ASSESSING CONCEPTUAL UNDERSTANDING IN MATHEMATICS:
Using Derivative Function to Solve Connected Problems

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ABSTRACT

Open and distance education plays an important role in the actualization of cultural goals as well as in societal developments. This is an independent teaching and learning method for mathematics which forms the dynamic of scientific thinking. Distance education is an important alternative to traditional teaching applications. These contributions brought by technology enable students to participate actively in having access to information and questioning it. Such an application increases students' motivation and teaches how mathematics can be used in daily life. Derivative is a mathematical concept which can be used in many areas of daily life.

The aim of this study is to enable the concept of derivatives to be understood well by using the derivative function in the solution of various problems. It also aims at interpreting difficulties theoretically in the solution of problems and determining mistakes in terms of teaching methods. In this study, how various aspects of derivatives are understood is emphasized. These aspects concern the explanation of concepts and process, and also their application to certain concepts in physics. Students' depth of understanding of derivatives was analyzed based on two aspects of understanding; theoretical analysis and contextual application. Follow-up interviews were conducted with five students.

The results show that the students preferred to apply an algebraic symbolic aspect instead of using logical meanings of function and its derivative. In addition, in relation to how the graph of the derivative function affects the aspect of function, it was determined that the students displayed low performance.

Keywords: **Mathematics education, learning, teaching, derivative, solving problems.**

INTRODUCTION

Effective learning is one of the important factors which increase students' success. Therefore, a number of researchers in mathematics are looking for ways to improve the quality of learning and to develop a rational understanding of the concept of calculus. As it is known, many concepts in mathematics do not mean much on their own, but they can be meaningful when they are correlated with other mathematical concepts and activities are conducted. In other words, conceptual in formation in mathematics teaching is meaningful when used in different areas by adaptation and when there is transition among concepts.

Understanding in calculus is the ability to explore the facts, rules and concepts and how they connect within the mathematical context. Students need to be encouraged and given the opportunity to reflect on the connections among various mathematical topics so that they can identify equivalent representations of the same concept (Berry and Nyman, 2003).

The concept of function is a concept which constitutes the core of mathematical subjects and supplies wholeness between subjects. At university level, the concepts of function and its derivative are used as the basic ideas in advanced mathematics courses. Research into mathematics education has shown that while the concepts of mathematics education were understood and they gave meaning to these concepts, students have difficulties (Vinner, 1989, Radatz, 1979, Klymchuk, et al. 2010). However, students can overcome these difficulties when they apply mathematical concepts in the correct manner.

Many researchers have found that students have a limited view of conceptual understanding of function and its derivative. Moreover, they report that students exhibit a predominant reliance on the use of and the need for algebraic formulas when dealing with the function concept (Breidenbach, Dubinsky, Hawks, Nichols 1992, Eisenberg, 1992, Eisenberg, Dreyfus, 1991).

Vinner and Dreyfus (1989) point out that there are many ways to learn to represent knowledge in mathematics and this makes the understanding of the concepts in calculus easier. The correlation among multiple representations provides the thought processes in the assessment of understanding and moves the obstacle when students solve unusual mathematical problems (Sigel, 1999). Orton (1983), Heid (1988) stated that there is a noticeable increase in the performance of students when they grasp concepts that can be represented in multiple ways in calculus. Kendal and Stacey (2000) found that the most capable students achieved the goal of developing facility with numerical, graphical, and symbolic representations of functions and derivatives. Kieran (1994) states in one of his studies that mathematical concepts should be perceived in different ways and should be evaluated in detail in order to make the learning permanent for students. One of the concepts which is suitable for the use of multiple representation is the concept of derivations. Derivations can be given as an example for this in order to use the mathematical concept in various areas and to explain it more effectively. The concept of the derivative is a difficult concept to understand for students. Learning the concept of derivatives, the relationship of derivatives with limit, continuity, tangent, and slope form a mathematical basis. However, students encounter certain difficulties.

In the conceptualization of derivations, the design of the educational process is also important (Amit and Vinner 1990, Leinhardt et al., 1990, Tall and Vinner 1981). The basic strategy in this concept is to form mathematical thinking of derivations by using the language of mathematics and to analyze the concepts and to form problem-solving. Graphs of derivations and function involve all the information concerning function or the behavior of the related problem. Students can overcome a lot of difficulties by using graphs. However, students rarely use them. Thus, conceptual knowledge stands on an operational level and it becomes impossible to coordinate between the concept of function and its graph. Generally, the derivative stands as the knowledge of formulas and rules. Unfortunately, these concepts are called operational knowledge that cannot always be understood by students.

On the other hand, Tall (1991) emphasizes that there was no consistency in students' success in multiple representations. Even though the fact that mathematical concepts, which include different displays, are understood by students seems to be an advantage, it can also be a disadvantage. The understanding of concepts requires the knowledge and the connection of relationships among different representations. Studies conducted indicate that students have difficulty in making connections among multiple representations of derivatives (Amoach and Laridon, 2004).

In order to understand derivations conceptually, it is not enough to know the relationships among the concept of derivatives and derivative-tangent, slope, the derivative-limit, and the derivative-the ratio of change (Zandieh, 2000). As Zandieh (2000) states, making connections among these concepts is necessary. This connection can only be accomplished with solutions of real-life related problems. In mathematics teaching, conceptual knowledge gains meaning as long as it is used in various areas and a transition between concepts is achieved (Turker, 1981; Vinner, 1989; Tall, 1991). Lesh (2000) has shown that there is a constitutive relationship between students' representational abilities and their mathematical understanding and problem solving proficiency.

Several researchers who explored calculus students' understanding of the derivative report that students have a variety of difficulties, such as a difficulty in geometrical interpretation and physical interpretation, and interpretation as the value of the limit of the derivative (Lauten et al 1994; Jong, &Brinkman, 1997). Asiala et al., (1997) examined students' conceptualization of the graphical implications for the derivative, continuity and the value of the limit. Zandieh, and Knapp (2006) analyzed in more detail the concept of the derivative at the freshman calculus level. They describe a framework for analyzing student understanding of the concept of the derivative. The derivative framework has two main components; multiple representations and layers of process-object pairs.

The first component of the derivative framework is: the concept of the derivative can be represented graphically as the slope of the tangent line to a curve at a point; verbally as the instantaneous rate of change; physically as speed or velocity; and symbolically as the limit of the difference quotient.

They describe how some students mentioned slope and rate of change, while other students mentioned velocity or the process of taking the derivative. The second component of the derivative framework is the aspects of the concept of the derivative.

The aspects of the concept of the derivative are ratio, limit, and function which are called layers of framework. We combine the layers to form the structure of the concept of the derivative.

Most studies have investigated students' understanding of the derivative in different aspects. However, a few researchers have investigated the use of the derivative concept in solving several problems.

The aim of this study is to investigate whether or not the concept of derivation is well understood, using the derivative function in the solution of various problems, and to compare the relationship between the function and its derivative function.

Students' depth of understanding was analyzed, based on two aspects of understanding; theoretical analysis and contextual application. The contextual application demonstrates knowledge of how and when to use the appropriate skills and strategies of calculus.

METHODOLOGY

The study was conducted with eighty-five freshman calculus students who had previously taken at least two semesters of single variable calculus in the Science Faculty during the spring semester in 2010-2011. The data for the research was collected from an examination consisting of fifteen open-ended questions. The test was designed to assess how well students had learned procedures for calculating derivative functions and solving related problems.

The target and behavior of the test were taken from the course program. The test consisted of two sections. Questions in the first section were related to developed conceptual knowledge of the derivative and questions in the second section were developed problems which are solvable using the derivative. Most of the questions consisted of non-routine problems; that is problems for which no method for solving had been taught. In the solution of these questions, the applications of function and derivative form a bridge

In addition, questions were designed to demonstrate graphically between a function and its derivative. Students were asked to explain how they arrived at their answers. Students' responses for each of these items were analyzed based on two aspects of understanding; theoretical analysis and contextual application.

RESULTS

The analysis of the results was to determine how students construct their understanding of the concept and how this may lead to changes in the instructional aspects.

We have seen that students have an inadequate reliance on solving real life problems which deal with the derivative function. Some of the questions that were asked of the students are detailed below:

Research Question: 1

"Provided that,

$$f(x) = 2x^2 + 3x + 1, \quad f'(x) = 4x + 3, \quad f'(0) = 3$$

Explain the meaning of number 3, geometrically."

This question was designed to assess how well the relationship between the derivative and the slope of a curve, and the graphic and algebraic forms of a function, were learned.

We wanted to see the geometrical definition of the derivative in the answer. Table: 1 indicates the performance of students in the first problem.

Table: 1
Categories of results in the problem 1

N=85	Correct Answer	Incorrect Support Work	Correct Support Work	Incorrect Answer	Partial Understanding	No response
	18%	10%	22%	31%	15%	4%

As can be seen in the Table 1 students performances are very poorly. Students were not able to give the expected performance in the first problem. Orton indicated that students have difficulties with graphical interpretations of the derivative. This conclusion is parallel to conclusions in the studies of Amit and Vinner (1990) and Aspinwall et al (1997). They report that, some students equate the derivative of a function for the line tangent to the graph of the function at a given point.

Research Question: 2

Show that if $f(x) = -x^2 + 4$ **then** $f'(-1) = 2$

This question was designed to assess the conceptual interpretation of the derivative. The aim of this question is to highlight the relationship between derivative and limit.

How can the slope of chord line to the approach of slope of the tangent line be expressed using limit conceptually? (See figure: 1). we wanted to see how students defined the derivative as a rate of change. This example illustrates that students were able to explain to a greater or lesser degree the differences of connection between the derivative and limit. Is the student able to think about the derivative as a rate of change? We wanted to see if students could associate the concepts of the derivative and limit?

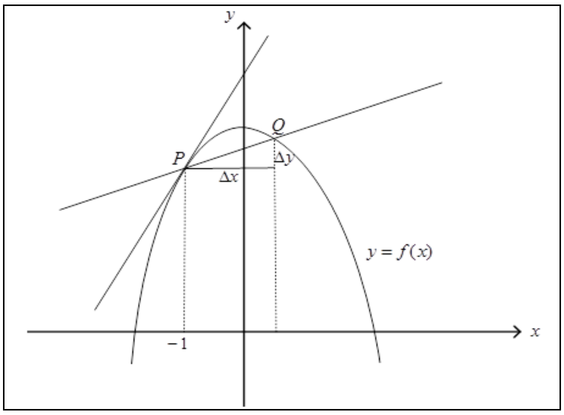

Figure: 1

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Table: 2 indicates the performance of students in the second problem.

Table: 2

Categories of results in the problem 2

N= 85	Correct Answer	Incorrect Support Work	Correct Support Work	Incorrect Answer	Partial Understanding	No response
	7%			40%	15%	6%

The majority of students answered this question using algebraic operations. This answer shows that the relationship between the derivative and the limit is static and memorized knowledge.

Plotting the tangent line from $x = -1$ to $f(x) = -x^2 + 4$, the slope was found to be 2 using algebraic operations 40%. So, using a graph was perceived as this form (7%).

$$f'(-1) = \frac{\Delta y}{\Delta x} = \lim_{x \to -1} \frac{f(x) - f(-1)}{x+1}$$

The above expression did not appear in the answers of students 93% Almost all the students gave similar explanations in their solutions. Hallet (1991) notes that the beliefs held by students are that applying calculus is manipulating symbols and numbers.

The results show that to calculate the derivative at a point we must give the relationship between the various systems of representation involved with the concept of the derivative function due to various difficulties, i.e. the connections among various mathematical topics have to be given to the students, so that they can identify equivalent representations of the same concept.

Research Question: 3

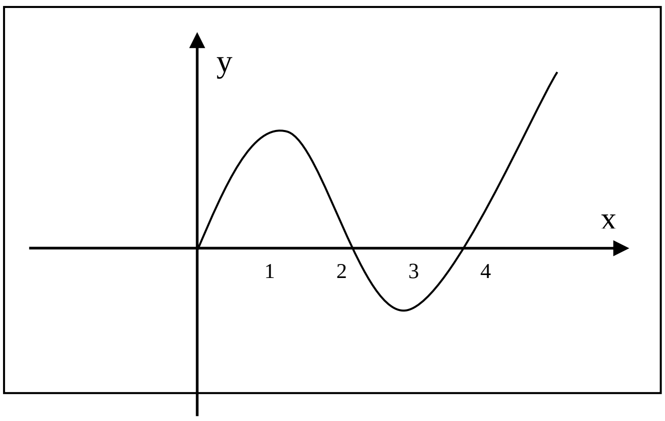

Figure: 2
The graph of the derivative function $f'(x)$

Figure: 2 shows the graph of the first derivative of a function $y = f(x)$. Use the graph of f' to estimate the intervals on which the function f is (a) increasing (b) decreasing. (c) Estimate where f has local extreme values.

The question was developed to construct the original function from the graphical properties of the derived function, i.e. this question related to visualization of the knowledge which the students could not understand algebraically or memorize. Table 3 indicates the performance of students in the third problem.

Table: 3
Categories of results in problem 3

N= 85	Correct Answer	Incorrect Support Work	Correct Support Work	Incorrect Answer	Partial Understanding	No response
	12%	18%	9%	26%	35%	----

We have seen that most students interpreted the derived function graph as the graph of the function that they could not make reversible. The results show that the most common incorrect responses are the extreme values and the intervals of increase and decrease of the function. As seen from the responses, the principal systematic error was to specify that function f(x) has extreme values at x=1 and x=3. This item was answered poorly. Another systematic error was to specify that the function increases at the domain of (0,1)and (3,∞). Most of the students did not identify the graph of the function correctly from the graph of its derivative.

Berry and Nyman (2003) investigated using an observational study on how students think about the relationship between the graph of a derived function and the original function from which it was formed. The results confirm that at the start of the activity the students demonstrated an algebraic symbolic view of calculus and found it difficult to make connections between the graphs of a derived function and the function itself.

Although visualization is basic in mathematical education, Tall (1991) and Vinner (1989) determine in a study that students tended towards algebraic symbolic rather than graphical understanding. One of the reasons for this is that a visual expression was not plausible through the mathematical proof. Commonly, graphs are being used more often in problem solving (Eisenberg and Dreyfus 1991, Vinner, 1989).

Many researchers investigated the problems of students relating to graphical interpretation of the derivative (Orton 1983; Ferrini-Mundy, Lauten 1994; Amit and Vinner, 1990; Leinhard et.al., 1990). The difficulties in this question may be a result of traditional instructional methods that tend to emphasize the construction of a derived function from the original function but in a reserve way from the derived function to an original function.

Research Question: 4
"The following graph shows the position function of any car (Figure: 3). on which points is the velocity of the car maximum and minimum? What is the velocity of the car between A and B?

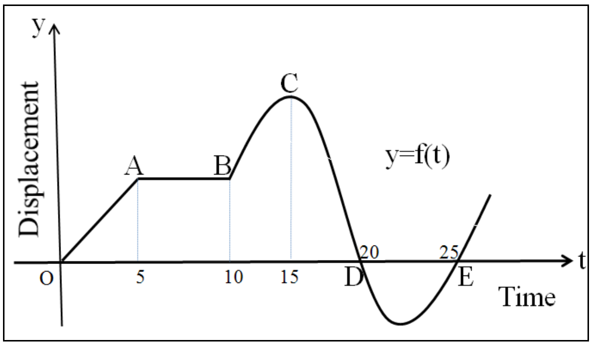

Figure : 3
The position function of any car

The aim of this question relates to knowledge of the relationship between the slope of a tangent line which passes through any point and the instantaneous physical variation ratio at the same point. In another words, it relates to how the slope of the tangent line via an instantaneous variation ratio could be determined. Also, physics makes calculus concrete for students and deepens students' understanding of the calculus concept. Table 4 indicates the performance of students in the fourth problem.

Table: 4
Categories of results in the problem 4

N= 85	Correct Answer	Incorrect Support Work	Correct Support Work	Incorrect Answer	Partial Understanding	No response
	5%	34%	6%	40%	15%	-----

In an analysis of the responses, we see that students have not explored the connection between the velocity and the position of the car. This item was also answered poorly. They could not explain their choice, although it was correct. Schwalbach and Dosemagen (2000), Lesh (2000), suggest that making connections between calculus and physics can yield a deep understanding of semantic as well as procedural knowledge.

These findings are consistent with those of Orton (1983), Heid (1988), and Hallet, D. H. (1991) who report that students have difficulty understanding derivation as an instantaneous variation ratio, and even have difficulty explaining instantaneous variation ratio.

Research Question: 5
"The number of telephone subscribers in an allocation area is given in the following Table 5. Find the increasing ratio from year 2007 to year 2010 and the instantaneous increasing ratio for the year 2007.
P(t): The number of mobile phones in year of t (x1000)

Table:5
The number of mobile phones in the years.

t	2007	2008	2009	2010
P(t)	509	547	735	1345

The aim of this question is to see, starting from any life problem, how an increasing ratio (mean variation ratio) and an instantaneous increasing ratio on any point (instantaneous variation ratio) can be found? Table 6 indicates the performance of students in the fifth problem.

Table: 6
Categories of results in the problem 5

N= 85	Correct Answer	Incorrect Support Work	Correct Support Work	Incorrect Answer	Partial Understanding	No response
	---	43%	8%		15%	---

About 72% of the 85 students did not demonstrate any conceptual knowledge of the problem. The item was answered poorly. Most of the students 32% demonstrated the ability to calculate the increasing ratio from the year 2007 to year 2010 and the instantaneous increasing ratio for the year 2007 algebraically. They could not explain their choice, although it was correct. So the students' answer was not the desired answer. Namely, they could not use the concept of derivation in solving real life problem.

The derivative as a rate of change,

$$p'(2007) = \lim_{t \to 2007} \frac{p(t) - p(2007)}{t - 2007}$$

Table: 7
Instantaneous rate of change

t	2008	2009	2010
$\frac{\Delta p}{\Delta t}$	38	114	279

According to the results from the answers, the reason for failure is the fact that the derivative with its physical definition is used to calculate the velocity and acceleration of a generally moving object at any given time. But, the relationship between the derivative and the variation ratio should not only be used to find the velocity and acceleration. Students must themselves discover how the concept of the derivative is used to solve life-related problems. Klymchuk et al., (2010) sought to find reasons why most students could not use their knowledge to construct the function in a familiar context. They identify students' difficulties and present their suggestions on how to improve their skills in solving application problems.

One method proposed for a conceptual understanding of the derivative is to use multiple denotes (Zandieh, and Knapp, 2006). Thus, different directions of the derivative concept could be perceived and could supply evaluations from a different angle.

For a conceptual understanding of the derivative, not only should individual relations of the derivative-tangent be known, but the slope of any line, derivative-limit, derivative-variation ratio and also mutual relationships for these concepts and the teaching of the derivative must be given enclosing these relationships.

One of the reasons for the difficulties students encounter while solving real life problems is traditional teaching methods or their inclination to solve the problems in a short time. It is believed that solving real life problems improves the quality of learning and develops a relational understanding of the concepts of calculus.

Table: 8
Evaluation of answers which students gave to the questions

N	Average	Standard Deviation	Questions
85	42.3	12.5	1st question
85	33.7	12.8	2 nd question
85	25.4	13.5	3 rd question
85	35.9	13.9	4 th question
85	14.4	2.6	5 th question
85	13.7	2.3	6 th question

As shown in Table 8 the averages of the students were fairly low. They were unsuccessful partly on the 1^{st}, 2^{nd}, and 4^{th} questions, and fairly unsuccessful on the 5^{th} and 6^{th} questions related with graphs and real life problems. In addition, the achievement of students did not differ on the 5^{th} and 6^{th} questions.

The interview was conducted with five students. Five students were selected as being representative of high-achievers, mid-achievers, and low-achievers, using the test. What we gathered from the interviews is described below. When we analyze the answers given to the first question, we can place the students into three categories. Those who answer the question correctly, those who lack in knowledge but answer the question correctly, and those give a wrong answer. Below is an interview with a student who seemed to have constructed the graphical interpretation of the derivative.

Inter. Can you explain what you understand from the first question?
S1 . 3 is the slope of tangent line to the function $y = f(x) = 2x^2 + 3x + 1$ at the point (0,1)
Inter. OK. How did you get 3 ?
Just find the slope of tangent line using the point (0,1).
S1. Then, he goes on to compute the slope of the tangent line and realizes that it is equal to the derivative. He does all of this easily.

Here is an interview with a student who answered the question correctly but lacked knowledge.

Inter. Could you explain the geometrical definition of the derivative?

S2. The geometrical meaning of the derivative is to draw a tangent line.
Inter. What is a tangent line?
S2. It means a line which touches a curve at one point.
Inter. This definition is not always correct. We will discuss it later. Let's go back to the question. How do you find the tangent line equation of a curve?
S2 I find a point and the slope of a line then write the equation.
Inter. OK. Can you find the point for this tangent line?
S2 Yes
Inter. Try it.
S2. Because f'(0) is given, I take x=0 .
Inter. How do you show a point on a plane?

S2. I show the point on the plane with (x, y).
Inter. Then find the point here.
S2. I have to find what y is to find x=0. So I put 0 instead of x in y= $f(x) = 2x^2 + 3x + 1$ then y is 1.
Inter. What happens then?
S2. (0, 1) is a point on the tangent line.
Inter. Now, let's analyze the equation of the tangent line. After finding the point, what is left?
S2. We need to find the slope.
Inter. Has the slope been given already?
S2. Yes, m=f' (0) it means it is 3
Inter. OK. Then can you find the equation for the line, sorry for the tangent line, at given point (0, 1)?
S2. Yes, if I say y-1=3(x-0), I get y=3x+1.
Inter. How do you express the line?
S2. Well, it is the equation of a line whose the slope is 3 and which is the tangent line of a curve at (0, 1) point.
Inter. What does f' (0) =3 mean?
S2. It is the slope of a line at (0, 1) point.
Inter. How do you express it as a whole?
S2. The derivative is the slope of a line
Inter. OK. What is the characteristic of this line?
S2. This line is the tangent line. Well, the derivative is the slope of a tangent line.
Inter. So, we have reached the correct answer step by step.

The student is unaware of the equivalence of the slope of the tangent line to the curve at a point and the value of the derivative at the same point. But he correctly finds the equation of the tangent line.In other interview, a lot of students said that they did not understand the connection of the fifth question with the derivative. Large number of students made no attempt to solve the problem.

Interviews provide the most fundamental information for this study.

DISCUSSION

According to the conclusions obtained from this study, it has been shown that students were unsuccessful in explaining the concept of the derivative from different dimensions. According to the students, the concept of the derivative is the operation of the derivative. Although a definition of the derivative via limit was known by the students, it is determined that they have difficulty using this relationship. The relationship between the concepts of the derivative and the variation ratio was not conceptualized. Similarly, although a geometric interpretation of the derivative was known, for the f(x) function, values of $f'(0) = 3$ and $f'(-1)$ could not be understood.

It was shown that students make conceptual mistakes among the concepts of the derivative- tangent-slope. From these conclusions, it seems that students have difficulties in establishing relationships between different presentations of the derivative. Most students could not apply the derivative concept to solve real life problems. According to Vinner (1983), if any student has conceptual definition knowledge, the student could not define a view of the concept.

Mistakes in this subject are nested with formal learning. In the concept of the derivative, in which visualization is important, it has been found that achievements of students in this direction are inadequate. While graphs of f and f′ could be constructed, it was determined that students could not interpret them effectively. Generally, for students, graphs do not mean much other than a visual object having meaning specified, besides determining linked functions. Students tend to direct to algebraic operations instead of logical implication of the function and the derivative.

Similarly, while a graph of the f function is obtained from a graph of the f′ function (or vice versa), students show low performance. The results show that in usual learning situations algebraic operations are weighed in mathematics education. In this study, the answers given to the questions match the results of most of the studies in the literature. Finally, the conceptual information on mathematical education gains importance in terms of how it has been used in various fields and how it relates to other concepts.

As with learning other mathematical concepts, establishing relationships with other mathematical concepts and examples from daily life, the concept of the derivative must be assimilated. The link between different representations of the same concept serves to an understanding and explanation of the concept of calculus. Also, making connections between the concepts in calculus and the solutions of related problems enables the concepts to be understood in detail.

CONCLUSIONS

In this paper we investigate whether or not the concept of derivation is well understood using the derivative function in the solution of various problems. There are many valid reasons for using communication technologies in mathematics teaching as in all subjects. Mathematics is a social and cultural product. Because of the importance of mathematics in all areas relating to information, the target mass is growing day by day.

Therefore, it seems that teaching mathematics in distance education is an applicable solution. Learning and teaching activities through distance education make the interaction of larger masses possible and learning becomes active through indivilization.

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