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INDEPENDENCE COMPLEXES OF STRONGLY ORDERABLE GRAPHS

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Abstract. We prove that for any finite strongly orderable (generalized strongly chordal) graph G, the independence complex $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to a wedge of spheres of dimension at least $\mathrm{bp}(G)-1$, where $\mathrm{bp}(G)$ is the biclique vertex partition number of G. In particular, we show that if G is a chordal bipartite graph, then $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to a sphere of dimension at least $\mathrm{bp}(G)-1$.

1. Introduction

An independent set in a graph is a subset of its vertices which are pairwise non-adjacent. The independence complex $\mathrm{Ind}(G)$ of a graph G is an abstract simplicial complex whose simplices correspond to independent sets of G. The topology of independence complexes of graphs has been the central subject of many papers (see, for instance [8, 9, 12, 13]). In the present work, we are mainly concerned with the homotopy type of independence complexes of strongly orderable graphs.

The class of strongly orderable graphs is firstly introduced by Dahlhaus [5] under the name "generalized strongly chordal graphs" as it constitutes a generalization of strongly chordal graphs and chordal bipartite graphs. Dragan [6] also provided vertex and edge elimination ordering characterizations of strongly orderable graphs. In our study, we benefit one of these characterizations of strongly orderable graphs, described in terms of quasi-simple vertex elimination schemes.

It turns out that the biclique vertex partititon number has a role to play in determining the homotopy type of independence complexes of strongly orderable graphs (and possibly of many other classes). Our main result is the following.

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Theorem 1. *If G is a strongly orderable graph, then* $\operatorname{Ind}(G)$ *is either contractible or homotopy equivalent to a wedge* $\bigvee S^{d_i}$ *of spheres, where* $d_i \geq \operatorname{bp}(G) - 1$ *for each* i.

Denote by $\gamma(G)$, the domination number of the graph G. A simple observation shows that $\operatorname{bp}(G) = \gamma(G)$ for every graph G which does not contain C_4 as a subgraph (see, [7]). Moreover, we show that this is also the case for C_4-free graphs. This naturally helps unifying several of earlier results regarding to the homotopy type of independence complexes of graphs. Recall from [12] that the independence complex of a chordal graph G is either contractible or homotopy equivalent to a wedge of spheres of dimension at least $\gamma(G) - 1$. Since $\operatorname{bp}(G) = \gamma(G)$ for every chordal graph G, it can be said that the independence complexes of strongly orderable graphs and chordal graphs have similar topological structure.

As Theorem 1 generalizes the current characterization for homotopy type of independence complexes of strongly chordal graphs, it is further possible to achieve an improvement in the case of bipartite graphs, since we have the advantage that any bipartite graph which is strongly orderable is a chordal bipartite graph.

Theorem 2. *If G is a chordal bipartite graph, then* $\operatorname{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere of dimension at least* $\operatorname{bp}(G) - 1$.

Theorem 2 also generalizes a result from [14]. In their seminal paper [14], Nagel and Reiner introduce some classes of graphs parametrized from shifted-skew shaped diagrams and determine the homotopy type of the independence complexes corresponding to these graphs via rectangular decompositions. As bipartite graphs related to such diagrams constitute a subclass of chordal bipartite graphs, we are also able to determine the homotopy type of their independence complexes.

Our paper is structured as follows: Section 2 provides the necessary background on graphs and simplicial complexes. In the subsequent section, we recall the structural properties of strongly orderable graphs and provide the characterization on the homotopy type of their independence complexes. In Section 4, we describe the bipartite graphs associated to shifted-skew diagrams from [14] and decide the homotopy type of their independence complexes.

2. Preliminaries

We start with recalling some basic notions from graph theory.

2.1. **Graphs.** All the graphs we study on are simple, i.e., do not have any loops or multiple edges. By writing $V(G)$ and $E(G)$, we mean the vertex set and the edge set of G, respectively. The edge $e := uv$ is contained in $E(G)$ if and only if u and v are adjacent in G. If $S \subset V(G)$, the graph induced by S is written $G[S]$. A graph G is said to be *H-free*, if it does not contain any induced subgraph isomorphic to H. We abbreviate $G[V \backslash S]$ to $G - S$, and write $G - x$ whenever $S = \{x\}$. The open and closed neighborhood of a vertex v are $N_G(v) = \{u \in V(G) : \ uv \in E(G)\}$ and

$N_G[v] = N_G(v) \cup \{v\}$, respectively. The cardinality of the set $N_G(v)$ is called the *degree* of the vertex v in G and denoted by $\deg_G(v)$.

A *bipartite* graph $G = (X, Y, E)$ is a graph with the vertex set $X \cup Y$ such that each of its edges is between a vertex of X and a vertex of Y. A bipartite graph $G = (X, Y, E)$ is called *convex* on Y if the vertices of Y can be ordered in such a way that the neighbours of any vertex $v \in X$ are consecutive. A bipartite graph G is called *convex bipartite* if it is convex on X or Y. If G is both convex on X and Y, then it is called *biconvex* or *doubly convex*.

Throughout, C_k denotes the cycle graph on $k \geq 3$ vertices and $K_{m,n}$ denotes the complete bipartite graph, for any $m, n \geq 1$. In particular, the complete bipartite graph $K_{1,n}$ is called a *star*. A graph is called *chordal* if it is C_k-free for $k \geq 4$. A bipartite graph is called *chordal bipartite* if it is C_k-free for $k \geq 6$.

A *biclique* in a graph G is a complete bipartite subgraph of G which is not necessarily induced. A set $\mathcal{B} = \{B_1, B_2, \dots, B_k\}$ of bicliques of a graph G is a *biclique vertex partition* of G of size k, if each vertex of G belongs to exactly one biclique in $\mathcal{B}$. Biclique vertex-partition number of a graph G, denoted by $\mathrm{bp}(G)$, is the smallest integer k such that G admits a biclique vertex-partition of size k.

A subset $S \subseteq V(G)$ is called a *dominating set* of G, if each vertex of G is either in S or adjacent to a vertex in S. The minimum size of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G.

We also use the notation $[n] := \{1, 2, \dots, n\}$ throughout, for any integer $n \geq 1$.

Let $G = (X, Y, E)$ be a bipartite graph with $|X| = m$ and $|Y| = n$. Then the $m \times n$ matrix $A(G) = [a_{ij}]$ is called the *bipartite adjacency (biadjacency) matrix* of G, where

$$a_{ij} = \begin{cases} 1, & x_i y_j \in E(G) \\ 0, & \textit{otherwise}. \end{cases}$$

Now let $A(G)$ be the biadjacency matrix of G indexed by any ordering of X and convex ordering $Y = [y_1 < y_2 < \dots < y_n]$. It is clear that $A(G)$ has consecutive 1's in each row (i.e, no induced submatrix [1 0 1]). Therefore, one may observe that if $A(G)$ is a biadjacency matrix of a convex bipartite graph G, then columns (or rows) of $A(G)$ can be permuted so that all the 1's in each row (or each column) appears consecutively in the resulting matrix.

2.2. **Simplicial Complexes.** An *(abstract) simplicial complex* Δ on a finite set of vertices V is a collection of subsets of V such that

(i) $\{v\} \in \Delta$ for every $v \in V$
(ii) if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of Δ are called *faces*. The dimension of a face $\sigma \in \Delta$ is $\dim(\sigma) := |\sigma| - 1$ and the dimension of Δ is $\dim(\Delta) := \max\{\dim(\sigma) : \sigma \in \Delta\}$. The *join* of two complexes Δ_1 and Δ_2 is defined by $\Delta_1 * \Delta_2 = \{\tau \cup \sigma : \tau \in \Delta_1, \sigma \in \Delta_2\}$.

In particular, the join of a simplicial complex Δ and zero-dimensional sphere $S^0 = \{\emptyset, \{a\}, \{b\}\}$ is called the *suspension* of Δ and denoted by $\Sigma\Delta = S^0 * \Delta$.

Similarly, the join of Δ and the simplicial complex $\{\emptyset, \{a\}\}$ is called the cone of Δ with apex a. A topological space is called *contractible* if its identity map is homotopic to a constant map. Note that a simplicial complex is contractible if it is a cone of another simplicial complex. It is also well-known that the suspension of a k-dimensional sphere is homotopy equivalent to $k+1$-dimensional sphere, that is, $\Sigma S^k \simeq S^{k+1}$.

One can associate to a graph G, the simplicial complex $\mathrm{Ind}(G)$, namely the *independence complex* of G, whose faces are independent sets of G.

We now provide some well-known facts from combinatorial topology, for which we use as a tool while computing the homotopy type of given graphs.

Theorem 3. *[9,13] Let G be a simple graph. If $N_G(u) \subseteq N_G(v)$ for some distinct vertices $u, v \in V(G)$, then the homotopy equivalence $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$ holds.*

Theorem 4. *[13] If v and u are distinct vertices with $N_G[v] \subseteq N_G[u]$, then $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - u) \vee \Sigma(\mathrm{Ind}(G - N_G[u])$.*

Note that a vertex v is called *simplicial* in G if $N_G[v]$ induces a complete graph in G. If v is a simplicial vertex in the graph G, then for any $u \in N_G(v)$, we have $N_G[v] \subseteq N_G[u]$. Since v remains simplicial in the graph $G - u$, applying Theorem 4 repeatedly for each neighbor of v leads us to the following property (see also [1]).

Corollary 1. *[9] If v is a simplicial vertex of the graph G, then*

$$\mathrm{Ind}(G) \simeq \bigvee_{u \in N_G(v)} \Sigma(\mathrm{Ind}(G - N_G[u]).$$

3. Homotopy Type of Strongly Orderable Graphs

In this section, we determine the homotopy type of independence complexes of strongly orderable graphs. We start with describing strongly orderable graphs.

Definition 1. *[6] Let $\sigma : v_1, v_2, \dots, v_n$ be an ordering of the vertices of a graph G. Then σ is called a* simplicial ordering *of G, if $i < j$, $i < k$ and $v_iv_j, v_iv_k \in E(G)$ implies that $v_iv_k \in E(G)$. On the other hand, σ is called a* strong ordering *of G, if $v_iv_j, v_iv_k, v_jv_l \in E(G)$, $i < l$ and $j < k$ implies that $v_jv_k \in E(G)$. The ordering σ is called a* strong simplicial ordering *of G if it is both strong and simplicial.*

Chordal graphs are well-known to be the class of graphs admitting a simplicial ordering [11], while graphs admitting a strong simplicial ordering are known as strongly chordal graphs, which is introduced by Farber [10].

A graph G is called a *strongly orderable* if G has a strong ordering of its vertices. Thus by definition, the class of strongly orderable graphs is a natural generalization of strongly chordal graphs. A strong simplicial ordering of a strongly chordal graph G is known to have further properties. A vertex v of G is called *simple* if $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$ for any $x, y \in N_G(v)$, that is, the closed neighborhoods of neighbors of v are linearly ordered under inclusion. Then an ordering

$\sigma : v_1, v_2, \ldots, v_n$ is called a *simple elimination ordering* if v_i is a simple vertex in the graph $G_i := G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for each $i \in [n]$. Farber [10] showed that a graph is strongly chordal if and only if it has a simple elimination ordering of its vertices. Dragan [6] gave a similar characterization for strongly orderable graphs.

Definition 2. *[6] Any two vertices u and v are said to be* comparable *in the graph G, if there holds $N_G(v)\backslash\{u\} \subseteq N_G(u)\backslash\{v\}$ or $N_G(u)\backslash\{v\} \subseteq N_G(v)\backslash\{u\}$, otherwise they are* noncomparable. *A vertex $w \in E(G)$ is called* quasi-simple *if for every $u, v \in N_G(w)$, the vertices u and v are comparable. An ordering $v_1, v_2, \ldots, v_n$ of the vertices of a graph G is called a* quasi-simple elimination ordering *if for each $i \in [n]$, the vertex v_i is a quasi-simple vertex in $G_i := G[\{v_i, v_{i+1}, \ldots, v_n\}]$.*

Theorem 5. *[6] A graph G is strongly orderable if and only if G has a quasi-simple elimination ordering.*

Now our task is to investigate the homotopy type of the independence complexes of strongly orderable graphs. In order to do that, we first need a structural property, namely hereditary property of strongly orderable graphs. We show that being a strongly orderable graph is closed under taking induced subgraphs.

Lemma 1. *If G is strongly orderable, then so is $G - x$ for any vertex x of G.*

Proof. Let $\alpha : v_1, v_2, \ldots, v_n$ be a strong ordering of the vertices of G. We show that removal of any vertex x from G still preserves the ordering. The case when $x = v_1$ is clear. Thus we assume that $x = v_i$ for some $i \in \{2, 3, \ldots, n\}$ and consider the graph $G^i := G - v_i$. We claim that the ordering $\beta : v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ is a strong ordering of $G - v_i$. For every $s \in \{2, 3, \ldots, n\}\backslash\{i\}$, we need to verify that the vertex v_s is quasi-simple in the graph $G^i_s := G^i\backslash\{v_1, \ldots, v_{s-1}\}$. Firstly, if $s > i$, then it is straightforward because of the strong ordering α of G. Now let $s < i$ and assume on the contrary that v_s is not quasi-simple. Then for some $v_k, v_l \in N_{G^i_s}(v_s)$ with $s < k < l$, the vertices v_k and v_l must be noncomparable in G^i_s, while they are comparable in G_s. Therefore the set $N_{G^i_s}(v_k)\backslash N_{G_s}(v_l)$ must contain a vertex v_r with $r > s$ and $r \neq i$. However, this is a contradiction, since $N_{G_s}(v_k)\backslash\{v_l\} \subseteq N_{G_s}(v_l)\backslash\{v_k\}$ because of the ordering α of G. $\square$

Lemma 2. *[4] Let G be a graph. If $N_G(u) \subseteq N_G(v)$ for some distinct vertices $u, v \in V(G)$, then $\mathrm{bp}(G) \leq \mathrm{bp}(G - v)$ holds.*

Proof. Let $\{B_1, B_2, \ldots, B_k\}$ be a biclique vertex partitioning of $G - v$. Assume without loss of generality that $u \in B_1$. Then observe that $\{B_1 \cup \{v\}, B_2, \ldots, B_k\}$ is a biclique vertex partitioning of G. $\square$

Remark 1. *The inequality $\mathrm{bp}(G) \geq \mathrm{bp}(G-v)$ is not true in general. For the graph G in Figure 1, we have $\mathrm{bp}(G) = 2 < \mathrm{bp}(G - v) = n + 1$ while $N_G(u_i) \subseteq N_G(v)$ for each $i \in [n]$.*

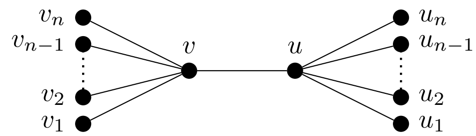


FIGURE 1. A graph G such that $\mathrm{bp}(G) = 2 < \mathrm{bp}(G - v) = n + 1$

Proof of Theorem 1. We use induction on the number of vertices. The theorem is trivial if G has fewer than 3 vertices. Let G be a strongly orderable graph and $\sigma : v_1, v_2, \ldots, v_n$ be a quasi-simple elimination ordering of the vertices of G. Suppose that the theorem is true for the graphs with fewer than n vertices. Since v_1 is quasi-simple in G, all of its neighbors are comparable. First assume that v_1 have two neighbors v_i, v_j (with $i < j$) which are not adjacent to each other. In this case, we must have $N_G(v_i) \subseteq N_G(v_j)$, since v_i and v_j are comparable in G. Then it follows that $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - y)$ by Theorem 3. Therefore, if $\mathrm{Ind}(G - y)$ is contractible, then $\mathrm{Ind}(G)$ is also contractible. If $\mathrm{Ind}(G - y)$ is not contractible, then by induction hypothesis, $\mathrm{Ind}(G - y) \simeq \bigvee S^{d_i}$, where $d_i \geq \mathrm{bp}(G - y) - 1$ for each i. Since $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - y)$ and $bp(G - y) - 1 \geq bp(G) - 1$ by Lemma 2, we obtain that $\mathrm{Ind}(G) \simeq \bigvee S^{d_i}$, where $d_i \geq \mathrm{bp}(G) - 1$. Thus we may further assume that all the neighbors of v_1 are pairwise adjacent. Notice that the vertex v_1 is simplicial in such a case. Following Theorem 4, we have

$$\mathrm{Ind}(G) \simeq \bigvee_{u \in N_G(v_1)} \Sigma(\mathrm{Ind}(G - N_G[u])) \tag{$*$}$$

Recall that $G - N_G[u]$ is a strongly orderable graph for each $u \in N_G(v_1)$, by Lemma 1. By the induction hypothesis, we know that for each $u \in N_G(v_1)$, the complex $\mathrm{Ind}(G - N_G[u])$ is either contractible or homotopy equivalent to a wedge sum $\bigvee S^{d_u}$ of spheres, where $d_u \geq \mathrm{bp}(G - N_G[u]) - 1$. Now, if $\mathrm{Ind}(G - N_G[u])$ is contractible for each $u \in N_G(v_1)$, then so is $\mathrm{Ind}(G)$. Therefore, we let u be an arbitrary neighbor of v_1 such that $\mathrm{Ind}(G - N_G[u]) \simeq \bigvee S^{d_u}$. Then it follows that $\Sigma(\mathrm{Ind}(G - N_G[u])) \simeq \bigvee S^{d_u+1}$. For any biclique vertex partition $\{B_1, B_2, \ldots, B_k\}$ of $G - N_G[u]$, observe that the collection $\{N_G[u], B_1, B_2, \ldots, B_k\}$ is a biclique partiton of G, which implies that $\mathrm{bp}(G) \leq \mathrm{bp}(G - N_G[u]) + 1$. Thus we have $d_u + 1 \geq \mathrm{bp}(G - N_G[u]) \geq \mathrm{bp}(G) - 1$. Hence the theorem follows from $(*)$. $\square$

As every strongly chordal graph is strongly orderable, we have the following.

Corollary 2. *If G is a strongly chordal graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a wedge $\bigvee S^{d_i}$ of spheres, where $d_i \geq \mathrm{bp}(G) - 1$ for each i.*

Note that Corollary 2 is well-known since every strongly chordal graph is a chordal graph and the biclique vertex partition number coincides with the domination number on C_4-free graphs. Although it is not hard to see, we include its proof for the completeness.

Proposition 1. *If G is a C_4-free graph, then* $\mathrm{bp}(G) = \gamma(G)$.

Proof. If S is a dominating set of G, then it is clear that $V(G)$ can be partitioned into stars each of which has its center from S, thus $\mathrm{bp}(G) \leq \gamma(G)$.

Conversely, let $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ be a biclique partition of G. We claim that for each $i \in [k]$, the subgraph B_i has at least one vertex v_i which is adjacent all other vertices in B_i so that $\{v_i : \ i \in [k]\}$ is a dominating set in G. Let $B_i \in \mathcal{B}$ be an arbitrary biclique of G, with the partitioning $X_i \cup Y_i$. Note that X_i and Y_i need not to be independent. The claim is trivial if $|X_i| = 1$ or $|Y_i| = 1$. Thus we let $\min\{|X_i|, |Y_i|\} \geq 2$ and assume on the contrary that there is no such vertex in B_i. This forces that there exist some vertices $x_{i_1}, x_{i_2} \in X_i$ and $y_{i_1}, y_{i_2} \in Y_i$ such that x_{i_1} and x_{i_2} (resp. y_{i_1} and y_{i_2}) are nonadjacent in G. This is a contradiction, since the set $\{x_{i_1} x_{i_2}, y_{i_1}, y_{i_2}\}$ induces a C_4 in G. This completes the proof. □

Since chordal graphs are C_4-free, Proposition 1 allows us interpret the homotopy type of independence complexes of chordal graphs in terms of biclique vertex partition number, when they are homotopy equivalent to a wedge sum of spheres. Recall from [12] that if the complex $\mathrm{Ind}(G)$ of a chordal graph G is homotopy equivalent to a wedge of spheres, then each of the spheres has dimension at least $\gamma(G) - 1$. Hence, Proposition 1 helps us unify the results for chordal and strongly orderable graphs.

Remark 2. *It is known that chordal graphs are vertex-decomposable, since they are codismantlable [3]. Therefore, the homotopy type of independence complexes of chordal graphs can also be inferred from vertex-decomposability [2]. However, unlike the class of chordal graphs, strongly orderable graphs are not vertex-decomposable. C_4 is an easy example of chordal bipartite (thus a strongly orderable) graph which is not vertex decomposable.*

Strongly orderable bipartite graphs coincide with the class of chordal bipartite graphs. Any quasi-simple vertex turns out to be a weak simplicial vertex in a chordal bipartite graph. A vertex x in G is said to be a *weak simplicial* if for any $u, v \in N_G(x)$, either $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$ holds [15]. This leads to a refinement of our main result on chordal bipartite graphs.

Lemma 3. *Every connected chordal bipartite graph with more than one edge has a pair x, y of vertices such that $N_G(x) \subseteq N_G(y)$.*

Proof. Let G be a chordal bipartite graph with more than one edge and let v be a weak simplicial vertex of G. First assume that $\deg_G(v) = 1$ and let $N_G(v) = \{w\}$. Then for every $u \in N_G(w) \backslash \{v\}$, we have $N_G(v) \subseteq N_G(u)$. If $\deg_G(v) \geq 2$, then any two neighbors of u form such a pair. □

Theorem 6. *If G is a chordal bipartite graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere of dimension at least* $\mathrm{bp}(G) - 1$.

Proof. Once again, we use the induction on the number of the vertices. Let G be a bipartite graph. We may assume that G has a component with more than one edge, since otherwise the claim is clear. Let H be such component of G. By Lemma 3, H has a pair of vertices x, y such that $N_G(x) \subseteq N_G(y)$. It follows that $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$, by Theorem 3. By induction hypothesis, the subcomplex $\mathrm{Ind}(G - v)$ is either contractible or homotopy equivalent to a sphere of dimension at least $\mathrm{bp}(G - v) - 1$. If $\mathrm{Ind}(G - v)$ is contractible, then so is $\mathrm{Ind}(G)$. Assume further that $\mathrm{Ind}(G - v)$ is homotopy equivalent to a sphere of dimension at least $\mathrm{bp}(G - v) - 1$. Since $\mathrm{bp}(G - v) \geq \mathrm{bp}(G)$ by Lemma 2 and $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$, the complex $\mathrm{Ind}(G)$ is homotopy equivalent to a sphere of dimension of at least $bp(G) - 1$. $\square$

We also have the following corollary, since every convex bipartite graph is a chordal bipartite graph.

Corollary 3. *If $G = (X, Y, E)$ is a convex bipartite graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere.*

4. Bipartite Graphs Related to Shifted-Skew Diagrams

In [14], Nagel and Reiner introduced graph classes associated to shifted-skew shaped diagrams. They also compute the homotopy type of such constructed graphs. In the case of bipartite graphs, our results from previous section generalize the mentioned classification. We first provide the necessary background about these diagrams and then conclude the homotopy type of independence complexes of bipartite graphs corresponding such diagrams. For more detailed description of shifted-skew shapes, we refer to [14].

Definition 3. *[14] A* shifted diagram *is an interpretation of the lattice points $\{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i < j\}$ by replacing each point with unit squares/cells where the first coordinate i (row index) increases from top to the bottom and the second coordinate j (column index) increases from left to the right, as in matrices.*

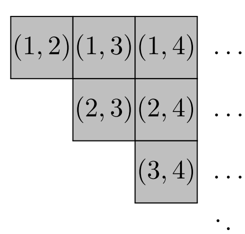
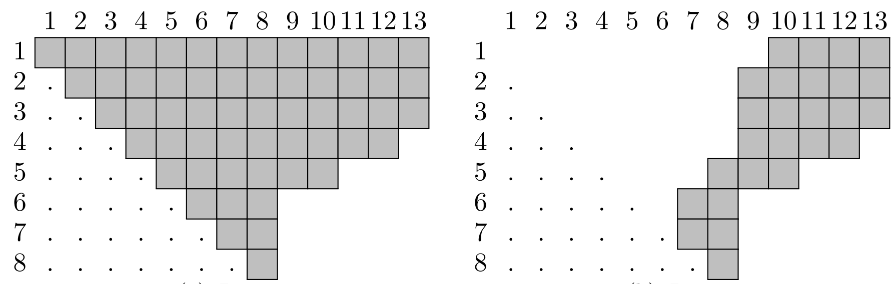


Figure 2. A shifted diagram

A *shifted Ferrers diagram* D_λ with respect to the strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ (where $\lambda_1 > \lambda_2 > \ldots > \lambda_t > 0$) is a finite shifted diagram consisting of λ_i cells in the row i. For instance, given partition $\lambda = (13, 12, 11, 9, 6, 3, 2, 1)$, corresponding diagram is depicted in the Figure 3-(a).

Now let λ and μ are such partitions with $\lambda \subseteq \mu$, that is $\mu_i \leq \lambda_i$ for all i, and possibly μ has less number of parts than λ has. Then one can form the *shifted skew diagram* $D := D_{\lambda/\mu}$ by removing the diagram D_μ from the diagram D_λ. An example with partitions $\lambda = (13, 12, 11, 9, 6, 3, 2, 1)$ and $\mu = (9, 7, 6, 5, 3, 1)$ given below (compare Figure 3-(a) with Figure 3-(b)).



(a) D_λ (b) $D_{\lambda/\mu}$

FIGURE 3. A shifted Ferrers diagram D_λ and shifted skew diagram $D_{\lambda/\mu}$.

For any shifted skew diagram and linearly ordered subsets $X = \{x_1 < x_2 < \ldots < x_m\}$ and $Y = \{y_1 < y_2 < \ldots < y_n\}$ of positive integers, let $D_{X,Y}$ denote the diagram consisting of cells in the position (i, j) whenever the cell (x_i, x_j) is present in D, i.e., we restrict the diagram D to the rows indexed by X and columns indexed by Y. For instance, if we set $X = \{x_1, x_2, x_3, x_4, x_5\} = \{1, 3, 5, 6, 7\}$ and $Y = \{y_1, y_2, y_3, y_4\} = \{8, 9, 11, 13\}$ for the diagram $D := D_{\lambda/\mu}$ in Figure 3, the corresponding diagram $D_{X,Y}$ is drawn as in the Figure 4.

Given shifted-skew diagram $D_{X,Y}$, Nagel and Reiner [14] define the bipartite graph $G(D_{X,Y}) = (X, Y; E)$ on the vertex $X \cup Y = \{x_1, x_2, \ldots x_m\} \cup \{y_1, y_2, \ldots, y_n\}$

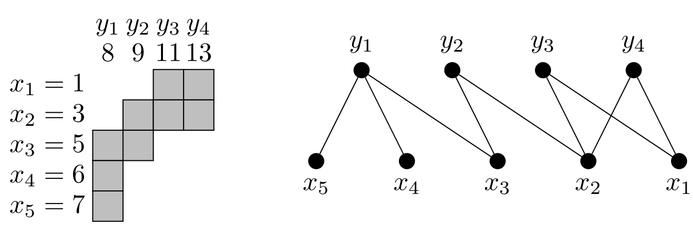


FIGURE 4. Shifted skew diagram $D_{X,Y}$ and the bipartite graph $G(D_{X,Y})$.

such that $(x_i, y_j) \in E(G)$ if the cell (i, j) is present in $D_{X,Y}$. One may observe that there is a one-to-one correspondence between the biadjacency matrix and the diagram of the graph $G(D_{X,Y})$, such that the cells in the diagram corresponds to 1's in the matrix (see, Figure 5).

$$A(G(D_{X,Y})) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 5. Biadjacency matrix of $G(D_{X,Y})$

Consequently, we deduce that all the 1's in each row (and each column) are consecutive, which in turn implies that the graph $G(D_{X,Y})$ is a doubly convex graph. Note that this fact is independent from the choice of the sets X and Y. In fact, the bipartite graph $G(D)$ corresponding to the diagram $D := D_{\lambda/\mu}$ is clearly a convex bipartite graph. Therefore, the choice of the sets X and Y will determine an induced subgraph $G(D_{X,Y})$ of $G(D)$ which is again convex bipartite. Hence the following fact is an immediate consequence of Corollary 3.

Corollary 4. *[14] Let $G(D_{X,Y})$ is the bipartite graph associated to a shifted-skew diagram $D_{X,Y}$. Then the complex $\mathrm{Ind}(G(D_{X,Y}))$ is either contractible or homotopy equivalent to a sphere.*

The same argument can be further applied to a bipartite graph $G(D)$ parametrized (in the similar fashion) from any diagram D whose cells in each row (or each column) appear consecutively, since $G(D)$ is a convex bipartite graph.

5. Conclusion

In our study, we characterize the homotopy type of the independence complexes of strongly orderable graphs. We further refine the mentioned characterization in the case of chordal bipartite graphs. These characterizations extend several known results and unify them in terms of biclique vertex partitions. There is, however, a natural question which arises in this context: "For which classes of graphs, the biclique vertex partitions is also relevant to the topology of independence complexes?".

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References

[1] Adamaszek, M., Splittings of independence complexes and the powers of cycles, *J. Comb. Theory Ser. A.*, 119(5) (2012), 1031–1047. https://doi.org/10.1016/j.jcta.2012.01.009

[2] Björner, A., Wachs, M., Shellable nonpure complexes and posets. I, *Trans. Amer. Math. Soc.*, 348(4) (1996), 1299–1327. https://doi.org/10.1090/s0002-9947-96-01534-6

[3] Bıyıkoğlu, T., Civan, Y., Vertex decomposable graphs, codismantlability, Cohen–Macaulayness and Castelnuovo–Mumford regularity, *Electron. J. Combin.*, 21(1) (2014). https://doi.org/10.37236/2387

[4] Civan, Y., Deniz, Z., Yetim, M. A., Chordal bipartite graphs, biclique vertex partitions and Castelnuovo-Mumford regularity of 1-subdivison graphs, Manuscript in preparation (2021).

[5] Dahlhaus, E., Generalized strongly chordal graphs, Technical Report, Basser Department of Computer Science, University of Sydney, 1993, 15 pp.

[6] Dragan, F. F., Strongly orderable graphs A common generalization of strongly chordal and chordal bipartite graphs, *Discrete Appl. Math.*, 99 (1-3) (2000), 427–442. https://doi.org/10.1016/S0166-218X(99)00149-3

[7] Duginov, O., Partitioning the vertex set of a bipartite graph into complete bipartite subgraphs, *Discrete Math. Theor. Comput. Sci.*, 16(3) (2014), 203–214. https://doi.org/10.46298/dmtcs.2090

[8] Ehrenborg, R., Hetyei, G., The topology of the independence complex, *European J. Combin.*, 27(6) (2006), 906–923. https://doi.org/10.1016/j.ejc.2005.04.010

[9] Engström, A., Complexes of directed trees and independence complexes, *Discrete Math.*, 309 (2009), 3299–3309. https://doi.org/10.1016/j.disc.2008.09.033

[10] Farber, M., Characterizations of strongly chordal graphs, *Discrete Math.*, 43(2-3) (1983),173–189. https://doi.org/10.1016/0012-365X(83)90154-1

[11] Golumbic, M. C., Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.

[12] Kawamura, K., Independence complexes of chordal graphs, *Discrete Math.*, 310 (2009), 2204–2211. https://doi.org/10.1016/j.disc.2010.04.021

[13] Marietti, M., Testa, D., A uniform approach to complexes arising from forests, *Electron. J. Comb.*, 15 (2008). https://doi.org/10.37236/825

[14] Nagel, U., Reiner, V., Betti numbers of monomial ideals and shifted skew shapes, *Electron. J. Comb.*, 16(2) (2009). https://doi.org/10.37236/69

[15] Uehara, R., Linear time algorithms on chordal bipartite and strongly chordal graphs, In: Automata, Languages and Programming, Lecture Notes in Computer Science, Volume 2380, Springer, 2002, pp. 993–1004. https://doi.org/10.1007/3-540-45465-9_85