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Some Identities For "Hyperbolic" Trigonometric Functions

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Abstract — In this article, we give proofs of some properties provided by "hyperbolic" trigonometric functions defined in [1].

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1. Introduction

An arithmetical function is a complex valued function defined on the set of positive integers and the set of these functions is denoted by A. The (Dirichlet) convolution $\left(g*h\right)$ of g and h is defined by $\left(g*h\right)(n) = \sum_{d|n} g(d)h\left(\frac{n}{d}\right)$ for all $g, h \in A$. Rearick [2] introduced the notions of Logarithm and Exponential operators of arithmetic functions. These operators were inverses of one another. The Logarithm operator takes Dirichlet products to sums in A, and the Exponential operator takes sums to Dirichlet products. Inspired by Rearick's work Li and MacHenry introduced LOG and EXP operators. The LOG operates on generalized Fibonacci polynomials($F_{k,n}(t)$) giving generalized Lucas polynomials($G_{k,n}(t)$). The EXP is the inverse of LOG[1]. Then Li and MacHenry defined the "Hyperbolic" SINE and "Hyperbolic" COSINE functions with the help of the EXP operator. First, let's give the definitions necessary to make sense of these definitions.

Definition 1.1. [1] An isobaric polynomial is a polynomial in the variables $t_1, t_2, \ldots, t_k$ for $k \in \{1, 2, \ldots\}$, with coefficients in $\mathbb{Z}$, of the form

$$P_{k,n}(t_1, t_2, \ldots, t_k) = \sum_{\alpha \vdash n} C_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k}$$

where $\alpha = \{\alpha_1, \alpha_2, \ldots \alpha_k\}$ and $\alpha \vdash n$ means that $\sum_{j=1}^{k} j\alpha_j = n$.

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Definition 1.2. [1] A weighted isobaric polynomial given by the following explicit expression:

$$P_{w,k,n}(t_1,t_2,\ldots,t_k)=\sum_{\alpha\vdash n}\binom{|a|}{\alpha_1,\alpha_2,\ldots,\alpha_k}\frac{\sum_{j=1}^{k}w_j\alpha_j}{|\alpha|}t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_k^{\alpha_k}$$

where w is the weight vector $(w_1,w_2,\ldots,w_k)$, $w_j\in\mathbb{Z}$ and $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_k$.

$F_{k,n}(t)$ and $G_{k,n}(t)$, are defined inductively by as follows:

$$F_{k,0}(t)=1,\ F_{k,n+1}(t)=t_1F_{k,n}(t)+\cdots+t_kF_{k,n-k+1}(t)(n>1),$$

and

$$\begin{aligned}G_{k,0}(t)&=k,\ G_{k,1}(t)=t_1,\ G_{k,n}(t)=G_{k-1,n}(t)(1\leq n\leq k),\\ G_{k,n}(t)&=t_1G_{k,n-1}(t)+\cdots+t_kG_{k,n-k}(t)(n>k),\end{aligned}$$

where the vector $t=(t_1,t_2,\ldots,t_k)$ and t_i $(1\leq i\leq k)$ are constant coefficients of the core polynomial

$$P(x;t_1,t_2,\ldots,t_k)=x^k-t_1x^{k-1}-\cdots-t_k.$$

Li and MacHenry [1] defined two operators $\mathscr{L}$(LOG) and $\mathscr{E}$(EXP).

Definition 1.3. [1] For a fixed k and $n\geq 1$,

$$\mathscr{L}(P_n)=-t_{n-1}P_1-2t_{n-2}P_2-\ldots-(n-1)t_1P_{n-1}+nP_n$$

where P_n is weighted isobaric polynomial and $t_i=0$ for $i>k$.

Definition 1.4. [1] For a fixed k, $\mathscr{E}(G_{k,0})=1$,

$$\mathscr{E}(G_{k,n})=\frac{1}{n}(F_{k,n-1}G_{k,1}+F_{k,n-2}G_{k,2}+\ldots+F_{k,1}G_{k,n-1}+G_{k,n}).$$

Lemma 1.5. [1] $\mathscr{L}$ and $\mathscr{E}$ are inverses of one another on F and G, i.e.,

$$\begin{aligned}\mathscr{L}(F_n)&=G_n\\ \mathscr{E}(G_n)&=F_n\end{aligned}$$

Definition 1.6. [1] "Hyperbolic" SINE and "Hyperbolic" COSINE functions are defined as;

$$\begin{aligned}C(G)&=\frac{1}{2}(\mathscr{E}(G)+\overline{\mathscr{E}(G)})\\ S(G)&=\frac{1}{2}(\mathscr{E}(G)-\overline{\mathscr{E}(G)}).\end{aligned}$$

2. "Hyperbolic" Trigonometric Operators

The purpose of this article is to give proofs of some properties provided by "Hyperbolic" trigonometric functions defined in [1].

Theorem 2.1. [1] Let δ be the function whose values are $(1,0,0,...,0,...)$,

$$C(G)^{*2} - S(G)^{*2} = \delta.$$

Theorem 2.2. [1] Let F and G be induced by the core $\left[t_{1,...,}t_k\right]$, $F^{'}$ and $G^{'}$ be induced by the core $\left[t^{'}_{1,...,}t'_k\right]$ with $\mathscr{L}(F) = G$ and $\mathscr{L}(F^{'}) = G^{'}$, then

$$\begin{aligned} C(G+G^{'}) &= C(G)*C(G^{'}) + S(G)*S(G^{'}), \\ S(G+G^{'}) &= S(G)*C(G^{'}) + C(G)*S(G^{'}). \end{aligned}$$

Theorem 2.3. Let F and G be induced by the core $\left[t_{1,...,}t_k\right]$, with $\mathscr{L}(F) = G$ then,

$$S(2G) = 2(S(G)*C(G))$$

Proof.

$$\begin{aligned} 2(S(G)*C(G)) &= 2(\frac{1}{2}(\mathscr{E}(G) - \overline{\mathscr{E}(G)}) * \frac{1}{2}(\mathscr{E}(G) + \overline{\mathscr{E}(G)})) \\ &= \frac{1}{2}(\mathscr{E}(G) - \overline{\mathscr{E}(G)}) * (\mathscr{E}(G) + \overline{\mathscr{E}(G)}) \\ &= \frac{1}{2}(\mathscr{E}(G)*\mathscr{E}(G) + \mathscr{E}(G)*\overline{\mathscr{E}(G)} - \overline{\mathscr{E}(G)}*\mathscr{E}(G) - \overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)}) \\ &= \frac{1}{2}(\mathscr{E}(2G) - \overline{\mathscr{E}(2G)}) \\ &= S(2G). \end{aligned}$$

Theorem 2.4. Let F and G be induced by the core $\left[t_{1,...,}t_k\right]$, with $\mathscr{L}(F) = G$ then,

$$C(2G) = 2(C(G))^{*2} - \delta.$$

Proof.

$$\begin{aligned} 2(C(G))^{*2} - \delta &= 2\left(\frac{1}{2}(\mathscr{E}(G) + \overline{\mathscr{E}(G)})\right)^{*2} - \delta \\ &= \frac{1}{2}\left(\mathscr{E}(G) + \overline{\mathscr{E}(G)}\right)^{*2} - \delta \\ &= \frac{1}{2}(\mathscr{E}(G)*\mathscr{E}(G) + 2\mathscr{E}(G)*\overline{\mathscr{E}(G)} + \overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)} - 2\delta) \\ &= \frac{1}{2}(\mathscr{E}(2G) + 2\delta + \overline{\mathscr{E}(2G)} - 2\delta) \\ &= \frac{1}{2}(\mathscr{E}(2G) + \overline{\mathscr{E}(2G)}) \\ &= C(2G). \end{aligned}$$

Theorem 2.5. Let F and G be induced by the core $[t_{1,...,}t_k]$, with $\mathscr{L}(F)=G$ then,

$$C(2G)=(C(G))^{*2}+(S(G))^{*2}$$

Proof.

$$\begin{aligned}
(C(G))^{*2}+(S(G))^{*2} &= \left(\frac{1}{2}(\mathscr{E}(G)+\overline{\mathscr{E}(G)})\right)^{*2}+\left(\frac{1}{2}(\mathscr{E}(G)-\overline{\mathscr{E}(G)})\right)^{*2} \\
&= \frac{1}{4}\left(\left(\mathscr{E}(G)+\overline{\mathscr{E}(G)}\right)^{*2}+\left(\mathscr{E}(G)-\overline{\mathscr{E}(G)}\right)^{*2}\right) \\
&= \frac{1}{4}(\mathscr{E}(G)*\mathscr{E}(G)+2\mathscr{E}(G)*\overline{\mathscr{E}(G)}+\overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)} \\
&\quad +\mathscr{E}(G)*\mathscr{E}(G)-2\mathscr{E}(G)*\overline{\mathscr{E}(G)}+\overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)}) \\
&= \frac{1}{4}(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}+\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}) \\
&= \frac{1}{4}(2\left(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}\right)) \\
&= \frac{1}{2}(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}) \\
&= C(2G).
\end{aligned}$$

Theorem 2.6. Let F and G be induced by the core $[t_{1,...,}t_k]$, with $\mathscr{L}(F)=G$ then,

$$C(2G)=2(S(G))^{*2}+\delta$$

Proof.

$$\begin{aligned}
2(S(G))^{*2}+\delta &= 2\left(\frac{1}{2}(\mathscr{E}(G)-\overline{\mathscr{E}(G)})\right)^{*2}+\delta \\
&= \frac{1}{2}(\left(\mathscr{E}(G)-\overline{\mathscr{E}(G)}\right)^{*2})+\delta \\
&= \frac{1}{2}(\mathscr{E}(G)*\mathscr{E}(G)-2\mathscr{E}(G)*\overline{\mathscr{E}(G)}+\overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)}+2\delta) \\
&= \frac{1}{2}(\mathscr{E}(2G)-2\delta+\overline{\mathscr{E}(2G)}+2\delta) \\
&= \frac{1}{2}(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}) \\
&= C(2G).
\end{aligned}$$

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