Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 2, 258–269.
https://doi.org/10.31197/atnaa.871010
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

An inexact operator splitting method for general mixed variational inequalities

Abdellah Bnouhachem[a]

[a] *Equipe MAISI, ENSA, BP 1136, Ibn Zohr University, Agadir, Morocco.*

Abstract

The present paper aims to deal with an inexact implicit method with a variable parameter for general mixed variational inequalities in the setting of real Hilbert spaces. Under standard assumptions, the global convergence of the proposed method is proved. Numerical example is presented to illustrate the proposed method and convergence result. The results and method presented in this paper generalize, extend and unify some known results in the literature.

Keywords: General mixed variational inequalities implicit methods resolvent operator inexact methods self-adaptive rules.
2010 MSC: 65K10, 90C25, 90C30.

1. Introduction

The theory of variational inequality problems has grown enormously in various branches of the pure and applied sciences, it has been widely studied in the literature. It provides a framework for many problems in finance, economics, networks analysis, optimization and others; see for example [1–26]. A useful and important generation of variational inequalities is in the general mixed variational inequality, denoted by GMVI, is to find $u^* \in H$ such that

$$\langle F(u^*), h(v) - h(u^*)\rangle + \varphi(h(v)) - \varphi(h(u^*)) \geq 0, \qquad \forall h(v) \in H, \tag{1}$$

where H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively and $F, h : H \to H$ are two nonlinear operators. Let $\partial\varphi$ denotes the subdifferential of function φ, where $\varphi : H \to R \cup \{\infty\}$ is a proper convex lower semi continuous function on H.
GMVI includes many important optimization problems as special cases:

Email address: babedallah@yahoo.com (Abdellah Bnouhachem)

Received January 30, 2021, Accepted February 23, 2022, Online February 26, 2022

- If K is closed convex set in H and $\varphi(v) = I_K(v)$ for all $v \in H$, where I_K is the indicator function of K defined by

$$I_K(v) = \begin{cases} 0, & \text{if} \quad v \in K; \\ \infty, & \text{otherwise.} \end{cases}$$

  Then the problem (1) is equivalent to the general variational inequality (GVI): find $u^* \in H$ such that $h(u^*) \in K$ and

$$\langle F(u^*), h(v) - h(u^*) \rangle \geq 0, \qquad \forall h(v) \in K. \tag{2}$$

  GVI has found many efficient applications in various application domains. We refer the readers to [3, 13, 19, 22] for some review papers.

- If $h = I$, then the problem (2) collapses to the classical variational inequality problem: find $u^* \in H$ such that

$$\langle F(u^*), v - u^* \rangle \geq 0, \qquad \forall v \in K, \tag{3}$$

  Variational inequality problem has been studied and treated in detail in several kinds of research due to its important role on the development of many problems; see for example [7, 8, 10, 14, 23].

Many researchers have concentrated on the development of GMVI on theoretical analysis, practical applications and algorithmic designs. For theoretical analysis, Lions and Stampacchia [17], Glowinski *et al.* [12] studied the existence of solution for GMVI. In general, the intrinsic complexity makes it impractical to find analytic solutions of GMVI. Therefore, it is particularly useful to design numerical algorithms approaching the solution set of GMVI. Using the resolvent operator technique, various iterative methods [1, 2, 4, 5, 9, 16, 20, 21, 24, 25, 26] have been well demonstrated in the literature to be very efficient for solving different scenarios of (1).

It is well-known ([20]) that u^* is solution of (1) if and only if $u^* \in H$ satisfies the relation:

$$h(u^*) = J_\varphi[h(u^*) - \alpha F(u^*)], \tag{4}$$

where $\alpha > 0$ and $J_\varphi = (I + \alpha\partial\varphi)^{-1}$ is the resolvent operator.
It is clear that u is solution of (1) if and only if u is a zero point of the function

$$g(u, \alpha) := h(u) - J_\varphi[h(u) - \alpha F(u)]. \tag{5}$$

Based on the Douglas-Peaceman-Rachford-Varga operator splitting (DPRV), Bnouhachem [4] proposed an implicit method with a variable parameter for solving (1). For a given u^k and $\beta \in (0, 2)$, the new iterative is obtained via solving the following system of nonlinear equations:

$$\Lambda_k(u) = 0, \tag{6}$$

where

$$\Lambda_k(u) := h(u) + \alpha_k F(u) - h(u^k) - \alpha_k F(u^k) + \beta g(u^k, \alpha_k). \tag{7}$$

Since in many cases solving problem (6) exactly is still too computationally expensive to obtain an exact solution. The more practical strategy is to get an approximate solution of (6) subject to some inexactness criteria. Obviously, variant inexactness criteria lead to different numerical algorithms. For example, Zeng and Yao [26] have presented an inexact implicit method for solving GMVI. For given u^k and $\alpha_k > 0$, the new iterate u^{k+1} satisfies the following condition

$$\|\Lambda_k(u^{k+1})\| \leq \varrho_k,$$

where ϱ_k is a nonnegative sequence satisfying $\sum_{k=0}^{\infty} \varrho_k < \infty$. Bnouhachem [4] proposed another criteria which it is more relax than that in [26]. For given u^k and $\alpha_k > 0$, the new iterate u^{k+1} satisfies the following condition

$$\|\Lambda_k(u^{k+1})\| \leq \varrho_k \|g(u^k, \alpha_k)\|,$$

where ϱ_k satisfies $\sum_{k=0}^{\infty} \varrho_k^2 < \infty$.

Based on the work of He *et al.* [15], Li and Bnouhachem [18] proposed another inexact method for solving (3), where the new iteration is generated via the following recursion:

$$\|\Lambda_k(u^{k+1})\| \leq \varrho_k \|e(u^k, \alpha_k) - e(u^{k+1}, \alpha_k)\|,$$

with

$$\sup \varrho_k < \frac{2-\beta}{2}, \quad \beta \in (0,2)$$

and

$$e(u, \alpha) := u - P_K[u - \alpha F(u)].$$

Latter, Z. Ge *et al.* [11] proposed an inexact operator splitting method for solving (3). At each iteration, it needs to find an approximate solution satisfying

$$\|\Lambda_k(u^{k+1})\|^2 \leq \varrho_k \alpha_k \langle F(u^{k+1}) - F(u^k), u^{k+1} - u^k \rangle,$$

with

$$\sup \varrho_k < 2 - \beta, \quad \beta \in (0,2).$$

In this paper, motivated by the work of Li and Bnouhachem [18] and Z. Ge *et al.* [11], and by the recent work going in this direction, we introduce and analyze an inexact operator splitting method for solving (1), where the new iteration is generated via the following recursion:

$$\begin{aligned} \|\Lambda_k(u^{k+1})\|^2 \quad &\leq \quad \varrho_k^2 \|g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\|^2 \\ &\quad + \varrho_k \alpha_k \langle F(u^{k+1}) - F(u^k), h(u^{k+1}) - h(u^k) \rangle, \end{aligned} \tag{8}$$

with

$$\sup \varrho_k < \frac{2-\beta}{2}, \quad \beta \in (0,2) \quad \text{and} \quad \liminf_{k\to\infty} (2 - \beta - 2\varrho_k) > 0. \tag{9}$$

Under appropriate conditions we derive the strong convergence results for this method. Preliminary numerical experiments are included to verify the theoretical assertions of the proposed method. Since the general mixed variational inequality problem includes the mixed variational inequality problem, the general variational inequality problem and the variational inequality problem as special cases, results presented in this paper continue to hold for these problems.

2. Preliminaries

This section states some preliminaries that are useful later. First, we need fundamental lemmas that are useful in the consequent analysis.

Lemma 2.1. *([6]) For a given $w \in H$, the inequality*

$$\langle w - z, z - v \rangle + \alpha\varphi(v) - \alpha\varphi(z) \geq 0, \quad \forall v \in H$$

holds if and only if $z = J_\varphi(w)$, where $J_\varphi = (I + \alpha\partial\varphi)^{-1}$ is the resolvent operator.

It follows from Lemma 2.1 that

$$\langle w - J_\varphi(w), J_\varphi(w) - v\rangle + \alpha\varphi(v) - \alpha\varphi(J_\varphi(w)) \geq 0, \qquad \forall v, w \in H \tag{10}$$

If φ is the indicator function of a closed convex set $K \subset H$, then the resolvent operator $J_\varphi(\cdot)$ reduces to the projection operator $P_K[\cdot]$ [20]. It is well-known that J_φ is nonexpansive i.e.,

$$\|J_\varphi(u) - J_\varphi(v)\| \leq \|u - v\|, \qquad \forall u, v \in H.$$

Lemma 2.2. *([1]) For all $u \in H$ and $\tilde{\alpha} > \alpha > 0$, it holds that*

$$\|g(u, \tilde{\alpha})\| \geq \|g(u, \alpha)\| \tag{11}$$

and

$$\frac{\|g(u, \alpha)\|}{\alpha} \geq \frac{\|g(u, \tilde{\alpha})\|}{\tilde{\alpha}}. \tag{12}$$

Lemma 2.3. *([18]) Let $\{b_k\}_{k=1}^{\infty}$ be a positive series and $b_k \in (0,1)$ for all k. If $\prod_{k=1}^{\infty}(1 - b_k) > 0$, then*

1. *$\sum_{k=1}^{\infty} b_k < \infty$ and thus $\lim_{k\to\infty} b_k = 0$;*
2. *$\prod_{k=1}^{\infty}(1 + tb_k) < \infty$ for any $t > 0$.*

In what follows we always assume that , h is homeomorphism on H, i.e., h is bijective, continuous and h^{-1} is continuous, and F is continuous and h-monotone operator on H i.e., $\langle F(u') - F(u), h(u') - h(u)\rangle \geq 0, \forall u', u \in H$ monotone, and the solution of (1) denoted by S^*, is nonempty.

3. The proposed method

Now, we introduce the inexact operator splitting method for solving (1). Let $\{\nu_k\}$ a non-negative sequence $\{\nu_k\}$ satisfying

$$\sum_{k=0}^{\infty} \nu_k < \infty. \tag{13}$$

Algorithm 3.1.

Step 0. Given $\epsilon > 0, \beta \in (0,2), \alpha_0 > 0$, $\delta \in (0,1)$, $\mu \in [0.5, 1)$, $\rho > 0$ $u^0, x^0 \in H$, and a non-negative sequence $\{\varrho_k\}$ satisfying

$$\sup \varrho_k < \frac{2-\beta}{2} \quad \text{and} \quad \liminf_{k\to\infty}(2 - \beta - 2\varrho_k) > 0.$$

Set $k = 0$ and $i = 0$.

Step 1. If $\|g(u^k, \rho)\| < \epsilon$, then stop, otherwise

Step 2.1. If $\|\Lambda_k(x^i)\|^2 \leq \varrho_k^2\|g(x^i, \rho_k) - g(u^k, \rho_k)\|^2 + \varrho_k\rho_k\langle F(x^i) - F(u^k), h(x^i) - h(u^k)\rangle$, set $u^{k+1} = x^i$ and then stop. Otherwise, go to Step 2.2.

Step 2.2. Find the smallest nonnegative integer l_i, such that $\rho_i = \mu^{l_i}\rho$ and

$$x^{i+1} = x^i - \rho_i\Lambda_k(x^i) \tag{14}$$

satisfies

$$s_i := \frac{\rho_i\|\Lambda_k(x^i) - \Lambda_k(x^{i+1})\|^2}{(x^i - x^{i+1})^T(\Lambda_k(x^i) - \Lambda_k(x^{i+1}))} \leq 2 - \delta. \tag{15}$$

Step 2.3. (Adjust ρ for the next step to avoid too small improvement)

$$\rho_i = \begin{cases} \rho_i * 1.5 & \text{if } s_i \le 0.5, \\ \rho_i & \text{otherwise.} \end{cases}$$

Set $i = i + 1$, and go to Step 2.1.

Step 3. Choose $\alpha_{k+1} \in [\frac{1}{1+\nu_k}\alpha_k, (1+\nu_k)\alpha_k]$ according to some self-adaptive rule, set $k =: k+1$, and go to Step 1.

Remark 3.1. *It follows from $\nu_k > 0$ and (13) that $\prod_{k=0}^{\infty}(1+\nu_k) < \infty$. Denote*

$$D_\nu := \prod_{k=0}^{\infty}(1+\nu_k).$$

Then, $\alpha_k \in [\frac{1}{D_\nu}\alpha_0, D_\nu\alpha_0]$ is bounded and

$$\alpha_l := \inf_k\{\alpha_k\} > 0 \qquad \text{and} \qquad \alpha_u := \sup_k\{\alpha_k\} < \infty.$$

We need the following lemmas to analyze the convergence for the proposed method.

Lemma 3.2. *Let $\{u^k\}$ be the sequence generated by Algorithm 3.1. Then for any $u^* \in S^*$ and $k > 0$, we have*

$$\begin{aligned} &\langle \alpha_k[F(u^k) - F(u^{k+1})], h(u^k) - h(u^{k+1})\rangle + \|g(u^k,\alpha_k) - g(u^{k+1},\alpha_k)\|^2 \\ &\le \langle g(u^k,\alpha_k) - g(u^{k+1},\alpha_k), h(u^k) - h(u^{k+1}) + \alpha_k[F(u^k) - F(u^{k+1})]\rangle \end{aligned} \tag{16}$$

and

$$\|g(u^k,\alpha_k)\|^2 \le \langle g(u^k,\alpha_k), h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\rangle. \tag{17}$$

Proof: Setting $w := h(u^k) - \alpha_k F(u^k)$ and $v := J_\varphi[h(u^{k+1}) - \alpha_k F(u^{k+1})]$ in (10), we get

$$\begin{aligned} &\langle h(u^k) - \alpha_k F(u^k) - J_\varphi[h(u^k) - \alpha_k F(u^k)], J_\varphi[h(u^k) - \alpha_k F(u^k)] - J_\varphi[h(u^{k+1}) - \alpha_k F(u^{k+1})]\rangle \\ &\quad + \alpha_k\varphi(J_\varphi[h(u^{k+1}) - \alpha_k F(u^{k+1})]) - \alpha_k\varphi(J_\varphi[h(u^k) - \alpha_k F(u^k)]) \ge 0, \end{aligned} \tag{18}$$

putting $w := h(u^{k+1}) - \alpha_k F(u^{k+1})$ and $v := J_\varphi[h(u^k) - \alpha_k F(u^k)]$ in (10), we have

$$\begin{aligned} &\langle h(u^{k+1}) - \alpha_k F(u^{k+1}) - J_\varphi[h(u^{k+1}) - \alpha_k F(u^{k+1})], \\ &\quad J_\varphi[h(u^{k+1}) - \alpha_k F(u^{k+1})] - J_\varphi[h(u^k) - \alpha_k F(u^k)]\rangle \\ &+ \alpha_k\varphi(J_\varphi[h(u^k) - \alpha_k F(u^k)]) - \alpha_k\varphi(J_\varphi[h(u^{k+1}) - \alpha_k F(u^{k+1})]) \ge 0. \end{aligned} \tag{19}$$

Analogously, we have the following inequalities:

$$\begin{aligned} &\langle h(u^*) - \alpha_k F(u^*) - J_\varphi[h(u^*) - \alpha_k F(u^*)], J_\varphi[h(u^*) - \alpha_k F(u^*)] - J_\varphi[h(u^k) - \alpha_k F(u^k)]\rangle \\ &\quad + \alpha_k\varphi(J_\varphi[h(u^k) - \alpha_k F(u^k)]) - \alpha_k\varphi(J_\varphi[h(u^*) - \alpha_k F(u^*)]) \ge 0 \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\langle h(u^k) - \alpha_k F(u^k) - J_\varphi[h(u^k) - \alpha_k F(u^k)], J_\varphi[h(u^k) - \alpha_k F(u^k)] - J_\varphi[h(u^*) - \alpha_k F(u^*)]\rangle \\ &\quad + \alpha_k\varphi(J_\varphi[h(u^*) - \alpha_k F(u^*)]) - \alpha_k\varphi(J_\varphi[h(u^k) - \alpha_k F(u^k)]) \ge 0, \end{aligned} \tag{21}$$

Adding (18) and (19), and using the definition of $g(u, \alpha)$, we get

$$\langle g(u^k, \alpha_k) - g(u^{k+1}, \alpha_k) - \alpha_k[F(u^k) - F(u^{k+1})], h(u^k) - h(u^{k+1}) + g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\rangle \geq 0,$$

i.e.,

$$\begin{aligned}&\langle \alpha_k[F(u^k) - F(u^{k+1})], h(u^k) - h(u^{k+1})\rangle + \|g(u^k, \alpha_k) - g(u^{k+1}, \alpha_k)\|^2 \\ &\leq \langle g(u^k, \alpha_k) - g(u^{k+1}, \alpha_k), h(u^k) - h(u^{k+1}) + \alpha_k[F(u^k) - F(u^{k+1})]\rangle\end{aligned}$$

and the first conclusion is proved. We now establish the proof of the second assertion. Adding (20) and (21), we have

$$\begin{aligned}&\langle \alpha_k[F(u^k) - F(u^*)], h(u^k) - h(u^*)\rangle + \|g(u^k, \alpha_k) - g(u^*, \alpha_k)\|^2 \\ &\leq \langle g(u^k, \alpha_k) - g(u^*, \alpha_k), h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\rangle.\end{aligned}$$

Since $g(u^*, \alpha_k) = 0$, we obtain

$$\begin{aligned}&\langle \alpha_k[F(u^k) - F(u^*)], h(u^k) - h(u^*)\rangle + \|g(u^k, \alpha_k)\|^2 \\ &\leq \langle g(u^k, \alpha_k), h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\rangle. \end{aligned} \tag{22}$$

Recall the h-monotonicity of F and (22), we obtain the inequality (17) and complete the proof. □

Lemma 3.3. *Let $\{u^k\}$ be the sequence generated by Algorithm 3.1, we have*

$$\begin{aligned}\|g(u^{k+1}, \alpha_k)\|^2 \leq \|g(u^k, \alpha_k)\|^2 - \frac{(2 - \beta - 2\varrho_k)}{\varrho_k}\|g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\|^2 \\ -\frac{\alpha_k}{\beta}\langle F(u^k) - F(u^{k+1}), h(u^k) - h(u^{k+1})\rangle.\end{aligned} \tag{23}$$

Proof: It follows from (16) and the definition of $\Lambda_k(u)$ that

$$\begin{aligned}&\langle \alpha_k[F(u^k) - F(u^{k+1})], h(u^k) - h(u^{k+1})\rangle + \|g(u^k, \alpha_k) - g(u^{k+1}, \alpha_k)\|^2 \\ &\qquad \leq \langle g(u^k, \alpha_k) - g(u^{k+1}, \alpha_k), \beta g(u^k, \alpha_k) - \Lambda_k(u^{k+1})\rangle.\end{aligned}$$

Then

$$\begin{aligned}&\langle g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k), \beta g(u^k, \alpha_k)\rangle \leq \langle g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k), \Lambda_k(u^{k+1})\rangle \\ &-\langle \alpha_k[F(u^k) - F(u^{k+1})], h(u^k) - h(u^{k+1})\rangle - \|g(u^k, \alpha_k) - g(u^{k+1}, \alpha_k)\|^2.\end{aligned} \tag{24}$$

Using (24), we get

$$\begin{aligned}\|g(u^{k+1}, \alpha_k)\|^2 &= \|g(u^k, \alpha_k) + (g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k))\|^2 \\ &= \|g(u^k, \alpha_k)\|^2 + 2\langle g(u^k, \alpha_k), g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\rangle \\ &\quad +\|g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\|^2 \\ &\leq \|g(u^k, \alpha_k)\|^2 + \frac{2}{\beta}\langle g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k), \Lambda_k(u^{k+1})\rangle \\ &\quad -\frac{(2 - \beta)}{\beta}\|g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\|^2 \\ &\quad -\frac{2\alpha_k}{\beta}\langle F(u^k) - F(u^{k+1}), h(u^k) - h(u^{k+1})\rangle.\end{aligned} \tag{25}$$

By using Cauchy-Schwarz inequality and (8), we have

$$\begin{aligned}&2\langle g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k), \Lambda_k(u^{k+1})\rangle \leq \varrho_k\|g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\|^2 + \frac{\|\Lambda_k(u^{k+1})\|^2}{\varrho_k} \\ &\leq 2\varrho_k\|g(u^{k+1}, \alpha_k) - g(u^k, \alpha_k)\|^2 + \alpha_k\langle F(u^k) - F(u^{k+1}), h(u^k) - h(u^{k+1})\rangle.\end{aligned} \tag{26}$$

Combining (25) and (26), we can get the assertion of this lemma. □

We can assume $\|g(u^k,\alpha_k)\| \neq 0$, otherwise u^k is a solution. We define

$$\chi_k^2 = \frac{\varrho_k^2\|g(u^{k+1},\alpha_k) - g(u^k,\alpha_k)\|^2 + \varrho_k\alpha_k\langle F(u^k) - F(u^{k+1}), h(u^k) - h(u^{k+1})\rangle}{\|g(u^k,\alpha_k)\|^2}. \tag{27}$$

It follows from (8) that

$$\|\Lambda_k(u^{k+1})\|^2 \leq \chi_k^2\|g(u^k,\alpha_k)\|^2. \tag{28}$$

4. Convergence of the proposed method

In this section, we prove the global convergence for the proposed method. Before proceeding, we need the following results, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method.

Lemma 4.1. *Let $\{u^k\}$ be the sequence generated by Algorithm 3.1 and $\{\chi_k\}$ be defined by (27), we have*

$$\sum_{k=0}^{\infty}\chi_k^2 < \infty, \quad \lim_{k\to\infty}\chi_k^2 = 0.$$

Proof: It follows from Lemma 2.2 and $\alpha_{k+1} \leq (1+\nu_k)\alpha_k$ that

$$\|g(u^{k+1},\alpha_{k+1})\|^2 \leq \|g(u^{k+1},(1+\nu_k)\alpha_k)\|^2 \quad \text{and} \quad \frac{\|g(u^{k+1},(1+\nu_k)\alpha_k)\|^2}{(1+\nu_k)^2\alpha_k^2} \leq \frac{\|g(u^{k+1},\alpha_k)\|^2}{\alpha_k^2}.$$

Then

$$\begin{aligned}
\|g(u^{k+1},\alpha_{k+1})\|^2 &\leq (1+\nu_k)^2\|g(u^{k+1},\alpha_k)\|^2 \\
&\leq (1+\nu_k)^2[\|g(u^k,\alpha_k)\|^2 - \frac{(2-\beta-2\varrho_k)}{\varrho_k}\|g(u^{k+1},\alpha_k) - g(u^k,\alpha_k)\|^2 \\
&\quad -\frac{\alpha_k}{\beta}\langle F(u^k) - F(u^{k+1}), h(u^k) - h(u^{k+1})\rangle] \\
&\leq (1+\nu_k)^2\|g(u^k,\alpha_k)\|^2 \\
&\leq \prod_{i=0}^{k}(1+\nu_i)^2\|g(u^0,\alpha_0)\|^2,
\end{aligned} \tag{29}$$

where the second inequality follows from (23) and the third inequality follows from the h-monotonicity of F. It follows from $\nu_k > 0$ and (13) that $\prod_{k=0}^{\infty}(1+\nu_k) < \infty$. Thus, the sequence $\{\|g(u^k,\alpha_k)\|\}$ is bounded, which implies that it has at least one cluster point. Assume that the sequence generated by Algorithm 3.1 is not convergent, then there exists a subsequence $\{u^{k_i}\}$ of $\{u^k\}$ satisfying

$$\lim_{i\to\infty}\|g(u^{k_i},\alpha_{k_i})\| = \eta > 0. \tag{30}$$

It follows from (27) and (29) that

$$0 < \frac{\|g(u^{k+1},\alpha_{k+1})\|^2}{\|g(u^k,\alpha_k)\|^2} \leq (1+\nu_k)^2\left(1 - \min\left\{\frac{2-\beta-2\varrho_k}{{\varrho_k}^3}, \frac{1}{\beta\varrho_k}\right\}\chi_k^2\right) \tag{31}$$

and consequently

$$\begin{aligned}
0 &< \frac{\eta^2}{\|g(u^{k_0},\alpha_{k_0}\|^2} \\
&= \prod_{k=k_0}^{\infty}\frac{\|g(u^{k+1},\alpha_{k+1})\|^2}{\|g(u^k,\alpha_k)\|^2} \\
&\leq \prod_{k=k_0}^{\infty}(1+\nu_k)^2\left(1 - \min\left\{\frac{2-\beta-2\varrho_k}{{\varrho_k}^3}, \frac{1}{\beta\varrho_k}\right\}\chi_k^2\right)
\end{aligned} \tag{32}$$

Since $\prod_{k=0}^{\infty}(1+\nu_k)^2 < \infty$. From (32), we obtain

$$\prod_{k=k_0}^{\infty}\left(1-\min\left\{\frac{2-\beta-2\varrho_k}{{\varrho_k}^3}, \frac{1}{\beta\varrho_k}\right\}\chi_k^2\right) > 0. \tag{33}$$

Then, it follows from Lemma 2.3 and $0 < \min\{\frac{2-\beta-2\varrho_k}{{\varrho_k}^3}, \frac{1}{\beta\varrho_k}\}\chi_k^2 < 1$ (see (9) and (31)) that

$$\sum_{k=0}^{\infty}\chi_k^2 < \infty, \quad \lim_{k\to\infty}\chi_k^2 = 0 \quad \text{and} \quad \prod_{k=0}^{\infty}(1+t\chi_k^2) < \infty, \quad \forall\, t > 0. \tag{34}$$

Theorem 4.2. *Let $\{u^k\}$ be the sequence generated by Algorithm 3.1. Then there exists a constant $k_0 \geq 0$, such that for all $k \geq k_0$*

$$\begin{aligned}&\|h(u^{k+1}) - h(u^*) + \alpha_{k+1}[F(u^{k+1}) - F(u^*)]\|^2\\ &\leq (1+\xi_k)\|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2 - \gamma\|g(u^k,\alpha_k)\|^2,\end{aligned} \tag{35}$$

where $\xi_k := (1+\nu_k)^2\left(1+\frac{4\chi_k^2}{\beta(2-\beta)}\right) - 1$ *and* $\gamma := \frac{\beta(2-\beta)}{2}$.

Proof: It follows from (7) that

$$\begin{aligned}&\|h(u^{k+1}) - h(u^*) + \alpha_k[F(u^{k+1}) - F(u^*)]\|^2\\ &= \|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)] - [\beta g(u^k,\alpha_k) - \Lambda_k(u^{k+1})]\|^2\\ &= \|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2 + \|\beta g(u^k,\alpha_k) - \Lambda_k(u^{k+1})\|^2\\ &\quad -2\beta\langle h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)], g(u^k,\alpha_k)\rangle\\ &\quad +2\langle h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)], \Lambda_k(u^{k+1})\rangle\\ &\leq \|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2 - 2\beta\|g(u^k,\alpha_k)\|^2\\ &\quad +2\langle h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)], \Lambda_k(u^{k+1})\rangle\\ &\quad +\|\beta g(u^k,\alpha_k) - \Lambda_k(u^{k+1})\|^2.\end{aligned} \tag{36}$$

where the inequality follows from (17).
Since $\|\Lambda_k(u^{k+1})\|^2 \leq \chi_k^2\|g(u^k,\alpha_k)\|^2$ and $\lim_{k\to\infty}\chi_k^2 = 0$, it is easy to show that there is a $k_0 > 0$, such that for all $k \geq k_0$

$$\|\beta g(u^k,\alpha_k) - \Lambda_k(u^{k+1})\|^2 \leq \beta^2\|g(u^k,\alpha_k)\|^2 + \frac{1}{4}\beta(2-\beta)\|g(u^k,\alpha_k)\|^2. \tag{37}$$

Using the Cauchy-Schwarz inequality and (28) we get

$$\begin{aligned}&2\langle h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)], \Lambda_k(u^{k+1})\rangle\\ &\leq \frac{4\chi_k^2}{\beta(2-\beta)}\|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2 + \frac{\beta(2-\beta)}{4\chi_k^2}\|\Lambda_k(u^{k+1})\|^2\\ &\leq \frac{4\chi_k^2}{\beta(2-\beta)}\|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2 + \frac{1}{4}\beta(2-\beta)\|g(u^k,\alpha_k)\|^2.\end{aligned} \tag{38}$$

Substituting (37) and (38) into inequality (36), we obtain

$$\begin{aligned}&\|h(u^{k+1}) - h(u^*) + \alpha_k[F(u^{k+1}) - F(u^*)]\|^2\\ &\leq \left(1+\frac{4\chi_k^2}{\beta(2-\beta)}\right)\|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2\\ &-\frac{1}{2}\beta(2-\beta)\|g(u^k,\alpha_k)\|^2.\end{aligned} \tag{39}$$

Since $0 < \alpha_{k+1} \le (1+\nu_k)\alpha_k$, using the h-monotonicity of F, it follows that

$$\begin{aligned}&\|h(u^{k+1}) - h(u^*) + \alpha_{k+1}[F(u^{k+1}) - F(u^*)]\|^2\\ &\le (1+\nu_k)^2\|h(u^{k+1}) - h(u^*) + \alpha_k[F(u^{k+1}) - F(u^*)]\|^2.\end{aligned} \tag{40}$$

Combining (40) with (39), we obtain the desired result. □

Now, we give the proof of global convergence of the proposed algorithm.

Theorem 4.3. *Suppose that H is finite dimension space. Then the whole sequence $\{u^k\}$ converges to a solution point of (1).*

Proof: From $\sum_{k=0}^{\infty} \nu_k < \infty$ and $\sum_{k=0}^{\infty} \chi_k^2 < \infty$, it follows that $\sum_{k=0}^{\infty} \xi_k < \infty$ and $\prod_{k=0}^{\infty}(1+\xi_k) < \infty$. Denote

$$D_s := \sum_{k=0}^{\infty} \xi_k < \infty \qquad \text{and} \qquad D_p := \prod_{k=0}^{\infty}(1+\xi_k).$$

Let $u^* \in S^*$, From (35) we get

$$\begin{aligned}&\|h(u^{k+1}) - h(u^*) + \alpha_{k+1}[F(u^{k+1}) - F(u^*)]\|^2\\ &\le \prod_{i=k_0}^{k}(1+\xi_i)\|h(u^{k_0}) - h(u^*) + \alpha_{k_0}[F(u^{k_0}) - F(u^*)]\|^2\\ &\le D_p\|h(u^{k_0}) - h(u^*) + \alpha_{k_0}[F(u^{k_0}) - F(u^*)]\|^2, \quad \forall k \ge k_0.\end{aligned}$$

Then, we can find a constant $D > 0$ such that

$$\|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2 \le D, \quad \forall k \ge k_0. \tag{41}$$

From(41), it is easy to verify that the sequence $\{u^k\}$ is bounded. It follows from (35) and (41) that

$$\begin{aligned}\gamma \sum_{k=k_0}^{\infty} \|g(u^k, \alpha_k)\|^2 \;&\le\; \|h(u^{k_0}) - h(u^*) + \alpha_{k_0}[F(u^{k_0}) - F(u^*)]\|^2\\ &\quad + \sum_{k=k_0}^{\infty} \xi_k\|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\|^2\\ &\le\; D + D\sum_{k=k_0}^{\infty} \xi_k\\ &\le\; (1+D_s)D.\end{aligned}$$

Then $\lim_{k\to\infty} \|g(u^k, \alpha_k)\| = 0$ and it follows from Lemma 2.2 that

$$\lim_{k\to\infty} \|g(u^k, \alpha_l)\| = 0.$$

Let u^* be a cluster point of $\{u^k\}$ then, there exists a subsequence $\{u_j^k\}$ of $\{u^k\}$ such that $u_j^k \longrightarrow u^*$. Since F and h are continuous then $g(u, \alpha_l)$ is continuous,

$$\|g(u^*, \alpha_l)\| = \lim_{j\to\infty} \|g(u^{k_j}, \alpha_l)\| = 0$$

and u^* is a solution point of (1). Assume that $\bar{u} \ne u^*$ is another cluster point of $\{u^k\}$. Since u^* is a cluster point of $\{u^k\}$, there exist a $k_0 > 0$ such that

$$\|h(u^{k_0}) - h(u^*) + \alpha_{k_0}[F(u^{k_0}) - F(u^*)]\| \le \frac{1}{2\sqrt{D_p}}\|h(\bar{u}) - h(u^*)\|. \tag{42}$$

Recall the h-monotonicity of F, for all $k \geq k_0$, it follows from (35) and (42) that

$$\begin{aligned}\|h(u^k) - h(u^*)\| &\leq \|h(u^k) - h(u^*) + \alpha_k[F(u^k) - F(u^*)]\| \\ &\leq \left(\prod_{i=k_0}^{k-1}(1+\xi_i)\right)^{\frac{1}{2}} \|h(u^{k_0}) - h(u^*) + \alpha_{k_0}[F(u^{k_0}) - F(u^*)]\| \\ &\leq \sqrt{D_p}\|h(u^{k_0}) - h(u^*) + \alpha_{k_0}[F(u^{k_0}) - F(u^*)]\| \\ &\leq \frac{1}{2}\|h(\bar{u}) - h(u^*)\|,\end{aligned}$$

Then

$$\|h(u^k) - h(\bar{u}) \geq \|h(\bar{u}) - h(u^*)\| - \|h(u^k) - h(u^*)\| \geq \frac{1}{2}\|h(\bar{u}) - h(u^*)\| > 0, \qquad \forall k \geq k_0.$$

This contradicts with the assumption that $\bar{u}$ is a cluster point of $\{u^k\}$, which means u^* is the unique cluster point of sequence $\{u^k\}$, i.e., $\lim_{k\to\infty} u^k = u^*$. □

5. Preliminary Computational Results

In order to illustrate the implementation and efficiency of the suggested method, we consider the nonlinear complementarity problems:
find $u \in R^n$ such that

$$u \geq 0, \quad F(u) \geq 0, \quad \langle u, F(u)\rangle = 0, \tag{43}$$

where $F(u) = D(u) + Mu + q$, $D(u)$ and $Mu + q$ are the nonlinear part and linear parts of $F(u)$ respectively. Problem (43) is a special case of Problem (1), by taking $h = I$ and

$$\varphi(v,u) = \begin{cases} 0, & \text{if} \quad v \in R^n_+; \\ +\infty, & \text{otherwise} . \end{cases}$$

The matrix $M = A^TA + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. In $D(u)$, the nonlinear part of $F(u)$, the components are $D_j(u) = a_j * \arctan(u_j)$ and a_j is a random variable in $(0, 1)$. We test the proposed method with the parameter adjusting to the following strategy. Set

$$\eta_k = \frac{\|\alpha_k[F(u^{k+1}) - F(u^k)]\|}{\|u^{k+1} - u^k\|},$$

and adjust the scaling parameter α_k

$$\alpha_k = \begin{cases} (1+\nu_k)\alpha_k & \text{if } \eta_k < 0.3, \\ \dfrac{1}{(1+\nu_k)}\alpha_k & \text{if } \eta_k > 3, \\ \alpha_k & \text{otherwise.} \end{cases}$$

All codes were written in Matlab. In all test we take $\alpha_0 = 0.001$ (in Table 1 and Table 2), $\varrho_k = 0.2$, $\delta = 0.2$, $\mu = 0.5$, $\rho_0 = 1$ and $\beta = 1.5$. For given initial value $u^0 = (0, \cdots, 0) \in R^n$ and setting $\|e(u^k, 1)\|_\infty \leq 10^{-7}$ as stop criterion. The comparison of the proposed method with those in [11, 18] are displayed in tables 1 and 2 with different dimensions, and with different initial parameter α_0 in Table 3. k is the numbers of iterations and l the numbers of mapping F evaluation.

Tables 1-3 show that the proposed method is very efficient algorithm even for large-scale classical nonlinear complementarity problems. Moreover, it demonstrates computationally that the superiority of the proposed method to the methods of [11, 18] in terms of number of the amount of computing the value of function F and CPU time.

Table 1: Numerical results for problem (43) with $q \in (-500, 0)$

Dimension of the problem	The proposed method			The method in [18]			The method in [11]		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)	k	l	CPU(Sec.)
n=200	90	1658	0.125	90	1709	0.138	87	2255	0.166
n=500	78	1655	0.389	78	1685	0.451	75	2125	0.612
n=700	94	2146	0.683	93	2167	0.719	93	2545	1.418
n=1000	120	2531	3.797	119	2576	3.917	118	3299	5.584
n=2000	99	2498	15.831	103	2617	17.395	104	2904	18.664
n=2500	126	3370	31.757	126	3431	36.705	125	3727	42.773

Table 2: Numerical results for problem (43) with $q \in (-500, 500)$

Dimension of the problem	The proposed method			The method in [18]			The method in [11]		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)	k	l	CPU(Sec.)
n=200	96	1075	0.064	95	1083	0.076	92	1704	0.111
n=500	95	1088	0.213	95	1108	0.234	91	1708	0.332
n=700	76	1039	0.358	76	1049	0.365	73	1613	0.467
n=1000	63	909	1.226	63	941	1.336	62	1595	2.587
n=2000	86	1167	7.011	86	1207	7.532	86	1807	12.368
n=2500	104	2038	18.396	104	2087	21.478	102	2913	29.143

Table 3: Numerical results for problem (43) with $n = 200$ and $q \in (-500, 0)$

α_0	The proposed method			The method in [18]			The method in [11]		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)	k	l	CPU(Sec.)
10^{-3}	90	1658	0.125	90	1709	0.138	87	2255	0.166
10^{-1}	128	2096	0.128	127	2117	0.161	123	2867	0.205
1	113	2131	0.135	113	2156	0.159	110	2797	0.214
10^{3}	102	2091	1.122	101	2107	0.145	97	2852	0.197
10^{4}	99	1979	0.115	99	2012	0.144	97	2223	0.214

6. Conclusions

In this paper, we proposed an inexact operator splitting method for general mixed variational inequalities under significantly relaxed accuracy criterion. Under standard assumptions, the global convergence of the proposed method is proved. And the numerical efficiency of our algorithm is verified compared with some existing algorithms. Our results could be viewed as significant extensions of the previously known results.

References

[1] A. Bnouhachem, A self-adaptive method for solving general mixed variational inequalities, J. Math. Anal. Appl. **309** (2005), 136-150.

[2] A. Bnouhachem, M.A. Noor and Th.M. Rassias, Three-step iterative algorithm for mixed variational inequalities, Appl. Math. and Comput. **183**(1) (2006), 436-446.

[3] A. Bnouhachem and M.A. Noor, Inexact proximal point method for general variational inequalities, J. Math. Anal. Appl. **324**(2) (2006), 1195-1212.

[4] A. Bnouhachem, An inexact implicit method for general mixed variatioanl inequalities, J. Comput. Appl. Math. **200** (2007), 377-387.

[5] A. Bnouhachem, M.A. Noor, Numerical methods for general mixed variational inequalities, App. Math. Comput. **204** (2008), 27-36.

[6] H. Brezis, Operateurs maximaux monotone et semigroupes de contractions dans les espace d'Hilbert, North-Holland, Amsterdam, Holland, 1973.

[7] W. Cholmajiak P. Kitisak and D. Yambangwai, An inertial parallel CQ subgradient extragradient method for variational inequalities application to signal-image recovery, Results in Nonlinear Analysis, **4**(4) (2021), 217-234.

[8] F. Facchinei and J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer Series in Operations Research. Springer, Berlin 2003.

[9] M. Farid, W. Cholamjiak, R. Ali, K.R. Kazmi, A new shrinking projection algorithm for a generalized mixed variational-like inequality problem and asymptotically quasi-ϕ-nonexpansive mapping in a Banach space, Revista de la Real Academia de Ciencias Exactas, Fï£¡sicas y Naturales. Serie A. Matemáticas, **115**(3) (2021), 1-28.

[10] F. Giannessi, A. Maugeri and P.M. Pardalos, (eds.), Equilibrium Problems and Variational Models, Kluwer Academic, Dordrecht (2001).

[11] Z. Ge, G. Qian and D.R. Han, Global convergence of an nexact operator splitting method for monotone variational inequalities, J. Indus. Manag. Optim. **7**(4) (2011), 1013-1026.

[12] R. Glowinski, J.L. Lions and R. Tremoliers, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, Holland, 1981.

[13] B.S. He, Inexact implicit methods for monotone general variational inequalities, Math. Program. **86** (1999), 199-217.

[14] B.S. He, H. Yang, Q. Meng and D.R. Han, Modified Goldstein-Levitin-Polyak projection method for asymmetric strongly monotone variational inequalities, J. Optim. Theory Appl. **112**(1) (2002), 129-143.

[15] B.S. He, L.Z. Liao and S.L. Wang, Self-adaptive operator splitting methods for monotone variational inequalities, Numer. Math. **94** (2003), 715-737.

[16] T. Kai and X. Fuquan, A projection type algorithm for solving generalized mixed variational inequalities, Act. Math. Scien. **36B**(6) (2016), 1619-1630.

[17] J.L. Lions and G. Stampacchia, Variational inequalities, Comm. Pure Apl. Math. **20** (1967), 493-512.

[18] M. Li and A. Bnouhachem, A modified inexact operator splitting method for monotone variational inequalities, J. Global Optim. **41**(3) (2008), 417-426.

[19] M. Li and X.M. Yuan, An improved Goldstein's type method for a class of variant variational inequalities, J. Comput. Appl. Math. **214**(1) (2008), 304-312.

[20] M.A. Noor, *An implicit method for mixed variational inequalities*, Appl. Math. Lett. **11** (1998), 109-113.

[21] M.A. Noor, Pseudomonotone general mixed variational inequalities, Appl. Math. Comput. **141** (2003), 529-540.

[22] M.A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. **152** (2004), 199-277.

[23] S. Suanta, P. Peeyad, D. Yambangwai and W. Cholamjiak, A parallel-viscosity-type subgradient extragradient-line method for finding the common solution of variational inequality problems applied to image restoration problems, Mathematics, **8**(2) (2020), 248.

[24] S. Ullah and M.A. Noor, An effeicient method solving new general mixed variational inequalities, J. Inequal. Special Funct. **11**(3) (2020), 1-9.

[25] S.L. Wang, H. Yang and B.S. He, Inexact implicit method with variable parameter for mixed monotone variational inequalities, J. Optim. Theory Appl. **111**(2) (2001), 431-443.

[26] L.C. Zeng and J.C. Yao, Convergence analysis of a modified inexact implicit method for general mixed monotone variational inequalities, Math. Meth. Oper. Res. **62** (2005), 211-224.