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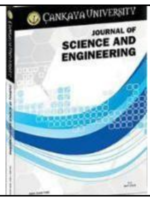

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A Fifth-Order Hybrid Block Integrator for Third-Order Initial Value Problems

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Keywords

Block Method, Chebyshev Polynomials, Linear Multistep Method.

Abstract

The formulation of hybrids block method as integrator of third-order Initial Value Problems in Ordinary Differential Equations is our focus in this paper. Chebyshev polynomials were used as trial function to develop a hybrid One-step Method (HBOSM3) adopting collocation and interpolation technique. The basic properties of HBOSM3 were integrated and findings revealed that the method was accurate and convergent. One of desirable features of these methods is the production of exact solutions at the grid points.

1. Introduction

This work is concerned with the class of the Problems

$$y^{(m)}(x) = f(x, y, y', y'' \ldots, y^{(m-1)})$$

$$y^{(s)}(x_0) = y_0(s) \quad s = 0,1,2,\ldots, m-1 \tag{1}$$

for the case $m = 3$.

The analytical solution of many of such problems does not exist. Thus, the need for formulation of numerical schemes to integrate (1) becomes necessary.

Researchers have reduced higher order of (1) to first order ODEs but with a set back see [1,2,3,4] and the inability of the method to utilize additional information associated with a specific ODE such as the oscillatory nature of the solution [5] occasioned by the increase dimension of the problem and low order of accuracy of the methods employed to solve the system of first-order IVPs of ODEs.

It has commonly been reported by scholars that implementation of linear multistep methods in predictor-corrector mode is very expensive.

To circumvent the setbacks encounter in predictor-corrector approach, the block methods have been introduced to solve IVPs in ODEs. [6,7] first proposed block methods as a means of obtaining stating values for predictor-corrector algorithms. The block methods however provide the advantage of being self starting, possess uniform order and they are obtained from a single continuous formula. Of recent, [8,9,10,11,12,13,14,15] developed different numerical methods and considered different trial functions.

Problems arising from ODEs can either be formulated as an IVP or a BVP. However, our concern shall be with IVP. Several researchers such as [16,17,18] attempt solving (1) directly using derived LMMs without reduction to system of first order ODEs. [6,18] developed new block methods which are self-starting using power series and newly constructed orthogonal polynomials as basis function. While [18] used power series as the basis

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function, [19] employed a newly derived orthogonal polynomials as trial function to develop one step hybrid block method for solution of general second order Initial value problem. Despite elegant properties of Chebyshev polynomials, it is rare seeing scholars considering it as basis function.

Thus, in this paper,we propose the development of one-step hybrid block linear multistep method for the solution of Initial Value Problems of Third Order Differential Equations with the use of Chebyshev polynomial as basis function.

2. Methodology

The approximation of analytical solution of problem (1) employing Chebyshev polynomial of the form

$$y(x)=\sum_{n=0}^{r+s-1}a_nT_n(x)\ , \tag{2}$$

as basis function on the partition $a=x_0<x_1<.......<x_k<x_{k+1}<......<x_k=b$ is considered here.

The function $y(x)$ is integrated in the interval [a, b], with a constant step size h, given by $h=x_{k+1}-x_k$; $k=0,1,....K-1$.

and $T_n(x)=Cos(nCos^{-1}x)\equiv\sum_{j=0}^{n}C_j^{(n)}x^n$, (3)

which is the nth degree Chebyshev polynomial which is valid in the range of definition of (2).

The Chebyshev polynomials $T_n(x)$ satisfied the recurrence relation

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\ \ n\ge 1\ (T_0(x)=1) \tag{4}$$

for interval [-1, 1].

Thus $T_1(x)=X=\dfrac{2x-x_{k+1}-x_k}{x_{k+1}-x_k}=\dfrac{2x-2x_k-h}{h}=t\ ,\ x_k\le x\le x_{k+1}$ (5)

where $t=t(x)$,a function of x, is given by (5).

The first, second and third derivative of (2) is given by

$$y'(x)=\sum_{n=0}^{r+s-1}a_nT_{n'}'(x) \tag{6}$$

$$y''(x)=\sum_{n=0}^{r+s-1}a_nT_n''(x) \tag{7}$$

$$y'''(x)=\sum_{n=0}^{r+s-1}a_nT_n'''(x) \tag{8}$$

where $x\in(a,b)$, the $a_n's$ are constant, **r** and **s** are points of collocation and interpolation respectively. Conventionally, we need to interpolate at least three points to be able to approximate the solution of (2) and, for this purpose, we proceed by arbitrarily selecting two off-step point,

x_{k+v}, $v\in(0,1)$ in (x_k,x_{k+1}) ensuring that the zero-stability of the main method is guaranteed.

Thus, equation (2) is interpolated at x_{k+i}, i=0, v and 1 and its third derivative is collocated at x_{k+i} i=0, v, and 1 so as to obtain a system of equations.

In what follows, we shall develop one step methods with three off step points $v=\frac{1}{4},\frac{1}{2}\,and\,\frac{3}{4}$.

2.1. Development of the Proposed Method

In this section, the derivation of continuous one-step method with three off-step point is considered. Using (2) with $r = 5$ and $s = 3$, we have a polynomial of degree seven as follow;

$$y(x) = \sum_{n=0}^{7} a_n T_n(x) \tag{9}$$

with third derivative given by

$$y'''(x) = \sum_{n=0}^{7} a_n T_n'''(x) \tag{10}$$

Collocating (10) at $x = x_{k+i}$, $i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and interpolating (9) at $x = x_{k+i}$, $i = 0, \frac{1}{4}$ and $\frac{1}{2}$ lead to a system of equations written in matrix form $AX = B$ as:

$$\begin{pmatrix} 0 & 0 & 0 & 192 & -1536 & 6720 & -21504 & 56448 \\ 0 & 0 & 0 & 192 & -768 & 960 & 768 & -4032 \\ 0 & 0 & 0 & 192 & 0 & -960 & 0 & 2688 \\ 0 & 0 & 0 & 192 & 768 & 960 & -768 & -4032 \\ 0 & 0 & 0 & 192 & 1536 & 6720 & 21504 & 56448 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} h^3 f_k \\ h^3 f_{k+\frac{1}{4}} \\ h^3 f_{k+\frac{1}{2}} \\ h^3 f_{k+\frac{3}{4}} \\ h^3 f_{k+1} \\ y_k \\ y_{k+\frac{1}{4}} \\ y_{k+\frac{1}{2}} \end{pmatrix} \tag{11}$$

The values of a_j, $j = 0,1,\ldots 7$ below are obtained using Maple software to solve (11)

$$\left.\begin{aligned}
a_0 &= \frac{1}{11520}h^3 f_k + \frac{143}{23040}h^3 f_{k+\frac{1}{4}} + \frac{21}{2560}h^3 f_{k+\frac{1}{2}} + \frac{5}{4608}h^3 f_{k+\frac{3}{4}} + \frac{1}{23040}h^3 f_{k+1} + y_k - 2y_{k+\frac{1}{4}} + 2y_{k+\frac{1}{2}} \\
a_1 &= \frac{13}{161280}h^3 f_k + \frac{793}{80640}h^3 f_{k+\frac{1}{4}} + \frac{377}{26880}h^3 f_{k+\frac{1}{2}} + \frac{163}{80640}h^3 f_{k+\frac{3}{4}} + \frac{13}{161280}h^3 f_{k+1} + y_k - 4y_{k+\frac{1}{4}} + 3y_{k+\frac{1}{2}} \\
a_3 &= \frac{1}{46080}h^3 f_k + \frac{7}{5760}h^3 f_{k+\frac{1}{4}} + \frac{7}{2560}h^3 f_{k+\frac{1}{2}} + \frac{7}{5760}h^3 f_{k+\frac{3}{4}} + \frac{1}{46080}h^3 f_{k+1} \\
a_4 &= -\frac{1}{46080}h^3 f_k - \frac{7}{11520}h^3 f_{k+\frac{1}{4}} + \frac{7}{11520}h^3 f_{k+\frac{3}{4}} + \frac{1}{46080}h^3 f_{k+1} \\
a_5 &= \frac{1}{46080}h^3 f_k + \frac{1}{5760}h^3 f_{k+\frac{1}{4}} - \frac{1}{2560}h^3 f_{k+\frac{1}{2}} + \frac{1}{5760}h^3 f_{k+\frac{3}{4}} + \frac{1}{46080}h^3 f_{k+1} \\
a_6 &= -\frac{1}{46080}h^3 f_k + \frac{1}{23040}h^3 f_{k+\frac{1}{4}} - \frac{1}{23040}h^3 f_{k+\frac{3}{4}} + \frac{1}{46080}h^3 f_{k+1} \\
a_7 &= \frac{1}{161280}h^3 f_k - \frac{1}{40320}h^3 f_{k+\frac{1}{4}} + \frac{1}{26880}h^3 f_{k+\frac{1}{2}} - \frac{1}{40320}h^3 f_{k+\frac{3}{4}} + \frac{1}{161280}h^3 f_{k+1}
\end{aligned}\right\} \tag{12}$$

Substituting the values of $a_i (0 \le i \le 7)$ into (2), we obtain a continuous scheme in the form

$$y(x)=\alpha_0(x)y_k+\alpha_{\frac{1}{4}}(x)y_{k+\frac{1}{4}}+\alpha_{\frac{1}{2}}(x)y_{k+\frac{1}{2}}+h^3\left(\sum_0^1\beta_j(x)f_{k+j}+\beta_{\frac{1}{4}}(x)f_{k+\frac{1}{4}}+\beta_{\frac{1}{2}}(x)f_{k+\frac{1}{2}}+\beta_{\frac{3}{4}}(x)f_{k+\frac{3}{4}}\right)\Bigg\} \quad (13)$$

Simplifying (13), we have

$$\left.\begin{aligned}
\alpha_0(t)&=t+2t^2\\
\alpha_{\frac{1}{4}}(t)&=-4t-4t^2\\
\alpha_{\frac{1}{2}}(t)&=1+3t+2t^2\\
\beta_0(t)&=h^3\left(\frac{13}{161280}t-\frac{1}{23040}t^2+\frac{1}{1152}t^4-\frac{1}{2880}t^5-\frac{1}{1440}t^6+\frac{1}{2520}t^7\right)\\
\beta_{\frac{1}{4}}(t)&=h^3\left(\frac{583}{80640}t+\frac{193}{11520}t^2-\frac{1}{144}t^4+\frac{1}{180}t^5+\frac{1}{720}t^6-\frac{1}{630}t^7\right)\\
\beta_{\frac{1}{2}}(t)&=h^3\left(\frac{97}{26880}t+\frac{21}{1280}t^2+\frac{1}{48}t^3-\frac{1}{96}t^5+\frac{1}{420}t^7\right)\\
\beta_{\frac{3}{4}}(t)&=h^3\left(-\frac{47}{80640}t-\frac{5}{2304}t^2+\frac{1}{144}t^4+\frac{1}{180}t^5-\frac{1}{720}t^6-\frac{1}{630}t^7\right)\\
\beta_1(t)&=h^3\left(\frac{13}{161280}t+\frac{7}{23040}t^2-\frac{1}{1152}t^4-\frac{1}{2880}t^5+\frac{1}{1440}t^6+\frac{1}{2520}t^7\right)
\end{aligned}\right\} \quad (14)$$

where $t=\dfrac{2x-2x_k-h}{h}$

Evaluating (13) with the expressions in (14) at $x=x_{k+\frac{3}{4}}$ and $x=x_{k+1}$ the following main methods are obtained as

$$y_{k+\frac{3}{4}}=y_k-3y_{k+\frac{1}{4}}+3y_{k+\frac{1}{2}}+\frac{h^3}{15360}\left(f_k+116f_{k+\frac{1}{4}}+126f_{k+\frac{1}{2}}-4f_{k+\frac{3}{4}}+f_{k+1}\right) \quad (15)$$

$$y_{k+1}=3y_k-8y_{k+\frac{1}{4}}+6y_{k+\frac{1}{2}}+\frac{h^3}{3840}\left(f_k+86f_{k+\frac{1}{4}}+126f_{k+\frac{1}{2}}+26f_{k+\frac{3}{4}}+f_{k+1}\right) \quad (16)$$

Differentiating (13), we obtain

$$\left.\begin{aligned}
&\alpha'_0(t)=\frac{1}{h}\left(2+8t\right)\\
&\alpha'_{\frac{1}{4}}(t)=-\frac{1}{h}\left(8+16t\right)\\
&\alpha'_{\frac{1}{2}}(t)=\frac{1}{h}\left(6+8t\right)\\
&\beta'_0(t)=h^2\left(\frac{13}{80640}-\frac{1}{5760}t+\frac{1}{144}t^3-\frac{1}{288}t^4-\frac{1}{120}t^5+\frac{1}{180}t^6\right)\\
&\beta'_{\frac{1}{4}}(t)=h^2\left(\frac{583}{40320}+\frac{193}{2880}t-\frac{1}{18}t^3+\frac{1}{18}t^4+\frac{1}{60}t^5-\frac{1}{45}t^6\right)\\
&\beta'_{\frac{3}{4}}(t)=h^2\left(-\frac{47}{40320}-\frac{5}{576}t+\frac{1}{18}t^3+\frac{1}{18}t^4-\frac{1}{60}t^5-\frac{1}{45}t^6\right)\\
&\beta'_1(t)=h^2\left(\frac{13}{80640}+\frac{7}{5760}t-\frac{1}{144}t^3-\frac{1}{288}t^4+\frac{1}{120}t^5+\frac{1}{180}t^6\right)
\end{aligned}\right\}\tag{17}$$

The second derivatives of continuous functions are given as

$$\left.\begin{aligned}
&\alpha''_0(t)=\frac{16}{h^2}\\
&\alpha''_{\frac{1}{4}}(t)=-\frac{32}{h^2}\\
&\alpha''_{\frac{1}{2}}(t)=\frac{16}{h^2}\\
&\beta''_0(t)=h\left(-\frac{1}{2880}+\frac{1}{24}t^2-\frac{1}{36}t^3-\frac{1}{12}t^4+\frac{1}{15}t^5\right)\\
&\beta''_{\frac{1}{4}}(t)=h\left(\frac{193}{1440}-\frac{1}{3}t^2+\frac{4}{9}t^3+\frac{1}{6}t^4-\frac{4}{15}t^5\right)\\
&\beta''_{\frac{1}{2}}(t)=h\left(\frac{21}{160}+\frac{1}{2}t-\frac{5}{6}t^3+\frac{2}{5}t^5\right)\\
&\beta''_{\frac{3}{4}}(t)=h\left(-\frac{5}{288}+\frac{1}{3}t^2+\frac{4}{9}t^3-\frac{1}{6}t^4-\frac{4}{15}t^5\right)\\
&\beta''_1(t)=h\left(\frac{7}{2880}-\frac{1}{24}t^2-\frac{1}{36}t^3+\frac{1}{12}t^4+\frac{1}{15}t^5\right)
\end{aligned}\right\}\tag{18}$$

The additional methods to be coupled with the main methods are obtained by evaluating the first and second derivatives of (13) at $x_k\ x_{k+\frac{1}{4}}$, $x_{k+\frac{1}{2}}\ x_{k+\frac{3}{4}}$ and x_{k+1}. Thus, we have the following discrete schemes:

$$hy'_k+6y_k-8y_{k+\frac{1}{4}}+2y_{k+\frac{1}{2}}=h^3\left(\frac{307}{80640}f_k+\frac{793}{40320}f_{k+\frac{1}{4}}-\frac{19}{4480}f_{k+\frac{1}{2}}+\frac{79}{40320}f_{k+\frac{3}{4}}-f_{k+1}\right)\tag{19}$$

$$hy'_{k+\frac{1}{4}}+2y_k-2y_{k+\frac{1}{2}}=h^3\left(-\frac{79}{161280}f_k-\frac{383}{40320}f_{k+\frac{1}{4}}-\frac{3}{8960}f_{k+\frac{1}{2}}-\frac{1}{8064}f_{k+\frac{3}{4}}+\frac{1}{32256}f_{k+1}\right)(20)$$

$$hy'_{k+\frac{1}{2}}-2y_k+8y_{k+\frac{1}{4}}-6y_{k+\frac{1}{2}}=h^3\left(\frac{13}{80640}f_k+\frac{583}{40320}f_{k+\frac{1}{4}}+\frac{97}{13440}f_{k+\frac{1}{2}}-\frac{47}{40320}f_{k+\frac{3}{4}}+\frac{13}{80640}f_{k+1}\right)(21)$$

$$hy'_{k+\frac{3}{4}}-6y_k+16y_{k+\frac{1}{4}}-10y_{k+\frac{1}{2}}=h^3\left(\frac{89}{161280}f_k+\frac{1801}{40320}f_{k+\frac{1}{4}}+\frac{117}{1792}f_{k+\frac{1}{2}}+\frac{163}{40320}f_{k+\frac{3}{4}}+\frac{1}{32256}f_{k+1}\right)(22)$$

$$hy'_{k+1}-10y_k+24y_{k+\frac{1}{4}}-14y_{k+\frac{1}{2}}=h^3\left(\frac{11}{16128}f_k+\frac{3061}{40320}f_{k+\frac{1}{4}}+\frac{569}{4480}f_{k+\frac{1}{2}}+\frac{503}{8064}f_{k+\frac{3}{4}}+\frac{391}{80640}f_{k+1}\right)(23)$$

$$h^2y''_k-16y_k+32y_{k+\frac{1}{4}}-16y_{k+\frac{1}{2}}=h^3\left(-\frac{233}{2880}f_k-\frac{101}{480}f_{k+\frac{1}{4}}+\frac{31}{480}f_{k+\frac{1}{2}}-\frac{41}{1440}f_{k+\frac{3}{4}}-\frac{1}{192}f_{k+1}\right)(24)$$

$$h^2y''_{k+\frac{1}{4}}-16y_k+32y_{k+\frac{1}{4}}-16y_{k+\frac{1}{2}}=h^3\left(\frac{1}{160}f_k+\frac{1}{72}f_{k+\frac{1}{4}}-\frac{13}{480}f_{k+\frac{1}{2}}+\frac{1}{120}f_{k+\frac{3}{4}}-\frac{1}{720}f_{k+1}\right)(25)$$

$$h^2y''_{k+\frac{1}{2}}-16y_k+32y_{k+\frac{1}{4}}-16y_{k+\frac{1}{2}}=h^3\left(-\frac{1}{2880}f_k+\frac{193}{1440}f_{k+\frac{1}{4}}+\frac{21}{160}f_{k+\frac{1}{2}}-\frac{5}{288}f_{k+\frac{3}{4}}+\frac{7}{2880}f_{k+1}\right)(26)$$

$$h^2y''_{k+\frac{3}{4}}-16y_k+32y_{k+\frac{1}{4}}-16y_{k+\frac{1}{2}}=h^3\left(\frac{1}{288}f_k+\frac{13}{120}f_{k+\frac{1}{4}}+\frac{139}{480}f_{k+\frac{1}{2}}+\frac{37}{360}f_{k+\frac{3}{4}}-\frac{1}{240}f_{k+1}\right)(27)$$

$$h^2y''_{k+\frac{1}{4}}-16y_k+32y_{k+\frac{1}{4}}-16y_{k+\frac{1}{2}}=h^3\left(-\frac{1}{320}f_k+\frac{209}{1440}f_{k+\frac{1}{4}}+\frac{19}{96}f_{k+\frac{1}{2}}+\frac{157}{480}f_{k+\frac{3}{4}}+\frac{239}{2880}f_{k+1}\right)(28)$$

Equations $(19),(20)\text{K}\ (28)$ are solved simultaneously to obtain the block method as shown below.

$$\left.\begin{aligned}
y_{k+\frac{1}{4}} &= y_k + \frac{1}{4}hy_k' + \frac{1}{32}h^2y_k'' + \frac{113}{71680}h^3f_k + \frac{107}{64512}h^3f_{k+\frac{1}{4}} - \frac{103}{107520}h^3f_{k+\frac{1}{2}} + \frac{43}{107520}h^3f_{k+\frac{3}{4}} - \frac{47}{645120}h^3f_{k+1}\\
y_{k+\frac{1}{2}} &= y_k + \frac{1}{2}hy_k' + \frac{1}{8}h^2y_k'' + \frac{331}{40320}h^3f_k k + \frac{83}{5040}h^3f_{k+\frac{1}{4}} - \frac{1}{168}h^3f_{k+\frac{1}{2}} + \frac{13}{5040}h^3f_{k+\frac{3}{4}} - \frac{19}{40320}h^3f_{k+1}\\
y_{k+\frac{3}{4}} &= y_k + \frac{3}{4}hy_k' + \frac{9}{32}h^2y_k'' + \frac{1431}{71680}h^3f_k + \frac{1863}{35840}h^3f_{k+\frac{1}{4}} - \frac{243}{35840}h^3f_{k+\frac{1}{2}} + \frac{45}{7168}h^3f_{k+\frac{3}{4}} - \frac{81}{71680}h^3f_{k+1}\\
y_{k+1} &= y_k + hy_k' + \frac{1}{2}h^2y_k'' + \frac{31}{840}h^3f_k + \frac{34}{315}h^3f_{k+\frac{1}{4}} + \frac{1}{210}h^3f_{k+\frac{1}{2}} + \frac{2}{105}h^3f_{k+\frac{3}{4}} - \frac{1}{504}h^3f_{k+1}\\
y'_{k+\frac{1}{4}} &= hy_k' + \frac{1}{4}h^2y_k'' + \frac{145}{9103}h^3f_k + \frac{3}{128}h^3f_{k+\frac{1}{4}} - \frac{47}{3840}h^3f_{k+\frac{1}{2}} + \frac{29}{5760}h^3f_{k+\frac{3}{4}} - \frac{7}{7680}h^3f_{k+1}\\
y'_{k+\frac{1}{2}} &= hy_k' + \frac{1}{2}h^2y_k'' + \frac{53}{1440}h^3f_k + \frac{1}{10}h^3f_{k+\frac{1}{4}} - \frac{1}{48}h^3f_{k+\frac{1}{2}} + \frac{1}{90}h^3f_{k+\frac{3}{4}} - \frac{1}{480}h^3f_{k+1}\\
y'_{k+\frac{3}{4}} &= hy_k' + \frac{3}{4}h^2y_k'' + \frac{147}{2560}h^3f_k + \frac{117}{640}h^3f_{k+\frac{1}{4}} + \frac{27}{1280}h^3f_{k+\frac{1}{2}} + \frac{3}{128}h^3f_{k+\frac{3}{4}} - \frac{9}{2560}h^3f_{k+1}\\
y'_{k+1} &= hy_k' + h^2y_k'' + \frac{7}{90}h^3f_k + \frac{4}{15}h^3f_{k+\frac{1}{4}} + \frac{1}{15}h^3f_{k+\frac{1}{2}} + \frac{4}{45}h^3f_{k+\frac{3}{4}}\\
y''_{k+\frac{1}{4}} &= h^2y_k'' + \frac{251}{2880}h^3f_k + \frac{323}{1440}h^3f_{k+\frac{1}{4}} - \frac{11}{120}h^3f_{k+\frac{1}{2}} + \frac{53}{1440}h^3f_{k+\frac{3}{4}} - \frac{19}{2880}h^3f_{k+1}\\
y''_{k+\frac{1}{2}} &= h^2y_k'' + \frac{29}{360}h^3f_k + \frac{31}{90}h^3f_{k+\frac{1}{4}} + \frac{1}{15}h^3f_{k+\frac{1}{2}} + \frac{1}{90}h^3f_{k+\frac{3}{4}} - \frac{1}{360}h^3f_{k+1}\\
y''_{k+\frac{3}{4}} &= h^2y_k'' + \frac{27}{320}h^3f_k + \frac{51}{160}h^3f_{k+\frac{1}{4}} + \frac{9}{40}h^3f_{k+\frac{1}{2}} + \frac{21}{160}h^3f_{k+\frac{3}{4}} - \frac{3}{320}h^3f_{k+1}\\
y''_{k+1} &= h^2y_k'' + \frac{7}{90}h^3f_k + \frac{16}{45}h^3f_{k+\frac{1}{4}} + \frac{2}{15}h^3f_{k+\frac{1}{2}} + \frac{16}{45}h^3f_{k+\frac{3}{4}} + \frac{7}{90}h^3f_{k+1}
\end{aligned}\right\} \quad (29)$$

3. Basic Properties of the Method

3.1. Order and Error Constant

Here under, the basic properties of the derived schemes are discussed.
The implicit schemes (15) and (16) belong to the class of LMM of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^3\sum_{j=0}^{k}\beta_j f_{n+j} \quad (30)$$

Following [21], the approach adopted in [9,20,3], we define the local truncation error associated with equation (30) by the difference operator

$$L[y(x):h] = \sum_{j=0}^{k}\left[\alpha_j y(x_n + jh) - h^3\beta_j f(x_n + jh)\right] \quad (31)$$

Where $y(x)$ is an arbitrary function, continuously differentiable on [a, b].
Expanding (31) in Taylor series about the point x, we obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \ldots + C_{p+3}h^{p+3}y^{p+3}(x)$$

Where the C_0, C_1, $C_2 \ldots C_p \ldots C_{p+2}$ are obtained as

$$C_0 = \sum_{j=0}^{k}\alpha_j \quad , \qquad C_1 = \sum_{j=1}^{k} j\alpha_j \, , \qquad C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right]$$

Definition 1:
The LMM (30) is said to be of order p if $C_0 = C_1 = C_2 = \ldots C_p = C_{p+2} = 0$. $C_{p+3} \neq 0$ is called the error constant and $C_{p+3} h^{p+3} y^{p+3}(x_n)$ is the principal local truncation error at the point x_n (see Lambert [3] and Henrici [21]).
The approach adopted above to obtain order and error constant can be further simplified as presented below. Defining the derived schemes in a generalized form as

$$y_{k+p} - uy_{k+q} - vy_{k+r} - wy_k - h^3 \{af_k + bf_{k+r} + cf_{k+q} + df_{k+p}\} = 0 \tag{32}$$

Where $p > q > n$ and p, q and n are desired positive real numbers.
Expanding (32) in Taylor series gives

$$\sum_{j=0}^{\infty} \frac{h^j}{j!} y_k^{(j)} \{(p)^j - u(q)^j - v(r)^j\} + wy_k - \sum_{j=0}^{\infty} \frac{h^{(j+3)}}{j!} y_k^{(j+3)} \{b(r)^j + c(q)^j + d(p)^j\} - ah^3 y_k^{(3)} = 0 \tag{33}$$

Thus, we have

$$L[y(x_k);h] = y\left(x_k + \frac{h}{4}\right) - y(x_k) - \frac{h}{4} y'(x_k) - \frac{h^2}{32} y''(x_k) - \frac{113h^3}{71680} y'''(x_k) - \frac{107h^3}{64512} y'''\left(x_k + \frac{h}{4}\right) + \frac{103h^3}{107520} y'''\left(x_k + \frac{h}{2}\right) - \frac{43h^3}{107520} y'''\left(x_k + \frac{3h}{4}\right) + \frac{47h^3}{645120} y'''(x_k + h)$$

$$L[y(x_k);h] = y\left(x_k + \frac{h}{2}\right) - y(x_k) - \frac{h}{2} y'(x_k) - \frac{h^2}{8} y''(x_k) - \frac{331h^3}{40320} y'''(x_k) - \frac{83h^3}{5040} y'''\left(x_k + \frac{h}{4}\right) + \frac{h^3}{168} y'''\left(x_k + \frac{h}{2}\right) - \frac{13h^3}{5040} y'''\left(x_k + \frac{3h}{4}\right) + \frac{19h^3}{40320} y'''(x_k + h)$$

$$L[y(x_k);h] = y\left(x_k + \frac{3h}{4}\right) - y(x_k) - \frac{3h}{4} y'(x_k) - \frac{9h^2}{32} y''(x_k) - \frac{1431h^3}{71680} y'''(x_k) - \frac{1863h^3}{35840} y'''\left(x_k + \frac{h}{4}\right) + \frac{243h^3}{35840} y'''\left(x_k + \frac{h}{2}\right) - \frac{45h^3}{7168} y'''\left(x_k + \frac{3h}{4}\right) + \frac{81h^3}{71680} y'''(x_k + h)$$

$$L[y(x_k);h] = y(x_k + h) - y(x_k) - hy'(x_k) - \frac{h^2}{2} y''(x_k) - \frac{31h^3}{840} y'''(x_k) - \frac{34h^3}{315} y'''\left(x_k + \frac{h}{4}\right) - \frac{h^3}{210} y'''\left(x_k + \frac{h}{2}\right) - \frac{2h^3}{105} y'''\left(x_k + \frac{3h}{4}\right) + \frac{h^3}{504} y'''(x_k + h)$$

In this way, we obtain the following local truncation errors

$$L[y(x_{k+\frac{1}{4}}), h] = \frac{139h^8 y^{(8)}(x_k)}{2642411520} + O(h^9), \qquad L[y'(x_{k+\frac{1}{4}}), h] = \frac{107h^9 y^{(9)}(x_k)}{165150720} + O(h^{10})$$

$$L[y(x_{k+\frac{1}{2}}), h] = \frac{1h^8 y^{(8)}(x_k)}{2949120} + O(h^9), \qquad L[y'(x_{k+\frac{1}{2}}), h] = \frac{h^9 y^{(9)}(x_k)}{645120} + O(h^{10})$$

$$L[y(x_{k+\frac{3}{4}}), h] = \frac{243h^8 y^{(8)}(x_k)}{293601280} + O(h^9), \qquad L[y'(x_{k+\frac{3}{4}}), h] = \frac{9h^9 y^{(9)}(x_k)}{3670016} + O(h^{10})$$

$$L[y(x_{k+1}), h] = \frac{h^8 y^{(8)}(x_k)}{645120} + O(h^9), \qquad L[y'(x_{k+1}), h] = \frac{h^9 y^{(9)}(x_k)}{322560} + O(h^{10})$$

$$L[y''(x_{k+\frac{1}{4}}), h] = \frac{3h^{10} y^{(10)}(x_k)}{655360} + O(h^{11}), \qquad L[y''(x_{k+\frac{1}{2}}), h] = \frac{h^{10} y^{(10)}(x_k)}{368640} + O(h^{11})$$

$$L[y''(x_{k+\frac{3}{4}}),h]=\frac{5h^{10}y^{(10)}(x_k)}{655360}+O(h^{11})$$

$$L[y''(x_{k+1}),h]=0+O(h^{11})$$

Thus, with the analysis above the method is of order $P=5$.

3.2. Zero Stability

Definition 2: The LMM (30) is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root of modulus one has multiplicity not greater than three.

All the roots of the derived schemes have been verified to be less than or equal to 1 and $|Z|=1$, simple.

Region of Absolute Stability

The region of absolute stability is as shown in Figure 1 below.

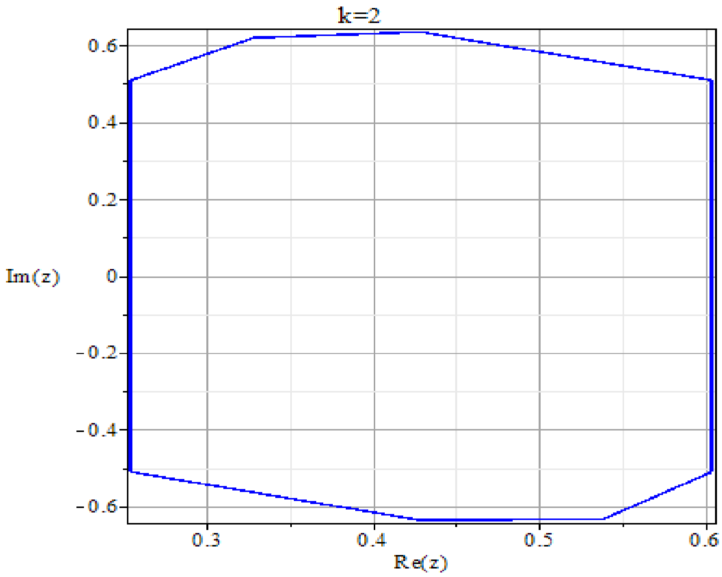

Figure 1. Region of Absolute Stability of the Proposed Scheme

3.3. Consistency

Definition 3: The LMM (30) is said to be consistent if it is of order $p\geq 1$ and its first and second characteristic polynomials defined as $\rho(z)=\sum_{j=0}^{k}\alpha_j z^j$ and $\sigma(z)=\sum_{j=0}^{k}\beta_j z^j$

where z satisfied (i) $\sum_{j=0}^{k}\alpha_j=0$ (ii) $\rho(1)=\rho'(1)=0$ and (iii) $\rho'''(1)=3!\sigma(1)$

The discrete schemes derived are all of order greater than one and satisfy the conditions (i)-(iii).

4. Numerical Applications

We consider here the application of the derived schemes to four test problems for the efficiency and accuracy of the method implemented as block method.

Problem 1. (A constant coefficient non-homogeneous problem)

$$y'''-2y''-3y'+10y=34xe^{-2x}-16e^{-2x}-10x^2+6x+34$$

$y(0) = 3\ y'(0) = y''(0) = 0\,,\quad x \in [0, b]$

Exact Solution: $y(x) = x^2 e^{-2x} - x^2 + 3$

Source: [22]

Problem 2. (System of Third Order Non- homogeneous Equations)

Consider linear system

$$y''' = \frac{1}{68}(817y + 1393z + 448w)$$

$$z''' = -\frac{1}{68}(1141y + 2837z + 896w)$$

$$w''' = \frac{1}{136}(3059y + 4319z + 1592w)$$

with initial conditions

$$y(0) = -2, z(0) = -2, w(0) = -12$$

$$y'(0) = -12, z'(0) = 28, w'(0) = -33$$

$$y''(0) = 20, z''(0) = -52, w''(0) = 5$$

The analytical solution of the problem is given by

$$y = \exp(x) - 2\exp(2x) + 3\exp(-3x)$$

$$z = 3\exp(x) + 2\exp(2x) - 7\exp(-3x)$$

$$w = -11\exp(x) - 5\exp(2x) + 4\exp(-3x)$$

Source: [23]

Problem 3. Non-linear Blasius Equation (Application Problem)

$$2y''' + yy'' = 0$$

$$y(0) = 0$$

$$y'(0) = 0$$

$$y''(0) = 1$$

The exact solution does not exist.

Source: [24]

Problem 4. Non-linear problem

$$y^2 y''' = 1$$

$$y(0) = 1\ y'(0) = 1\ y''(0) = 1\ h = 0.1$$

Source: [2]

The above problem was derived by [25] to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface.

4.1. Table of Results for Derived Schemes

Table 1. Comparing the Exact and Approximate Solutions for Problem 1

S/N	Exact Solution	HBOSM3	Error in HBOSM3 Order $p=5$
0.1	2.99818730753077981860	2.99818730747705733110	5.37224875×10^{-11}
0.2	2.98681280184142557200	2.98681280161700194330	2.24423629×10^{-10}
0.3	2.95939304724846237890	2.95939304672226093300	5.26201446×10^{-10}
0.4	2.91189263425875545460	2.91189263327710326110	9.81652194×10^{-10}
0.5	2.84196986029286058040	2.84196985867073284320	1.62212774×10^{-09}
0.6	2.74842991628839275480	2.74842991380017445720	2.48821829×10^{-09}
0.7	2.63083251233138717370	2.63083250870099794350	3.63038923×10^{-09}
0.8	2.48921377151657946140	2.48921376640688782660	5.10969163×10^{-09}
0.9	2.32389209945948509600	2.32389209246103497220	6.99845012×10^{-09}
1.0	2.13533528323661269190	2.13533527385580657200	9.38080612×10^{-09}

Table 2a. Comparing the Exact and Approximate Solutions Problem 2

S/N	$Y(X)$	$Z(X)$	$W(X)$	$y(x)$	$z(x)$	$w(x)$
0.1	-3.1231894926	0.5837759655	-15.3156161567	-3.1231894560	0.5837758783	-15.3156161090
0.2	-4.1798720039	2.8955946767	-18.8192683596	-4.1798719797	2.8955946098	-18.8192683368
0.3	-5.2907280101	5.1490391846	-22.7376338078	-5.2907279628	5.1490390677	-22.7376337494
0.4	-6.5671473768	7.5295418360	-27.2927163439	-6.5671472799	7.5295416251	-27.2927161981
0.5	-8.1153095717	10.2017225511	-32.7093811319	-8.1153095365	10.2017223859	-32.7093811583
0.6	-10.0394579136	13.3141947486	-39.2223944883	-10.0394578721	13.3141945579	-39.2223945116
0.7	-12.4447423776	17.0026290626	-47.0835436967	-12.4447425605	17.0026291561	-47.0835442368
0.8	-15.4400130476	21.3918511265	-56.5694498912	-15.4400134215	21.3918516601	-56.56945068415
0.9	-19.1409691671	26.5977797475	-67.9907375934	-19.1409702416	26.5977812383	-67.9907399319
1.0	-23.6741293029	32.7300983533	-81.7033541871	-23.6741307084	32.7300983533	-81.7033568756

Table 2b. Absolute Errors Comparing the Exact and Numerical Solution of HBOSM3 for Problem 2

x	$Y(x)-y(x)$	$Z(x)-z(x)$	$W(x)-w(x)$
	Order $p=5$	Order $p=5$	Order $p=5$
0.1	3.65973272×10^{-08}	8.71808821×10^{-08}	4.76871627×10^{-08}
0.2	2.41625333×10^{-08}	6.69145923×10^{-08}	2.28771554×10^{-08}
0.3	4.73871817×10^{-08}	1.16903887×10^{-07}	5.84087227×10^{-08}
0.4	9.68805796×10^{-08}	2.10923016×10^{-07}	1.45746882×10^{-07}
0.5	3.52409556×10^{-08}	1.65185369×10^{-07}	2.63339165×10^{-08}
0.6	4.15722789×10^{-08}	1.90594603×10^{-07}	2.33019014×10^{-08}
0.7	1.82932181×10^{-07}	9.34770509×10^{-08}	5.40070027×10^{-07}
0.8	3.73898906×10^{-07}	5.33604181×10^{-07}	7.92936188×10^{-07}

0.9	$1,07452850\times10^{-06}$	1.49079207×10^{-06}	2.33857116×10^{-06}
1.0	1.40552304×10^{-06}	2.35365180×10^{-06}	2.68851230×10^{-06}

Table 3. Comparing the Solution of the Approximate and the Existing Method for Problem 3

S/N	**Exact Solution**	**HBOSM3**	**Error in HBOSM3** **Order** $p=5$	**Error in [3]** **Order** $p=6$
0.1	0.00499995518745601000	0.00499995833397275076	3.14693822×10^{-09}	4.27300000×10^{-08}
0.2	0.01999865908023810000	0.01999866684107568891	7.76083759×10^{-09}	1.20759000×10^{-06}
0.3	0.04498987410259470000	0.04498987947242828080	5.36983358×10^{-09}	8.60719000×10^{-06}
0.4	0.07995737735167610000	0.07995737798187473660	6.30198637×10^{-10}	3.40900400×10^{-05}
0.5	0.12487004764653700000	0.12487005751675742098	9.87022042×10^{-09}	9.74068000×10^{-05}
0.6	0.17967712636121700000	0.17967714132520461733	1.49639876×10^{-08}	2.25711000×10^{-04}
0.7	0.24430361290038500000	0.24430361709211550733	4.19173051×10^{-09}	4.51454700×10^{-04}
0.8	0.31864597946467400000	0.31864600945693486117	2.99922609×10^{-08}	8.08472900×10^{-04}
0.9	0.40256860621313400000	0.40256862074667307803	1.45335391×10^{-08}	1.32622070×10^{-03}
1.0	0.49590033762933700000	0.49590038304760480831	4.54182678×10^{-08}	2.0220546×10^{-03}

Table 4. Comparing the Solution of the Approximate and the Existing Method for Problem 4

S/N	**Exact Solution**	**HBOSM3**	**Error in HBOSM3** **Order** $p=5$	**Error in [7]** **Order** $p=4$
0.2	1.22121001337746352620	1.22121000453347653350	8.84398699×10^{-09}	2.40500000×10^{-05}
0.4	1.48883473296637175650	1.48883477988252287300	4.69161511×10^{-09}	7.71670000×10^{-05}
0.6	1.80736134919720764840	1.80736139771131630540	4.85141087×10^{-09}	7.94945000×10^{-06}
0.8	2.17981922624938085950	2.17981923396911604190	7.71973518×10^{-08}	4.34949000×10^{-03}
1.0	2.60827491835217941000	2.60827486766264780410	5.06895316×10^{-07}	1.83199620×10^{-02}

KEY:
HBOSM3: HYBRID BLOCK ONE STEP METHOD WITH THREE OFF-STEP POINTS

5. Discussion of Results

Problems 1 and 2 are constant coefficient non- homogeneous equations and system of third order non-homogeneous equations respectively. The results were displayed in Tables 1 and 2 respectively. The absolute errors obtained revealed that low errors resulted from the comparison of the solutions obtained from the implementation of the derived schemes with the corresponding exact solutions.
Problem 3 however considered Blassius equation in Thermodynamics, while another non-linear differential equation was considered in problem 4. Their exact solutions were not available. Hence, they were generated directly using Maple software environment. Tables 3 and 4 presented the solutions of problems 3 and 4 as comparison of our developed order 5 HBOSM3 with order 6 method of [24] and order 4 method of [2] respectively. The superiority of the method has been established numerically.

6. Conclusions

Initial value problem solver had been developed by interpolation and collocation techniques using Chebyshev polynomials as basis function. Four test problems have been considered to show the efficiency and accuracy of the method. Tables 1, 2, 3 and 4 display the accuracy and comparison of the numerical results of the HBOSM3 with the exact solution and existing methods. The method's desirability and superiority have been established by the numerical results. With little extension, the approach adopted in this paper is viable for the solution of higher order initial value problems of ordinary differential equations.

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Declaration of Competing Interest

The authors declare that there is no competing financial interest or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Rotimi Oluwasegun Folaranmi: Conceptualization, Methodology, Writing - Original Draft
Abayomi Ayotunde Ayoade: Writing, Reviewing and Editing.
Tolulope Latunde: Software, Validation.

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