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Parameter Estimation on Geometric Distribution of Order k with Different Reward Laws

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Abstract

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent trials with two possible outcomes, "0" and "1" where "1" represents the success of Type-I, and "0" denotes the success of Type-II. For nonnegative integers k_r and k_l using a reward scheme, we obtained the distribution of the number of trials (W) until the sum of consecutive rewards of Type-I is equal to or exceeds the level k_r, or the sum of consecutive rewards of Type-II is equal to or exceeds the level k_l. The geometric distributed rewards are studied by Eryılmaz et al. in [1]. In this study, the survival function of W is obtained for binary sequence with Bernoulli and exponential rewards as well as geometric rewards. A simulation study is performed to compare the theoretical and simulated probabilities. Proportion estimates are also discussed for distribution of W with geometric rewards.

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Keywords: Geometric distribution of order k; Bernoulli reward; Exponential reward; Estimation.

Farklı Ödül Kuralları ile Ödüllü k Ardıl Geometrik Dağılımın Parametre Tahmini

Öz

$\xi_1, \xi_2, \ldots$ "0" ve "1" olmak üzere iki olası sonuca sahip bağımsız denemelerin bir dizisi olsun, burada "1", I. tip başarıyı ve "0", II. tip başarıyı temsil etmektedir. Ödül şemasında kullanan negatif olmayan k_r ve k_l tamsayıları için, ardışık I. tip başarılardan elde edilen toplam ödüllerin k_r yi veya ardışık II. tip başarılardan elde edilen ödüllerin toplamı k_l'yi aşana kadar yapılan deneme sayısının (W) dağılımı elde edilmiştir. Geometrik dağılımlı ödüller Eryılmaz ve arkadaşları [1] tarafından çalışılmıştır. Bu çalışmada, W'nun yaşam fonksiyonu geometrik ödülün yanı sıra Bernoulli ve üstel ödüller için de elde edilmiştir. Teorik ve simüle edilmiş olasılıkları karşılaştırmak için bir simülasyon çalışması yapılmıştır. Ayrıca W'nun oran tahmini, ödüllerin geometrik olduğu durum için tartışılmıştır.

Anahtar Kelimeler: Ödüllü k ardıl geometrik dağılım; Bernoulli ödül; Üstel ödül; Tahmin.

1. Introduction

The geometric distribution is a well-known discrete distribution in the probability theory. Therefore, distribution with many generalized geometric sub-models are constructed. Some generalization of the geometric distribution can be found in [1-4].

In this paper, the survival function of the waiting time introduced by Eryılmaz et al. [1] is presented in a different way. Eryılmaz et al. [1] use the phase type distribution with matrix notations to obtaining waiting time distribution. However, we use the longest run's distribution without matrix calculation (it is introduced by Demir and Eryılmaz [5]) to get the distribution of W. It is only a different way and it is not main contribution. The main contribution of the paper is to provide the different kinds of rewards such as exponential and binomial and to discuss the estimation for the parameters of the distribution of waiting time W. In Section 2, the survival function of W is presented for binary sequence in a different way with Bernoulli and exponential rewards besides geometric one. A simulation study is also performed to control the true probabilities. In Section 3, point estimation is discussed by the proportion method. In section 4, a numerical example is given. Some concluding remarks are provided in Section 5.

2. Distribution of Waiting Time W with Different Rewards

In this section, we present the distribution of waiting time proposed by Eryılmaz et al. [1] under independent binary trials with different rewards. It is noticed that survival probabilities of [1] is obtained in a different way.

Let $\{\xi_n, n \geq 1\}$ be a sequence of independent trials that can take values 0 and 1. Let "1" and "0" indicate the Type-I and Type-II success, Y and Z denote the random earned rewards associated with the Type-I and Type-II success, respectively. Furthermore, $P\{Y > 0\} = 1$ and $P\{Z > 0\} = 1$. Throughout the paper, we suppose that $Y_1, Y_2, \ldots$ and $Z_1, Z_2, \ldots$ are independent (also independent with trials) random rewards with cumulative distribution functions $F(x) = P\{Y_i \leq x\}$ and $G(x) = P\{Z_i \leq x\}$, $i = 1, 2, \ldots$, respectively. Mallor and Santos [6] proposed a scheme for giving rewards $\{Y_i, i \geq 1\}$ and $\{Z_i, i \geq 1\}$. The random variable $Y_i(Z_i)$ is associated with Type-I (Type-II) success when it occupies the *i*th place in a run of successes of Type-I (Type-II).

Let us consider the sequence of binary trials

101011001001011.

According to [6] the reward scheme, reward sequence is as following:

$$y_1, z_1, y_1, z_1, y_1, y_2, z_1, z_2, y_1, z_1, z_2, y_1, z_1, y_1, y_2$$

where $y(z)$, is a realization of $Y(Z)$. This scheme is also used by [1].

Eryılmaz et al. [1] consider the trinary independent sequence and obtained the distribution of the number of trials W until either the sum of consecutive rewards of Type-I is equal to or exceeds the level k_r, or the sum of consecutive rewards of Type-II is equal to or exceeds the level k_l.

We give the following two lemmas in order to obtain another representation for the distribution of W. The following lemma is a consequence of Theorem 2 in [1].

Lemma 1. Let $\{\xi_n, n \geq 1\}$ arbitrarily dependent binary trials and $L_n^{(1)}$ and $L_n^{(2)}$ denote the number of longest runs for Type-I and Type-II success. Then survival function of W is given by

$$
\begin{aligned}
P\{W>n\} &= \sum_{m_1=0}^{n}\sum_{m_2=0}^{n} P\left\{\sum_{i=1}^{m_1} Y_i < k_r, \sum_{i=1}^{m_2} Z_i < k_l\right\} P\left\{L_n^{(1)} = m_1, L_n^{(2)} = m_2\right\} \\
&= \sum_{m_1=0}^{n}\sum_{m_2=0}^{n} \left\{1 - P\left(\sum_{i=1}^{m_1} Y_i \ge k_r\right)\right\}\left\{1 - P\left(\sum_{i=1}^{m_2} Z_i \ge k_l\right)\right\} \\
&\times\Big[P\left\{L_n^{(1)} < m_1, L_n^{(2)} < m_2\right\} - P\left\{L_n^{(1)} < m_1+1, L_n^{(2)} < m_2\right\} \\
&- P\left\{L_n^{(1)} < m_1, L_n^{(2)} < m_2+2\right\} + P\left\{L_n^{(1)} < m_1+1, L_n^{(2)} < m_2+1\right\}\Big].
\end{aligned}
$$

Lemma 2. [5] Let $\{\xi_n, n \ge 1\}$ independent binary trials with $P(\xi_1 = 1) = p$ and $P(\xi_1 = 0) = 1 - p$. In addition, $L_n^{(1)}$ and $L_n^{(2)}$ denote the number of longest runs for Type-I and Type-II success. Then

$$
P\left\{L_n^{(1)} < k_1, L_n^{(2)} < k_2\right\} = \sum_{r_1}\sum_{r_2}\sum_{n_1} N(r_1, k_1, n_1) N(r_2, k_2, n - n_1) b(r_1, r_2, n_1, n),
$$

where

$$
N(a,b,c) = \sum_{j=0}^{a} (-1)^j \binom{a}{j}\binom{c - j(b-1) - 1}{a-1} \tag{1}
$$

and

$$
b(r_1, r_2, n_1, n) = \begin{cases} 2p^{n_1}(1-p)^{(n-n_1)}, & r_1 = r_2 \\ p^{n_1}(1-p)^{(n-n_1)}, & r_2 = r_1 + 1 \, veya \, r_1 = r_2 + 1 \\ 0, & \text{otherwise} \end{cases} \tag{2}
$$

The following theorem is another form of the Theorem 2 of [1].

Theorem 1. Let $\{\xi_n, n \ge 1\}$ independent binary trials and $\{\xi_n, n \ge 1\}$ independent from rewards $\{Y_i, i \ge 1\}$ and $\{Z_i, i \ge 1\}$. Then survival function of W is given by

$$
\begin{aligned}
P\{W>n\} = \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}&\left\{1-P\left(\sum_{i=1}^{m_1}Y_i \ge k_r\right)\right\}\left\{1-P\left(\sum_{i=1}^{m_2}Z_i \ge k_l\right)\right\} \\
&\times\left\{\sum_{r_1=1}^{\left[\frac{n}{2}+1\right]}\sum_{r_2=1}^{\left[\frac{n}{2}+1\right]}\sum_{n_1=0}^{n}N(r_1,m_1,n_1)N(r_2,m_2,n-n_1)b(r_1,r_2,n_1,n)\right. \\
&-\sum_{r_1=1}^{\left[\frac{n}{2}+1\right]}\sum_{r_2=1}^{\left[\frac{n}{2}+1\right]}\sum_{n_1=0}^{n}N(r_1,m_1+1,n_1)N(r_2,m_2,n-n_1)b(r_1,r_2,n_1,n) \\
&-\sum_{r_1=1}^{\left[\frac{n}{2}+1\right]}\sum_{r_2=1}^{\left[\frac{n}{2}+1\right]}\sum_{n_1=0}^{n}N(r_1,m_1,n_1)N(r_2,m_2+2,n-n_1)b(r_1,r_2,n_1,n) \\
&\left.\sum_{r_1=1}^{\left[\frac{n}{2}+1\right]}\sum_{r_2=1}^{\left[\frac{n}{2}+1\right]}\sum_{n_1=0}^{n}N(r_1,m_1+1,n_1)N(r_2,m_2+1,n-n_1)b(r_1,r_2,n_1,n)\right\},
\end{aligned}
\tag{3}
$$

where $N(a,b,c)$ and $b(r_1,r_2,n_1,n)$ are defined as in Eqn. (1) and Eqn. (2), respectively.

Proof. Proof follows from using Lemma 2 in Lemma 1.

Let gives some numerical computation for the survival function of W with the geometric rewards. $Y_1,Y_2,\ldots$ iid random variables with probability mass function (pmf)

$$P\{Y_i = y\} = p_r(1-p_r)^{y-1},\ y=1,2,\ldots \tag{4}$$

and $Z_1,Z_2,\ldots$ iid random variables with

$$P\{Z_i = z\} = p_l(1-p_l)^{z-1},\quad z=1,2,\ldots \tag{5}$$

for $i=1,2,\ldots$. In this case for $k_r > n > 0$, Eryılmaz et al. [1] gives

$$P\left\{\sum_{i=1}^{n}Y_i \ge k_r\right\} = 1-\sum_{y=n}^{k_r-1}\binom{y-1}{n-1}p_r^n(1-p_r)^{y-n}, \tag{6}$$

and for $k_l > n > 0$

$$P\left\{\sum_{i=1}^{n}Z_i \ge k_l\right\} = 1-\sum_{z=n}^{k_l-1}\binom{z-1}{n-1}p_l^n(1-p_l)^{z-n}. \tag{7}$$

We consider two different kinds of rewards as well as geometric ones.

Let us consider $Y_1, Y_2, \ldots$ are iid Bernoulli random variables with pmf $f_{Y_i}(y) = P(Y_i = y) = p_r^{\,y}(1-p_r)^{1-y}$, $y = 0,1$ and $Z_1, Z_2, \ldots$ denote the iid random variables with pmf $f_{Z_i}(z) = P(Z_i = z) = p_l^{\,z}(1-p_l)^{1-z}$, $z = 0,1$. For $k_r > n > 0$,

$$P\left\{\sum_{i=1}^{n} Y_i \geq k_r\right\} = \sum_{j=k_r}^{n} \binom{n}{j} p_r^{\,j}(1-p_r)^{n-j}$$
$$= \frac{\binom{n}{k_r}(1-p_r)^{(n-k_r)}\, hypergeom\left([1,-n+k_r],[k_r+1],\frac{p_r}{-1+p_r}\right)\left(p_r^{\,k_r} - p_r^{\,(k_r+1)}\right)}{1-p_r} \tag{8}$$

and $k_l > n > 0$

$$P\left\{\sum_{i=1}^{n} Z_i \geq k_l\right\} = \sum_{j=k_l}^{n} \binom{n}{j} p_l^{\,j}(1-p_l)^{n-j}$$
$$= \frac{\binom{n}{k_l}(1-p_l)^{(n-k_l)}\, hypergeom\left([1,-n+k_l],[k_l+1],\frac{p_l}{-1+p_l}\right)\left(p_l^{\,k_l} - p_l^{\,(k_l+1)}\right)}{1-p_l}, \tag{9}$$

where hypergeom is a hypergeometric function, which is a particular case represented by the hypergeometric series.

Let us consider $Y_1, Y_2, \ldots$ are iid random variables with pdf $f_{Y_i}(y) = \frac{1}{\beta_r}\exp\left(-\frac{y}{\beta_r}\right)$, $y > 0$ and $Z_1, Z_2, \ldots$ are iid random variables with pdf $f_{Z_i}(z) = \frac{1}{\beta_l}\exp\left(-\frac{z}{\beta_l}\right)$, $z > 0$. For $k_r > n > 0$

$$P\left\{\sum_{i=1}^{n} Y_i \geq k_r\right\} = \int_{k_r}^{\infty} \frac{1}{\Gamma(n)\beta_r^{\,n}} y^{n-1} \exp\left(-\frac{y}{\beta_r}\right) dx = \exp\left(-\frac{k_r}{\beta_r}\right) \sum_{j=0}^{n-1} \frac{\left(\frac{k_r}{\beta_r}\right)^j}{j!} \tag{10}$$

and for $k_l > n > 0$

$$P\left\{\sum_{i=1}^{n} Z_i \geq k_l\right\} = \int_{k_l}^{\infty} \frac{1}{\Gamma(n)\beta_l^{\,n}} y^{n-1} \exp\left(-\frac{y}{\beta_l}\right) dx = \exp\left(-\frac{k_l}{\beta_l}\right) \sum_{j=0}^{n-1} \frac{\left(\frac{k_l}{\beta_l}\right)^j}{j!}. \tag{11}$$

Table 1: $P(W > n)$ for p = 0.3, p_l = 0.6, p_r = 0.4 when the rewards from geometric distribution

k_l	k_r	n	$P(W = n)$	Simulated values	$P(W > n)$
5	3	1	0.1259	0.1252	0.8741
		2	0.1956	0.1945	0.6785
		3	0.2105	0.2086	0.4680
		4	0.1637	0.1654	0.3043
		5	0.0836	0.0801	0.2208
		6	0.0510	0.0550	0.1698
		7	0.0355	0.0352	0.1343
		8	0.0301	0.0302	0.1042
		9	0.0214	0.0222	0.0828
		10	0.0168	0.0170	0.0660
5	4	1	0.0827	0.0832	0.9173
		2	0.1626	0.1619	0.7547
		3	0.1972	0.1961	0.5576
		4	0.1656	0.1730	0.3919
		5	0.0938	0.0917	0.2981
		6	0.0629	0.0627	0.2352
		7	0.0450	0.0425	0.1903
		8	0.0385	0.0378	0.1517
		9	0.0282	0.0302	0.1235
		10	0.0226	0.0226	0.1009
5	5	1	0.0568	0.0579	0.9432
		2	0.1376	0.1430	0.8056
		3	0.1852	0.1832	0.6204
		4	0.1642	0.1669	0.4563
		5	0.0988	0.0973	0.3575
		6	0.0700	0.0668	0.2875
		7	0.0511	0.0487	0.2364
		8	0.0442	0.0404	0.1922
		9	0.0331	0.0327	0.1591
		10	0.0269	0.0283	0.1322
5	6	1	0.0412	0.0382	0.9588
		2	0.1195	0.1227	0.8393
		3	0.1751	0.1693	0.6642
		4	0.1612	0.1628	0.5030
		5	0.1008	0.0984	0.4023
		6	0.0741	0.0787	0.3282
		7	0.0550	0.0576	0.2732
		8	0.0480	0.0514	0.2252
		9	0.0365	0.0360	0.1887
		10	0.0300	0.0283	0.1586

Table 2: $P(W > n)$ for p = 0.3, k_l = 5, k_r = 3 when the rewards from geometric distribution

p_l	p_r	n	$P(W = n)$	Simulated values	$P(W > n)$
0.6	0.3	1	0.1649	0.1650	0.8351
		2	0.2161	0.2103	0.6190
		3	0.2175	0.2142	0.4015
		4	0.1583	0.1586	0.2431
		5	0.0732	0.0777	0.1700
		6	0.0404	0.0389	0.1295
		7	0.0279	0.0287	0.1016
		8	0.0236	0.0263	0.0780
		9	0.0166	0.0167	0.0614
		10	0.0130	0.0129	0.0484
0.6	0.5	1	0.0929	0.0915	0.9071
		2	0.1755	0.1713	0.7316
		3	0.2035	0.2079	0.5281
		4	0.1673	0.1690	0.3608
		5	0.0922	0.0907	0.2686
		6	0.0600	0.0620	0.2086
		7	0.0422	0.0445	0.1664
		8	0.0359	0.0357	0.1305
		9	0.0258	0.0256	0.1046
		10	0.0205	0.0204	0.0842
0.6	0.7	1	0.0449	0.0492	0.9551
		2	0.1364	0.1342	0.8187
		3	0.1897	0.1926	0.6290
		4	0.1694	0.1609	0.4596
		5	0.1040	0.1017	0.3556
		6	0.0737	0.0751	0.2819
		7	0.0530	0.0551	0.2289
		8	0.0455	0.0471	0.1834
		9	0.0337	0.0358	0.1497
		10	0.0270	0.0264	0.1227
0.6	0.9	1	0.0209	0.0204	0.9791
		2	0.0988	0.1011	0.8802
		3	0.1763	0.1741	0.7039
		4	0.1646	0.1673	0.5394
		5	0.1085	0.1118	0.4308
		6	0.0815	0.0785	0.3494
		7	0.0604	0.0607	0.2890
		8	0.0523	0.0517	0.2367
		9	0.0401	0.0368	0.1966
		10	0.0327	0.0346	0.1639

Table 3: $P(W>n)$ for $p = 0.9$, $p_l = 0.6$, $p_r = 0.4$ when the rewards from geometric distribution

k_l	k_r	n	$P(W=n)$	Simulated values	$P(W>n)$
5	3	1	0.3266	0.3215	0.6734
		2	0.4234	0.4267	0.2501
		3	0.1587	0.1598	0.0913
		4	0.0493	0.0474	0.0420
		5	0.0178	0.0205	0.0241
		6	0.0150	0.0140	0.0091
		7	0.0038	0.0045	0.0053
		8	0.0026	0.0032	0.0028
		9	0.0015	0.0015	0.0013
		10	0.0006	0.0005	0.0007
5	4	1	0.1970	0.1962	0.8030
		2	0.3722	0.3635	0.4308
		3	0.2478	0.2532	0.1830
		4	0.0970	0.1004	0.0860
		5	0.0304	0.0317	0.0556
		6	0.0277	0.0278	0.0279
		7	0.0100	0.0099	0.0179
		8	0.0082	0.0083	0.0096
		9	0.0038	0.0040	0.0058
		10	0.0021	0.0014	0.0037
5	5	1	0.1192	0.1215	0.8808
		2	0.2948	0.2907	0.5860
		3	0.2828	0.2958	0.3031
		4	0.1536	0.1487	0.1495
		5	0.0544	0.0509	0.0951
		6	0.0382	0.0376	0.0569
		7	0.0180	0.0187	0.0389
		8	0.0160	0.0150	0.0228
		9	0.0076	0.0071	0.0152
		10	0.0053	0.0052	0.0100
5	6	1	0.0725	0.0705	0.9275
		2	0.2204	0.2258	0.7070
		3	0.2760	0.2657	0.4311
		4	0.1977	0.1981	0.2333
		5	0.0887	0.0908	0.1446
		6	0.0509	0.0521	0.0937
		7	0.0262	0.0268	0.0675
		8	0.0242	0.0242	0.0432
		9	0.0126	0.0129	0.0306
		10	0.0100	0.0103	0.0206

Table 4: $P(W > n)$ for $p = 0.9$, $k_l = 5$, $k_r = 3$ when the rewards from geometric distribution

p_l	p_r	n	$P(W = n)$	Simulated values	$P(W > n)$
0.6	0.3	1	0.4436	0.4456	0.5564
		2	0.3859	0.3857	0.1706
		3	0.1036	0.1047	0.0669
		4	0.0396	0.0373	0.0273
		5	0.0121	0.0136	0.0152
		6	0.0097	0.0080	0.0056
		7	0.0024	0.0023	0.0032
		8	0.0016	0.0012	0.0016
		9	0.0009	0.0008	0.0008
		10	0.0003	0.0003	0.0004
0.6	0.5	1	0.2276	0.2187	0.7724
		2	0.4302	0.4387	0.3423
		3	0.2255	0.2300	0.1168
		4	0.0574	0.0526	0.0594
		5	0.0245	0.0241	0.0349
		6	0.0215	0.0202	0.0134
		7	0.0054	0.0064	0.0080
		8	0.0038	0.0036	0.0042
		9	0.0022	0.0029	0.0020
		10	0.0009	0.0009	0.0011
0.6	0.7	1	0.0836	0.0859	0.9164
		2	0.3517	0.3534	0.5647
		3	0.3936	0.3878	0.1711
		4	0.0685	0.0725	0.1025
		5	0.0403	0.0391	0.0623
		6	0.0377	0.0375	0.0246
		7	0.0096	0.0098	0.0150
		8	0.0069	0.0070	0.0081
		9	0.0042	0.0035	0.0039
		10	0.0017	0.0015	0.0022
0.6	0.9	1	0.0116	0.0113	0.9884
		2	0.1505	0.1457	0.8379
		3	0.6082	0.6162	0.2298
		4	0.0730	0.0700	0.1568
		5	0.0596	0.0597	0.0972
		6	0.0581	0.0579	0.0391
		7	0.0149	0.0146	0.0242
		8	0.0110	0.0117	0.0132
		9	0.0068	0.0061	0.0064
		10	0.0027	0.0029	0.0037

Table 5: $P(W > n)$ for $p = 0.3$, $\beta_l = 3$, $\beta_r = 1$ when the rewards from exponential distribution

k_l	k_r	n	$P(W = n)$	Simulated values	$P(W > n)$
5	3	1	0.1471	0.1471	0.8529
		2	0.2139	0.2120	0.6390
		3	0.1608	0.1632	0.4782
		4	0.1166	0.1160	0.3616
		5	0.0723	0.0704	0.2893
		6	0.0576	0.0581	0.2317
		7	0.0393	0.0402	0.1924
		8	0.0325	0.0323	0.1599
		9	0.0248	0.0257	0.1351
		10	0.0200	0.0188	0.1152
		11	0.0162	0.0158	0.0990
		12	0.0133	0.0142	0.0857
		13	0.0110	0.0112	0.0747
		14	0.0092	0.0089	0.0655
		15	0.0078	0.0075	0.0577
5	4	1	0.1377	0.1388	0.8623
		2	0.2029	0.2035	0.6594
		3	0.1553	0.1556	0.5041
		4	0.1156	0.1159	0.3885
		5	0.0735	0.0725	0.3150
		6	0.0596	0.0592	0.2555
		7	0.0414	0.0403	0.2141
		8	0.0345	0.0351	0.1796
		9	0.0266	0.0260	0.1530
		10	0.0216	0.0205	0.1314
		11	0.0176	0.0189	0.1137
		12	0.0147	0.0153	0.0991
		13	0.0121	0.0115	0.0869
		14	0.0103	0.0099	0.0766
		15	0.0087	0.0092	0.0679
5	5	1	0.1342	0.1357	0.8658
		2	0.1978	0.1993	0.6679
		3	0.1518	0.1506	0.5161
		4	0.1143	0.1153	0.4018
		5	0.0735	0.0720	0.3284
		6	0.0602	0.0597	0.2681
		7	0.0422	0.0419	0.2259
		8	0.0354	0.0360	0.1905
		9	0.0275	0.0283	0.1631
		10	0.0224	0.0229	0.1406
		11	0.0184	0.0181	0.1223
		12	0.0154	0.0153	0.1069
		13	0.0128	0.0121	0.0941
		14	0.0109	0.0105	0.0833
		15	0.0093	0.0092	0.0740
5	6	1	0.1330	0.1295	0.8670
		2	0.1956	0.1981	0.6715
		3	0.1500	0.1518	0.5215
		4	0.1132	0.1102	0.4083
		5	0.0732	0.0722	0.3351
		6	0.0603	0.0597	0.2748
		7	0.0425	0.0409	0.2322

		8	0.0358	0.0358	0.1965
		9	0.0279	0.0294	0.1686
		10	0.0228	0.0231	0.1458
		11	0.0188	0.0194	0.1270
		12	0.0157	0.0160	0.1113
		13	0.0131	0.0129	0.0982
		14	0.0112	0.0113	0.0871
		15	0.0095	0.0098	0.0775

Table 6: $P(W > n)$ for $p = 0.3$, $k_l = 5$, $k_r = 2$ when the rewards from exponential distribution

β_l	β_r	n	$P(W = n)$	Simulated values	$P(W > n)$
3	1	1	0.1728	0.1747	0.8272
		2	0.2360	0.2369	0.5912
		3	0.1681	0.1644	0.4232
		4	0.1147	0.1130	0.3085
		5	0.0677	0.0681	0.2408
		6	0.0520	0.0521	0.1888
		7	0.0345	0.0342	0.1543
		8	0.0281	0.0291	0.1262
		9	0.0211	0.0215	0.1052
		10	0.0167	0.0177	0.0884
		11	0.0134	0.0134	0.0751
		12	0.0109	0.0110	0.0642
		13	0.0088	0.0084	0.0554
		14	0.0074	0.0079	0.0480
		15	0.0061	0.0056	0.0419
3	2	1	0.2426	0.2421	0.7574
		2	0.2751	0.2744	0.4823
		3	0.1730	0.1725	0.3094
		4	0.1024	0.1018	0.2070
		5	0.0537	0.0542	0.1533
		6	0.0379	0.0386	0.1153
		7	0.0237	0.0243	0.0917
		8	0.0190	0.0186	0.0727
		9	0.0137	0.0142	0.0590
		10	0.0106	0.0113	0.0484
		11	0.0083	0.0081	0.0401
		12	0.0066	0.0077	0.0335
		13	0.0052	0.0050	0.0282
		14	0.0043	0.0037	0.0240
		15	0.0035	0.0036	0.0205
2	3	1	0.2115	0.2129	0.7885
		2	0.2387	0.2360	0.5498
		3	0.1801	0.1803	0.3697
		4	0.1188	0.1201	0.2510
		5	0.0671	0.0672	0.1838
		6	0.0430	0.0419	0.1408
		7	0.0267	0.0261	0.1141
		8	0.0210	0.0204	0.0931
		9	0.0154	0.0156	0.0777
		10	0.0125	0.0125	0.0652
		11	0.0100	0.0103	0.0552
		12	0.0082	0.0081	0.0471
		13	0.0066	0.0069	0.0404
		14	0.0055	0.0055	0.0349

		15	0.0046	0.0045	0.0303
3	3	1	0.2862	0.2863	0.7138
		2	0.2918	0.2930	0.4220
		3	0.1722	0.1694	0.2497
		4	0.0931	0.0932	0.1567
		5	0.0450	0.0452	0.1117
		6	0.0297	0.0303	0.0820
		7	0.0178	0.0166	0.0642
		8	0.0141	0.0145	0.0501
		9	0.0100	0.0102	0.0401
		10	0.0077	0.0072	0.0324
		11	0.0059	0.0063	0.0265
		12	0.0047	0.0050	0.0218
		13	0.0036	0.0037	0.0182
		14	0.0030	0.0036	0.0152
		15	0.0024	0.0024	0.0129

Table 7: $P(W > n)$ for $p = 0.9$, $\beta_l = 3$, $\beta_r = 1$ when the rewards from exponential distribution

k_l	k_r	n	$P(W = n)$	Simulated values	$P(W > n)$
5	3	1	0.0637	0.0633	0.9363
		2	0.1439	0.1432	0.7924
		3	0.1886	0.1860	0.6038
		4	0.1802	0.1807	0.4236
		5	0.1317	0.1311	0.2920
		6	0.0902	0.0896	0.2018
		7	0.0526	0.0549	0.1492
		8	0.0380	0.0401	0.1112
		9	0.0232	0.0241	0.0880
		10	0.0196	0.0194	0.0684
		11	0.0131	0.0121	0.0553
		12	0.0116	0.0114	0.0437
		13	0.0079	0.0079	0.0358
		14	0.0070	0.0075	0.0288
		15	0.0051	0.0055	0.0237
5	4	1	0.0354	0.0343	0.9646
		2	0.0805	0.0819	0.8841
		3	0.1287	0.1283	0.7555
		4	0.1548	0.1549	0.6007
		5	0.1443	0.1432	0.4564
		6	0.1173	0.1191	0.3391
		7	0.0813	0.0820	0.2578
		8	0.0588	0.0601	0.1990
		9	0.0382	0.0381	0.1608
		10	0.0306	0.0305	0.1303
		11	0.0213	0.0205	0.1089
		12	0.0190	0.0188	0.0899
		13	0.0138	0.0133	0.0761
		14	0.0126	0.0132	0.0635
		15	0.0095	0.0090	0.0540
5	5	1	0.0250	0.0250	0.9750
		2	0.0478	0.0473	0.9272
		3	0.0814	0.0807	0.8458
		4	0.1142	0.1158	0.7316
		5	0.1281	0.1290	0.6035
		6	0.1225	0.1231	0.4811

		7	0.0998	0.0982	0.3813
		8	0.0778	0.0773	0.3035
		9	0.0548	0.0549	0.2487
		10	0.0425	0.0422	0.2063
		11	0.0304	0.0303	0.1759
		12	0.0263	0.0260	0.1496
		13	0.0198	0.0189	0.1298
		14	0.0182	0.0185	0.1116
		15	0.0142	0.0152	0.0973
5	6	1	0.0211	0.0208	0.9789
		2	0.0323	0.0322	0.9465
		3	0.0516	0.0515	0.8949
		4	0.0781	0.0780	0.8168
		5	0.0999	0.1007	0.7169
		6	0.1096	0.1098	0.6073
		7	0.1030	0.1025	0.5043
		8	0.0889	0.0903	0.4154
		9	0.0690	0.0697	0.3464
		10	0.0544	0.0534	0.2920
		11	0.0402	0.0405	0.2518
		12	0.0334	0.0330	0.2184
		13	0.0257	0.0254	0.1927
		14	0.0233	0.0234	0.1694
		15	0.0187	0.0179	0.1508

Table 8: $P(W > n)$ for $p = 0.9$, $k_l = 5$, $k_r = 2$ when the rewards from exponential distribution

β_l	β_r	n	$P(W = n)$	Simulated values	$P(W > n)$
3	1	1	0.1407	0.1396	0.8593
		2	0.2470	0.2472	0.6123
		3	0.2275	0.2288	0.3848
		4	0.1579	0.1570	0.2269
		5	0.0839	0.0839	0.1429
		6	0.0511	0.0514	0.0918
		7	0.0253	0.0248	0.0666
		8	0.0201	0.0199	0.0465
		9	0.0114	0.0115	0.0351
		10	0.0095	0.0094	0.0256
		11	0.0060	0.0060	0.0196
		12	0.0050	0.0053	0.0146
		13	0.0032	0.0030	0.0114
		14	0.0026	0.0028	0.0088
		15	0.0019	0.0018	0.0069
3	2	1	0.3500	0.3492	0.6500
		2	0.3387	0.3365	0.3113
		3	0.1660	0.1687	0.1453
		4	0.0765	0.0760	0.0688
		5	0.0271	0.0266	0.0417
		6	0.0185	0.0194	0.0232
		7	0.0073	0.0072	0.0159
		8	0.0062	0.0068	0.0097
		9	0.0031	0.0027	0.0066
		10	0.0022	0.0025	0.0044
		11	0.0013	0.0012	0.0031

		12	0.0010	0.0010	0.0021
		13	0.0006	0.0006	0.0015
		14	0.0004	0.0005	0.0011
		15	0.0003	0.0002	0.0008
2	3	1	0.4703	0.4710	0.5297
		2	0.3253	0.3246	0.2044
		3	0.1140	0.1145	0.0904
		4	0.0500	0.0490	0.0404
		5	0.0159	0.0162	0.0245
		6	0.0121	0.0121	0.0124
		7	0.0043	0.0043	0.0081
		8	0.0035	0.0031	0.0047
		9	0.0017	0.0021	0.0030
		10	0.0010	0.0012	0.0019
		11	0.0007	0.0006	0.0013
		12	0.0004	0.0005	0.0008
		13	0.0002	0.0003	0.0006
		14	0.0002	0.0003	0.0004
		15	0.0001	0.0001	0.0003
3	3	1	0.4810	0.4826	0.5190
		2	0.3261	0.3262	0.1929
		3	0.1118	0.1090	0.0811
		4	0.0462	0.0487	0.0349
		5	0.0140	0.0129	0.0209
		6	0.0105	0.0100	0.0104
		7	0.0036	0.0031	0.0068
		8	0.0030	0.0031	0.0039
		9	0.0014	0.0016	0.0025
		10	0.0009	0.0010	0.0016
		11	0.0005	0.0006	0.0010
		12	0.0004	0.0004	0.0007
		13	0.0002	0.0003	0.0005
		14	0.0001	0.0002	0.0003
		15	0.0001	0.0001	0.0002

Table 9. $P(W > n)$ for $p = 0.3$, $p_l = 0.6$, $p_r = 0.4$ when the rewards from binomial distribution

k_l	k_r	n	$P(W = n)$	Simulated values	$P(W > n)$
5	3	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0017	0.0018	0.9983
		4	0.0021	0.0021	0.9961
		5	0.0153	0.0154	0.9809
		6	0.0244	0.0247	0.9564
		7	0.0269	0.0274	0.9295
		8	0.0262	0.0258	0.9033
		9	0.0246	0.0242	0.8787
		10	0.0234	0.0248	0.8553
		11	0.0219	0.0208	0.8334
		12	0.0213	0.0200	0.8121
		13	0.0202	0.0195	0.7919
		14	0.0196	0.0195	0.7723
		15	0.0187	0.0185	0.7536
5	4	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0000	0.0000	1.0000

		4	0.0002	0.0002	0.9998
		5	0.0134	0.0124	0.9864
		6	0.0225	0.0227	0.9639
		7	0.0251	0.0253	0.9388
		8	0.0244	0.0254	0.9144
		9	0.0230	0.0238	0.8914
		10	0.0219	0.0217	0.8696
		11	0.0205	0.0201	0.8491
		12	0.0200	0.0202	0.8291
		13	0.0189	0.0191	0.8102
		14	0.0185	0.0185	0.7917
		15	0.0176	0.0187	0.7741
5	5	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0000	0.0000	1.0000
		4	0.0000	0.0002	1.0000
		5	0.0131	0.0124	0.9869
		6	0.0223	0.0227	0.9646
		7	0.0248	0.0253	0.9398
		8	0.0241	0.0254	0.9157
		9	0.0227	0.0238	0.8930
		10	0.0216	0.0217	0.8714
		11	0.0203	0.0201	0.8511
		12	0.0197	0.0202	0.8314
		13	0.0187	0.0191	0.8127
		14	0.0183	0.0185	0.7944
		15	0.0175	0.0187	0.7770
5	6	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0000	0.0000	1.0000
		4	0.0000	0.0000	1.0000
		5	0.0131	0.0128	0.9869
		6	0.0222	0.0221	0.9647
		7	0.0248	0.0261	0.9399
		8	0.0241	0.0237	0.9159
		9	0.0227	0.0227	0.8932
		10	0.0216	0.0210	0.8716
		11	0.0202	0.0202	0.8514
		12	0.0197	0.0194	0.8317
		13	0.0187	0.0188	0.8130
		14	0.0182	0.0182	0.7948
		15	0.0174	0.0175	0.7773

Table 10: $P(W>n)$ for $p = 0.3$, $k_l = 5$, $k_r = 3$ when the rewards from binomial distribution

p_l	p_r	n	$P(W=n)$	Simulated values	$P(W>n)$
0.6	0.3	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0007	0.0007	0.9992
		4	0.0009	0.0011	0.9983
		5	0.0140	0.0139	0.9842
		6	0.0232	0.0229	0.9609
		7	0.0258	0.0253	0.9351
		8	0.0250	0.0250	0.9101
		9	0.0235	0.0243	0.8865
		10	0.0224	0.0240	0.8640
		11	0.0210	0.0216	0.8430

		12	0.0204	0.0202	0.8225
		13	0.0193	0.0187	0.8031
		14	0.0188	0.0190	0.7842
		15	0.0180	0.0175	0.7662
0.6	0.5	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0034	0.0032	0.9966
		4	0.0039	0.0039	0.9927
		5	0.0170	0.0178	0.9758
		6	0.0261	0.0251	0.9497
		7	0.0286	0.0271	0.9212
		8	0.0277	0.0273	0.8934
		9	0.0261	0.0256	0.8673
		10	0.0247	0.0242	0.8426
		11	0.0232	0.0230	0.8194
		12	0.0224	0.0225	0.7970
		13	0.0212	0.0208	0.7758
		14	0.0206	0.0212	0.7552
		15	0.0196	0.0198	0.7356
0.6	0.7	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.9261	0.9340	0.9907
		4	0.8983	0.8760	0.9818
		5	0.2175	0.2222	0.9600
		6	0.3083	0.3152	0.9292
		7	0.3321	0.3448	0.8960
		8	0.3213	0.3306	0.8638
		9	0.3018	0.2996	0.8337
		10	0.2850	0.2816	0.8052
		11	0.2664	0.2672	0.7785
		12	0.2560	0.2540	0.7529
		13	0.2414	0.2524	0.7288
		14	0.2325	0.2254	0.7055
		15	0.2206	0.2168	0.6835
0.6	0.9	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0197	0.0187	0.9803
		4	0.0155	0.0148	0.9648
		5	0.0282	0.0281	0.9366
		6	0.0373	0.0381	0.8993
		7	0.0395	0.0408	0.8598
		8	0.0380	0.0382	0.8217
		9	0.0356	0.0365	0.7862
		10	0.0334	0.0332	0.7528
		11	0.0311	0.0314	0.7216
		12	0.0297	0.0302	0.6920
		13	0.0279	0.0281	0.6641
		14	0.0266	0.0263	0.6374
		15	0.0252	0.0242	0.6123

Table 11: $P(W > n)$ for $p = 0.9$, $p_l = 0.6$, $p_r = 0.4$ when the rewards from binomial distribution

k_l	k_r	n	$P(W = n)$	Simulated values	$P(W > n)$
5	3	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0467	0.0462	0.9533
		4	0.0802	0.0796	0.8731

		5	0.0939	0.0920	0.7792
		6	0.0939	0.0934	0.6854
		7	0.0838	0.0838	0.6015
		8	0.0749	0.0749	0.5266
		9	0.0615	0.0625	0.4651
		10	0.0540	0.0545	0.4110
		11	0.0436	0.0421	0.3675
		12	0.0389	0.0396	0.3286
		13	0.0314	0.0320	0.2972
		14	0.0287	0.0283	0.2685
		15	0.0237	0.0245	0.2448
5	4	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0000	0.0000	1.0000
		4	0.0168	0.0178	0.9832
		5	0.0380	0.0386	0.9452
		6	0.0543	0.0546	0.8910
		7	0.0631	0.0616	0.8279
		8	0.0655	0.0674	0.7624
		9	0.0626	0.0603	0.6998
		10	0.0586	0.0599	0.6412
		11	0.0518	0.0512	0.5894
		12	0.0473	0.0470	0.5421
		13	0.0407	0.0403	0.5014
		14	0.0374	0.0386	0.4640
		15	0.0322	0.0330	0.4319
5	5	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0000	0.0000	1.0000
		4	0.0000	0.0000	1.0000
		5	0.0060	0.0059	0.9940
		6	0.0169	0.0160	0.9770
		7	0.0287	0.0279	0.9483
		8	0.0382	0.0384	0.9101
		9	0.0442	0.0438	0.8659
		10	0.0468	0.0473	0.8191
		11	0.0464	0.0473	0.7727
		12	0.0448	0.0456	0.7279
		13	0.0415	0.0407	0.6864
		14	0.0388	0.0397	0.6477
		15	0.0349	0.0340	0.6127
5	6	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0000	0.0000	1.0000
		4	0.0000	0.0000	1.0000
		5	0.0000	0.0000	1.0000
		6	0.0022	0.0020	0.9978
		7	0.0073	0.0073	0.9906
		8	0.0143	0.0142	0.9763
		9	0.0215	0.0226	0.9548
		10	0.0275	0.0278	0.9273
		11	0.0317	0.0310	0.8957
		12	0.0340	0.0350	0.8617
		13	0.0345	0.0341	0.8272
		14	0.0340	0.0342	0.7932
		15	0.0324	0.0327	0.7608

Table 12: $P(W > n)$ for $p = 0.9$, $k_l = 5$, $k_r = 3$ when the rewards from binomial distribution

p_l	p_r	n	n	$P(W = n)$	Simulated values
0.6	0.3	1	0.0000	0	1.0000
		2	0.0000	0	1.0000
		3	0.0197	0.0200	0.9803
		4	0.0392	0.0397	0.9411
		5	0.0526	0.0520	0.8886
		6	0.0596	0.0584	0.8290
		7	0.0604	0.0631	0.7686
		8	0.0594	0.0597	0.7092
		9	0.0543	0.0551	0.6549
		10	0.0509	0.0507	0.6040
		11	0.0447	0.0444	0.5594
		12	0.0413	0.0427	0.5181
		13	0.0357	0.0368	0.4824
		14	0.0330	0.0326	0.4494
		15	0.0286	0.0285	0.4207
0.6	0.5	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.0911	0.0907	0.9089
		4	0.1321	0.1323	0.7767
		5	0.1321	0.1310	0.6446
		6	0.1155	0.1148	0.5291
		7	0.0902	0.0899	0.4389
		8	0.0744	0.0766	0.3645
		9	0.0551	0.0559	0.3094
		10	0.0468	0.0453	0.2626
		11	0.0351	0.0338	0.2275
		12	0.0311	0.0324	0.1964
		13	0.0238	0.0232	0.1726
		14	0.0216	0.0225	0.1509
		15	0.0172	0.0164	0.1337
0.6	0.7	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.2500	0.2471	0.7500
		4	0.2275	0.2289	0.5224
		5	0.1546	0.1544	0.3678
		6	0.1054	0.1069	0.2624
		7	0.0628	0.0640	0.1995
		8	0.0506	0.0520	0.1490
		9	0.0314	0.0309	0.1175
		10	0.0268	0.0260	0.0908
		11	0.0180	0.0184	0.0728
		12	0.0154	0.0158	0.0574
		13	0.0106	0.0107	0.0468
		14	0.0091	0.0085	0.0378
		15	0.0068	0.0069	0.0310
0.6	0.9	1	0.0000	0.0000	1.0000
		2	0.0000	0.0000	1.0000
		3	0.5314	0.5316	0.4686
		4	0.1966	0.1964	0.2719
		5	0.0933	0.0929	0.1786
		6	0.0740	0.0736	0.1047
		7	0.0322	0.0324	0.0724
		8	0.0280	0.0285	0.0444
		9	0.0146	0.0146	0.0298

		10	0.0098	0.0099	0.0200
		11	0.0063	0.0063	0.0137
		12	0.0044	0.0049	0.0093
		13	0.0026	0.0024	0.0067
		14	0.0019	0.0018	0.0047
		15	0.0014	0.0015	0.0034

We also use the rewards Eqn. (6)-(7) in our study. [1] gives some numerical computation for survival function for rewards Eqn. (6) and Eqn. (7). Some extended numerical computations are presented in Tables 1-4 for survival probability $P(W > n)$ by using Eqn. (6)-(7) in Eqn. (3) for selected values of p, p_r, p_l, k_r, k_l. The simulated probabilities with 10000 trials are also included in the tables. From Tables 1-4, our theoretical and simulated probabilities can be observed as almost identical. This indicates that the survival function given in Eqn. (3) is correct.

Some numerical computations are presented in Tables 5-8 for survival probability $P(W > n)$ by using Eqn. (8)-(9) in Eqn. (3), for selected values of p, p_r, p_l, k_r, k_l. The simulated probabilities with 10000 trials are also included in the tables. From Tables 5-8, our theoretical and simulated probabilities can be observed almost identical.

Some numerical computations are presented in Tables 9-12 for survival probability $P(W > n)$ by using Eqn. (10)-(11) in Eqn. (3) for selected values of $p, \beta_r, \beta_l, k_r, k_l$. The simulated probabilities with 10000 trials are also included in the tables. From the Tables 9-12, our theoretical and simulated probabilities can be observed as almost identical. Matlab codes for theoretical probabilities can be requested from the authors.

3. Proportion Estimates for Generalized Geometric Distribution Parameters

In this section, we consider the estimation of parameters. The maximum likelihood estimation is not discussed here since the pmf of W is intractable. For this reason, we discuss the proportion type estimate for the parameters. We need the following probabilities to obtain the proportion estimation of the parameters.

$$P(W = 1) = pP(Y_1 \geq k_r) + (1 - p)P(Z_1 \geq k_l), \tag{12}$$

$$
\begin{aligned}
P(W=2) &= (1-p)P(Z_1<k_l)\,pP(Y_1\ge k_r) \\
&+pP(Y_1<k_r)(1-p)P(Z_1\ge k_l) \\
&+pP(Y_1<k_r)\,pP(Y_1+Y_2\ge k_r) \\
&+(1-p)P(Z_1<k_l)(1-p)P(Z_1+Z_2\ge k_l),
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
P(W=3) &= (1-p)P(Z_1<k_l)(1-p)P(Z_1+Z_2<k_l)\,pP(Y_1\ge k_r) \\
&+(1-p)P(Z_1<k_l)\,pP(Y_1<k_r)\,pP(Y_1+Y_2\ge k_r) \\
&+pP(Y_1<k_r)(1-p)P(Z_1<k_l)\,pP(Y_2\ge k_r) \\
&+pP(Y_1<k_r)(1-p)P(Z_1<k_l)(1-p)P(Z_1+Z_2\ge k_l) \\
&+pP(Y_1<k_r)\,pP(Y_1+Y_2<k_r)(1-p)P(Z_1\ge k_l) \\
&+(1-p)P(Z_1<k_l)\,pP(Y_1<k_r)(1-p)P(Z_2\ge k_l) \\
&+pP(Y_1<k_r)\,pP(Y_1+Y_2<k_r)\,pP(Y_1+Y_2+Y_3\ge k_r) \\
&+(1-p)P(Z_1<k_l)(1-p)P(Z_1+Z_2<k_l)(1-p) \\
&\times P(Z_1+Z_2+Z_3\ge k_l),
\end{aligned} \tag{14}
$$

Since the probabilities are too complicated, the estimation is performed under the conditions $p_l = p_r = p_J$ and $p_i = p,\ i = 1,2,\ldots$. Let $Y_1, Y_2, \ldots$ be the independent rewards with cumulative distribution function (cdf)

$$F_i(y) = P\{Y_i \le y\} = 1-(1-p_r)^y,\ \ y = 1,2,\ldots$$

and $Z_1, Z_2, \ldots$ independent Type-II rewards with cdf

$$G_i(z) = P\{Z_i \le z\} = 1-(1-p_l)^z,\ \ z = 1,2,\ldots.$$

for $i = 1,2,\ldots$.

Under condition the $p_l = p_r = p_J$, $p_i = p$, it can be written by

$$
\begin{aligned}
P(Y_1\ge k_r) &= 1-F_1(k_r-1) = (1-p_J)^{k_r-1}, \\
P(Z_1\ge k_l) &= 1-G_1(k_l-1) = (1-p_J)^{k_l-1},
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
P(Y_1+Y_2\ge k_r) &= nbincdf(k_r-3,2,p_J), \\
P(Z_1+Z_2\ge k_l) &= nbincdf(k_l-3,2,p_J),
\end{aligned} \tag{16}
$$

where

$$nbincdf(a,b,c) = \sum_{i=0}^{a}\binom{b+i+1}{i}c^b(1-c)^i \tag{17}$$

which is cdf of negative binomial distribution. These probabilities can be calculated with Matlab function *nbincdf.* Using Eqn. (12)-(13) and Eqn. (15)-(17), we have

$$P(W=1)=p(1-p_J)^{k_r-1}+(1-p)(1-p_J)^{k_l-1} \tag{18}$$

and

$$\begin{aligned} P(W=2)&=p(1-p_J)^{k_r-1}(1-p)p\left(1-(1-p_J)^{k_l-1}\right)\\ &+\left(1-nbincdf\left(k_r-3,2,p_J\right)-(1-p_J)^{k_l-1}\right)p^2\\ &+(1-p_J)^{k_l-1}(1-p)p\left(1-(1-p_J)^{k_r-1}\right)\\ &+\left(1-nbincdf\left(k_l-3,2,p_J\right)-(1-p_J)^{k_l-1}\right)(1-p)^2. \end{aligned} \tag{19}$$

Khan et al. [7] proposed the proportion type estimates for discrete Weibull distribution. In this section, we now use their methodology to get the estimates for parameters of distribution with survival function given in Eqn. (3).

Let $W_1, W_2, \ldots, W_n$ be a random sample from a distribution with survival function Eqn. (3). Furthermore, two types of rewards are independently distributed with pmfs Eqn. (4) and Eqn. (5), respectively.

Let us define the following indicator functions, for $i=1,2,\ldots n$,

$$\nu_1(W_i)=\begin{cases}1, & W_i=1\\ 0, & W_i>1\end{cases} \tag{20}$$

and

$$\nu_2(W_i)=\begin{cases}1, & W_i=2\\ 0, & W_i\neq 2\end{cases}. \tag{21}$$

It is noticed that the $\frac{1}{n}\sum_{i=1}^{n}\nu_1(W^i)$ and $\frac{1}{n}\sum_{i=1}^{n}\nu_2(W^i)$ are unbiased and consistent estimates for $P(W=1)$ and $P(W=2)$ respectively. Hence, the proportion estimates for parameters p and p_J are obtained by solving the following equations simultaneously

$$p(1-p_J)^{k_r-1}+(1-p)(1-p_J)^{k_l-1}=\tfrac{1}{n}\sum_{i=1}^{n}\nu_1(W_i) \tag{22}$$

$$\begin{aligned} h(p,p_J) &= p(1-p_J)^{k_r-1}(1-p)p\left(1-(1-p_J)^{k_l-1}\right) \\ &+\left(1-nbincdf(k_r-3,2,p_J)-(1-p_J)^{k_l-1}\right)p^2 \\ &+(1-p_J)^{k_l-1}(1-p)p\left(1-(1-p_J)^{k_r-1}\right) \\ &+\left(1-nbincdf(k_l-3,2,p_J)-(1-p_J)^{k_l-1}\right)(1-p)^2 = \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\nu_2(W_i) \end{aligned} . \tag{23}$$

From Eqn. (22) the parameter p can be expressed by

$$p^* = \frac{\frac{1}{n}\sum_{i=1}^{n}\nu_1(W_i)-(1-p_J)^{k_L-1}}{(1-p_J)^{k_R-1}-(1-p_J)^{k_L-1}}. \tag{24}$$

Using Eqn. (24) in Eqn. (23), we obtain the profile equation as

$$h(p^*,p_J) = \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\nu_2(W_i), \tag{25}$$

which is a function of p_J. Eqn. (25) can be solved by using one-dimensional root searching methods Newton-Raphson or secant method. The approximate value of the iterative root finder is a proportion estimate of parameter p_J and we denote it by $\hat{p}_J$. Then the proportion estimate of p is given by

$$\hat{p} = \frac{\frac{1}{n}\sum_{i=1}^{n}\nu_1(W_i)-(1-\hat{p}_J)^{k_L-1}}{(1-\hat{p}_J)^{k_R-1}-(1-\hat{p}_J)^{k_L-1}}. \tag{26}$$

In our trials, we observe that the equation in Eqn. (25) has two roots. In the example section, we give a suggestion to overcome this issue. As a final comment, If the constraint $p_l = p_r = p_J$ is relaxed, then Eqn. (14) should be taken in consideration and three equations should also be solved simultaneously to get the proportion estimates p_r, p_l and p.

4. Illustrative Example

In this section, we give an example to illustrate the methodology given in the previous section. We generate random data $n = 30$ sample of size from distribution in Eqn. (3) with the parameters $k_R = 8$, $k_L = 5$, $p_J = 0.4$ and $p = 0.3$. The generated data is presented in Table 13.

Table 13: Generated data

i	1	2	3	4	5	6	7
x_i	1	2	3	4	5	7	12
o_i	3	7	9	3	5	2	1

In Table 13, x_i and o_i denote the observed value and observed frequency, respectively. Using Eqn. (24)-(25), the proportion estimates of (p_J, p) are obtained by 0.4093 and 0.2252, respectively. The initial value is used as $p_J^{(0)}=0.9$ to solve the equation in Eqn. (25) and reach these estimates. It is noticed that these estimates are reasonable and they are around the true values of parameters. If the initial value is fixed by $p_J^{(0)}=0.1$, in this case, the proportion estimates of (p_J, p) are obtained by 0.2973 and 0.9031. These estimates are far from the true values of parameters. At this point, the authors have a suggestion which is given as follows: Calculated the Chi-Square goodness of fit statistic for the two different estimates. Then the estimates can be accepted, which gives minimum Chi-squares statistic. From Fig.1, we can observe the function $h(p^*, p_J)-\frac{1}{n}\sum_{i=1}^{n}\nu_2(W^i)$ is non-monotone and the equation $h(p^*, p_J)=\frac{1}{n}\sum_{i=1}^{n}\nu_2(W^i)$ has two roots.

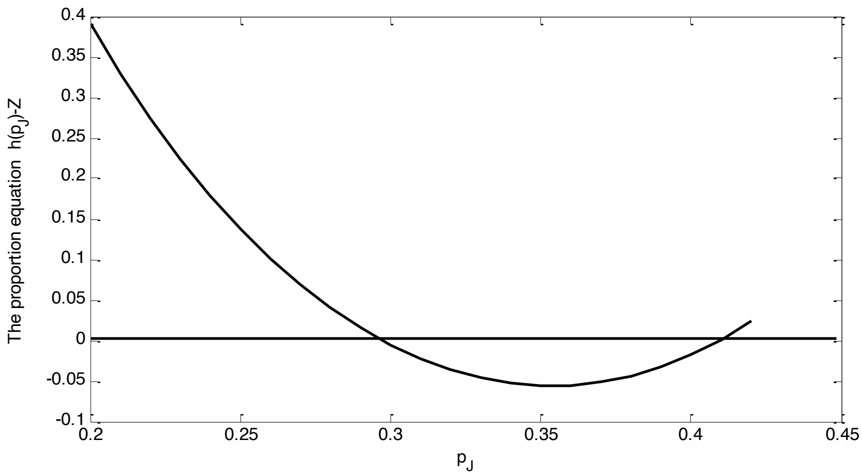

Figure 1: Graph of the $h(p_J)-\frac{1}{n}\sum_{i=1}^{n}\nu_2(W_i)$

5. Concluding Remarks and Future Researches

In this paper, the waiting time given by [1] is obtained in a different way. In this study, the survival function obtained by Eryılmaz et al. [1] based on geometric reward is extended for exponential and geometric rewards. An extensive Monte Carlo simulation study is carried for different parameter settings when the rewards are geometric, exponential and binomial. An estimation method is provided for the parameters of waiting time distribution under some constraints. An illustrative example is also carried out for sample of size $n=30$. As a future study, the proportion estimates can be used without any constraints.

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