant mean curvature through a given timelike curve.

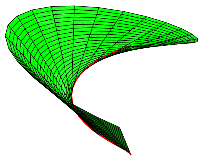

Figure. A surface with constant mean curvature along a given timelike curve (red in colour)

Aim

The aim of this paper is to construct surfaces with constant mean curvature through a given timelike curve

Design &Methodology

We construct surfaces using the Frenet frame of the given timelike curve and obtain conditions.

Originality

The study is original.

Findings

We find constraints on surfaces to have a constant mean curvature along a given timelike curve.

Conclusion

It is possible to obtain surfaces with constant mean curvature along a given timelike curve.

Declaration of Ethical Standards

The author of this article declares that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Construction of Surfaces with Constant Mean Curvature Along a Timelike Curve

Araştırma Makalesi / ResearchArticle

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ABSTRACT

We construct surfaces with constant mean curvature through a given timelike curve. We show that, it is possible to obtain such surfaces for any given timelike curve. The validity of the method supported with illustrative examples.

Keywords: Timelike curve, constant mean curvature surfaces, Minkowski 3-spac

1. INTRODUCTION

The mathematical model of the relativity theory is the Lorentz-Minkowski space time and it is an attractive area for researchers. The trajectory of a moving particle can be represented by a null curve if it travels at the speed of light and by a spacelike or timelike curve if it moves faster or slower than light, respectively.

Another important notion in Lorentz-Minkowski space time is surfaces. We see surfaces almost in every differential geometry book [1-3]. A constant mean curvature surface is a surface whose mean curvature is constant everywhere. It can be physically modeled by a soap bubble. There are several techniques to characterize surfaces. However, the construction of a surface is also an important issue. Current studies on surfaces have focused on finding surfaces with a common special curve [4 - 14]. Recently, Coşanoğlu and Bayram [15] obtained sufficient conditions for surfaces with constant mean curvature through a given curve in Euclidean 3-space. In the present paper, analogous to Coşanoğlu and Bayram [15], we obtain parametric surfaces with constant mean curvature through a given timelike curve. We present conditions for these types of surfaces. The method is validated with several examples.

2. MATERIAL and METHOD

The real vector space R^3 equipped with the metric tensor

$$\langle X,Y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

is called the Minkowski 3-space and denoted by R_1^3, where $X=(x_1,x_2,x_3)$, $Y=(y_1,y_2,y_3)\in R^3$ [1]. The Lorentzian vectorial product is defined by

$$X\times Y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2).$$

A vector $X\in R_1^3$ is called timelike, spacelike or lightlike (null) if

$$\begin{cases} \langle X,X\rangle < 0, \\ \langle X,X\rangle > 0 \text{ or } X=\vec{0}, \\ \langle X,X\rangle = 0, \end{cases}$$

respectively. Similarly, a curve in R_1^3 is called a timelike, spacelike or lightlike curve if its tangent vector field is always timelike, spacelike or lightlike, respectively.

The Frenet frame of a curve α is denoted by $\{T(s),N(s),B(s)\}$, where T, N and B are the tangent vector field, the principal normal vector field and the binormal vector field, respectively.

Assume that α is a unit speed timelike curve with curvature κ and torsion τ. Hence, tangent vector field is a timelike vector field, principal and binormal vector fields are spacelike. For these vectors, we have

$$T\times N = -B, \quad N\times B = T, \quad B\times T = -N.$$

The binormal vector field $B(s)$ is the unique spacelike unit vector field perpendicular to the timelike plane $\{T(s),N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T,N,B\}$ has the same orientation as R_1^3. Then, Frenet formulas are given by [16]

$$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = -\tau N.$$

The mean curvature of the surface $P(s,t)$ is given as

$$H(s,t) = -\frac{\det(P_s,P_t,P_{ss})G - 2\det(P_s,P_t,P_{st})F + \det(P_s,P_t,P_{tt})E}{2(EG-F^2)^{\frac{3}{2}}},$$

where E, F, G are the coefficients of the first fundamental form of the surface $P(s,t)$ [17].

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3. CONSTRUCTION OF SURFACES WITH CONSTANT MEAN CURVATURE ALONG A TIMELIKE CURVE

Let $\alpha(s)$, $L_1 \le s \le L_2$ be a timelike unit speed regular curve with curvature $\kappa(s)$ and torsion $\tau(s)$. Also, assume that $\alpha''(s) \neq 0$, $\forall s$. Parametric surfaces possessing $\alpha(s)$ can be written as

$$P(s,t)=\alpha(s)+u(s,t)T(s) + v(s,t)N(s)+w(s,t)B(s), \quad (1)$$

$L_1 \le s \le L_2$, $T_1 \le t \le T_2$, where $\{T(s),N(s),B(s)\}$ is the Frenet frame of $\alpha(s)$. C^2 functions $u(s,t)$, $v(s,t)$, $w(s,t)$ are called marching-scale functions. Observe that, choosing different marching-scale functions yields different surfaces along the curve $\alpha(s)$.

To simplify the calculations, we suppose that the curve $\alpha(s)$ is a parameter curve on the surface $P(s,t)$ in Eqn. (1). So, we have

$$u(s,t_0)= v(s,t_0)= w(s,t_0)\equiv 0,$$

for some $t_0 \in [T_1,T_2]$.

The mean curvature of the surface $P(s,t)$ is given as

$$H(s,t)=-\frac{\det(P_s,P_t,P_{ss})G-2\det(P_s,P_t,P_{st})F+\det(P_s,P_t,P_{tt})E}{2\left(EG-F^2\right)^{\frac{3}{2}}},$$

where E,F,G are the coefficients of the first fundamental form of the surface $P(s,t)$ [17] .

We make the following calculations required for the mean curvature.

$$\begin{aligned}P_s(s,t)&=\big(1+u_s(s,t)+\kappa(s)v(s,t)\big)T(s)\\&+\big(\kappa(s)u(s,t)+v_s(s,t)-\tau(s)w(s,t)\big)N(s)\\&+\big(\tau(s)v(s,t)+w_s(s,t)\big)B(s),\end{aligned}$$

$$P_t(s,t)=u_t(s,t)T(s)+v_t(s,t)N(s)+w_t(s,t)B(s),$$

$$P_s(s,t_0)=T(s),$$

$$P_t(s,t_0)=u_t(s,t_0)T(s)+v_t(s,t_0)N(s)+w_t(s,t_0)B(s),$$

$$P_{ss}(s,t_0)=\kappa(s)N(s),$$

$$\begin{aligned}P_{st}(s,t_0)=P_{ts}(s,t_0)&=\big(u_{ts}(s,t_0)+\kappa(s)v_t(s,t_0)\big)T(s)\\&+\big(\kappa(s)u_t(s,t_0)+v_{ts}(s,t_0)-\tau(s)w_t(s,t_0)\big)N(s)\\&+\big(\tau(s)v_t(s,t_0)+w_{ts}(s,t_0)\big)B(s),\end{aligned}$$

$$P_{tt}(s,t_0)=u_{tt}(s,t_0)T(s)+v_{tt}(s,t_0)N(s)+w_{tt}(s,t_0)B(s),$$

$$\det(P_s,P_t,P_{ss})(s,t_0)=-\kappa(s)w_t(s,t_0),$$

$$\begin{aligned}\det(P_s,P_t,P_{st})(s,t_0)&=\big[v_t\big(v_t\,\tau(s)+w_{ts}\big)\\&-w_t\big(u_t\,\kappa(s)+v_{ts}-\tau(s)w_t\big)\big](s,t_0),\end{aligned}$$

$$\begin{aligned}\det\big(P_s(s,t_0),P_t(s,t_0),P_{tt}(s,t_0)\big)&=v_t(s,t_0)w_{tt}(s,t_0)\\&-w_t(s,t_0)v_{tt}(s,t_0),\end{aligned}$$

where subscript denotes the partial derivative with respect to the parameter in question. Hence, we have the mean curvature of the surface $P(s,t)$ in Eqn. (1) along the curve $\alpha(s)$ as

$$\begin{aligned}H(s,t_0)=\frac{1}{2\left(v_t^2+w_t^2\right)^{\frac{3}{2}}}\Big[&\kappa w_t\left(-u_t^2+v_t^2+w_t^2\right)+v_tw_{tt}-w_tv_{tt}\\&-2u_t\big[w_t\left(\kappa u_t+v_{ts}-\tau w_t\right)-v_t\left(v_t\tau+w_{ts}\right)\big]\Big](s,t_0).\end{aligned}$$

Theorem : The surface $P(s,t)$ in Eqn. (1) has constant mean curvature along the timelike curve $\alpha(s)$ if one of the following conditions is satisfied:

i)
$$\begin{cases}u_t(s,t_0)= v_t(s,t_0)\neq 0\\ u(s,t_0)=v(s,t_0)=w(s,t_0)=w_t(s,t_0)=w_{tt}(s,t_0)\equiv 0\\ \tau(s)=\text{constant},\end{cases}$$

ii)
$$\begin{cases}u_t(s,t_0)=w_t(s,t_0)\neq 0\\ u(s,t_0)=v(s,t_0)=w(s,t_0)=v_t(s,t_0)=v_{tt}(s,t_0)\equiv 0\\ \kappa(s)-\tau(s)=\text{constant},\end{cases}$$

iii)
$$\begin{cases}u_t(s,t_0)=v_t(s,t_0)=w_t(s,t_0)\neq 0\\ u(s,t_0)=v(s,t_0)=w(s,t_0)\equiv 0\\ 4\tau(s)-\kappa(s)=\text{constant},\end{cases}$$

iv)
$$\begin{cases}v_t(s,t_0)\neq 0\\ u(s,t_0)=v(s,t_0)=w(s,t_0)=u_t(s,t_0)\equiv 0\\ w_t(s,t_0)=w_{tt}(s,t_0)\equiv 0,\end{cases}$$

v)
$$\begin{cases}v_t(s,t_0)=w_t(s,t_0)\neq 0\\ u(s,t_0)=v(s,t_0)=w(s,t_0)=u_t(s,t_0)\equiv 0.\\ \kappa(s)=\text{constant}.\end{cases}$$

Example : In this example, we construct surfaces with constant mean curvature along a given timelike curve. The unit speed timelike curve $\alpha(s)=\left(\frac{5}{3}s,\frac{4}{9}\cos(3s),\frac{4}{9}\sin(3s)\right)$ has the following Frenet apparatus

$$\begin{cases} T(s)=\left(\frac{5}{3},-\frac{4}{3}\sin(3s),\frac{4}{3}\cos(3s)\right), \\ N(s)=(0,-\cos(3s),-\sin(3s)), \\ B(s)=\left(-\frac{4}{3},\frac{5}{3}\sin(3s),-\frac{5}{3}\cos(3s)\right), \\ \kappa(s)=4,\ \ \tau(s)=5. \end{cases}$$

Choosing marching-scale functions as $u(s,t)=v(s,t)=t,\ \ w(s,t)\equiv 0$ and $t_0=0,$ Theorem (i) is satisfied and we obtain the surface

$$P_1(s,t)=\left(\frac{5}{3}(s+t),\frac{4}{9}\cos(3s)-\frac{4}{3}t\sin(3s),\left(\frac{4}{9}-t\right)\sin(3s)+\frac{4}{3}t\cos(3s)\right),$$

$-1\le s\le 1,\ \ 0\le t\le 1$ with constant mean curvature $H(s,0)=5$ along the timelike curve $\alpha(s)$ (Figure 1) .

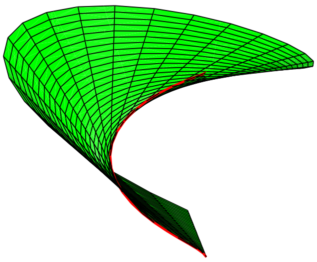

Figure 1. $P_1(s,t)$ with constant mean curvature along the timelike curve $\alpha(s)$.

For the same curve, if we choose marching-scale functions as $u(s,t)=w(s,t)=t,\ v(s,t)\equiv 0$ and $t_0=0,$ Theorem (ii) is satisfied and we obtain the surface

$$P_2(s,t)=\left(\frac{5s+t}{3},\frac{4}{9}\cos(3s)+\frac{t}{3}\sin(3s),\frac{4}{9}\sin(3s)-\frac{t}{3}\cos(3s)\right),$$

$-1\le s\le 1,\ \ 0\le t\le 1$ with constant mean curvature $H(s,0)=1$ along the timelike curve $\alpha(s)$ (Figure 2) .

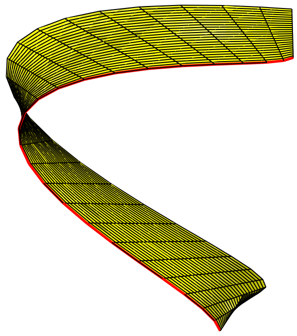

Figure 2. $P_2(s,t)$ with constant mean curvature along the timelike curve $\alpha(s)$.

Choosing marching-scale functions as $u(s,t)=v(s,t)=w(s,t)=t$ and $t_0=0,$ Theorem (iii) is satisfied and we obtain the surface

$$P_3(s,t)=\left(\frac{5s+t}{3},\left(\frac{4}{9}-t\right)\cos(3s)+\frac{t}{3}\sin(3s),\left(\frac{4}{9}-t\right)\sin(3s)-\frac{t}{3}\cos(3s)\right),$$

$-1\le s\le 1,\ \ 0\le t\le 1$ with constant mean curvature $H(s,0)=2\sqrt{2}$ along the timelike curve $\alpha(s)$ (Figure 3) .

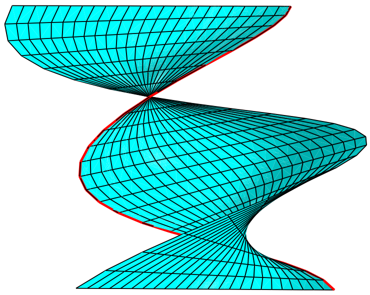

Figure 3. $P_3(s,t)$ with constant mean curvature along the timelike curve $\alpha(s)$.

If we choose $u(s,t)=w(s,t)\equiv 0,\ \ v(s,t)=t$ and $t_0=0,$ Theorem (iv) is satisfied and we obtain the surface

$$P_4(s,t)=\left(\frac{5s}{3},\left(\frac{4}{9}-t\right)\cos(3s),\left(\frac{4}{9}-t\right)\sin(3s)\right),$$

$-1\le s\le 1,\ 0\le t\le 1$ with constant mean curvature $H(s,0)=0$ along the timelike curve $\alpha(s)$ (Figure 4).

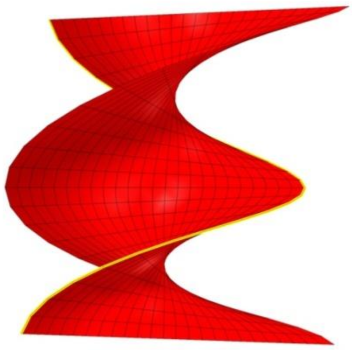

Figure 4. $P_4(s,t)$ with constant mean curvature along the timelike curve $\alpha(s)$.

Letting $u(s,t)\equiv 0,\ v(s,t)=w(s,t)=t$ and $t_0=0$, Theorem (v) is satisfied and we obtain the surface

$$P_5(s,t)=\left(\frac{5s-4t}{3},\left(\frac{4}{9}-t\right)\cos(3s)+\frac{5t}{3}\sin(3s),\left(\frac{4}{9}-t\right)\sin(3s)-\frac{5t}{3}\cos(3s)\right)$$

$-1\le s\le 1,\ 0\le t\le 1$ with constant mean curvature $H(s,0)=\sqrt{2}$ along the timelike curve $\alpha(s)$ (Figure 5) .

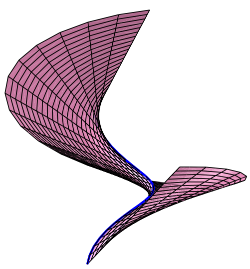

Figure 5. $P_5(s,t)$ with constant mean curvature along the timelike curve $\alpha(s)$.

6. CONCLUSION

In this study, we showed that it is possible to construct surfaces with constant mean curvature along a given timelike curve.

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DECLARATION OF ETHICAL STANDARDS

The author of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHOR'S CONTRIBUTION

Ergin BAYRAM : Handled the related work and wrote the paper.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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