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Research Article

On the Solutions of Linear Elliptic Biquaternion Equations*

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Abstract
The real and complex quaternion algebras are isomorphic to real matrix algebras including the special types 4×4 and 8×8 real matrices, respectively. These situations are based on the fact that a finite dimensional associative algebra L over any field K is isomorphic to a subalgebra of $M_n(K)$ where dimension of L equals n over the field K. Considering this fact and using the left Hamilton operator, we get 8×8 real matrix representations of elliptic biquaternions in this study. Then a numerical method is developed to solve the linear elliptic biquaternion equations with the aid of the aforesaid representations. Also, an illustrative example and an algorithm are provided to show how this method works.

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1. Introduction

Real quaternions were introduced in 1843 by Hamilton [1]. The set of real quaternions is represented as

$$H=\{q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}:q_0,q_1,q_2,q_3\in\mathbb{R}\},$$

where the quaternionic units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the following rules:

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1\,,\ \ \mathbf{ij}=-\mathbf{ji}=\mathbf{k}\,,\ \ \mathbf{jk}=-\mathbf{kj}=\mathbf{i},\ \ \mathbf{ki}=-\mathbf{ik}=\mathbf{j}. \tag{1}$$

Hamilton also discovered the biquaternions(complex quaternions) [2]. The set of biquaternions is given as in the following:

$$H_{\mathbb{C}}=\{A=A_0+A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}:\ A_0,A_1,A_2,A_3\in\mathbb{C}\},$$

where quaternionic units satisfy the same rules in (1).

The algebra of real quaternions is isomorphic to a real matrix algebra including special type 4×4 real matrices. Also, from [3], we know that there is an algebra isomorphism between the biquaternion algebra and a real matrix algebra including special type 8×8 real matrices. These representations can be seen as useful tools for discussing the solutions of linear real quaternion equations and linear biquaternion equations. There can be found some studies [4, 5, 6, 7], which investigate the solutions of the aforesaid equations, in the literature.

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In recent years, the set of elliptic biquaternions, that includes the set of biquaternions and real quaternions as a special case, has been introduced. The set of elliptic biquaternions is given in the cartesian form as

$$H\mathbb{C}_p = \left\{U = U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \,:\, U_0, U_1, U_2, U_3 \in \mathbb{C}_p\right\},$$

where $\mathbb{C}_p = \left\{x + Iy : x, y \in \mathbb{R}, I^2 = p, p \in \mathbb{R}^-\right\}$ states the system of elliptical complex numbers [8]. Some interesting studies [9, 10, 11, 12, 13] on elliptic biquaternions can be found in the literature.

Investigating the solutions of linear elliptic biquaternion equations is a new and interesting topic. A general method, derived from the right Hamilton operator, was given for solving linear elliptic biquaternion equations by Özen & Tosun in [12]. This study has motivated us in the process of preparing the present paper.

8×8 real matrix representations of elliptic biquaternions, derived from left Hamilton operator, are obtained in this study. Also, we develop a numerical method to discuss the solutions of linear elliptic biquaternion equations with the aid of these representations. Finally, we provide an illustrative example and an algorithm to show how this method works.

Throughout this study, the set of all matrices on real numbers and the set of all matrices on elliptical complex numbers are shown with $M_{m\times n}(\mathbb{R})$ and $M_{m\times n}(\mathbb{C}_p)$, respectively.

2. Preliminaries

Let $U = U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k},\ V = V_0 + V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k} \in H\mathbb{C}_p$ be any elliptic biquaternions and $\lambda \in \mathbb{C}_p$ be any elliptical complex number. Then, the operations of addition, scalar multiplication and multiplication are defined as in the following [8]:

$$U + V = (U_0 + V_0) + (U_1 + V_1)\mathbf{i} + (U_2 + V_2)\mathbf{j} + (U_3 + V_3)\mathbf{k}$$

$$\lambda U = (\lambda U_0) + (\lambda U_1)\mathbf{i} + (\lambda U_2)\mathbf{j} + (\lambda U_3)\mathbf{k}$$

$$\begin{aligned} UV = {} & [(U_0V_0) - (U_1V_1) - (U_2V_2) - (U_3V_3)] \\ & + [(U_0V_1) + (U_1V_0) + (U_2V_3) - (U_3V_2)]\mathbf{i} \\ & + [(U_0V_2) - (U_1V_3) + (U_2V_0) + (U_3V_1)]\mathbf{j} \\ & + [(U_0V_3) + (U_1V_2) - (U_2V_1) + (U_3V_0)]\mathbf{k}. \end{aligned}$$

In the matrix set $M_{m\times n}(\mathbb{C}_p)$, the ordinary matrix multiplication and addition are defined. On the other hand, the scalar multiplication is stated as

$$\lambda A = \lambda\,(a_{ij}) = (\lambda a_{ij}) \in M_{m\times n}(\mathbb{C}_p),$$

where $A = (a_{ij}) \in M_{m\times n}(\mathbb{C}_p)$ and $\lambda \in \mathbb{C}_p$ [14].

4×4 elliptic matrix representations of elliptic biquaternions is another concept that can be of importance. For $U = U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \in H\mathbb{C}_p$, the following matrix

$$H^-(U) = \begin{bmatrix} U_0 & -U_1 & -U_2 & -U_3 \\ U_1 & U_0 & U_3 & -U_2 \\ U_2 & -U_3 & U_0 & U_1 \\ U_3 & U_2 & -U_1 & U_0 \end{bmatrix},$$

which corresponds to left Hamilton operator

$$h^- : H\mathbb{C}_p \to H\mathbb{C}_p$$

$$V \ \to\ h^-(V) = VU,$$

is a faithful elliptic matrix representation of U. The properties

$$\left(H^-(U) = H^-(V)\right) \Leftrightarrow U = V$$

$$H^-(U + V) = H^-(U) + H^-(V)$$

$$H^{-}(UV) = H^{-}(V)H^{-}(U)$$

are satisfied for $U, V \in H\mathbb{C}_p$ [13].

Any elliptic matrix $B = B_1 + IB_2 \in M_{m\times n}(\mathbb{C}_p)$ has a faithful real matrix representation as in the following:

$$\gamma_p(B) = \begin{bmatrix} B_1 & -\sqrt{|p|}B_2 \\ \sqrt{|p|}B_2 & B_1 \end{bmatrix} \tag{2}$$

since the function

$$\begin{aligned} &\gamma_p : M_{m\times n}(\mathbb{C}_p) \to M^{\Omega}_{2m\times 2n}(\mathbb{R}) \\ &B = B_1 + IB_2 \to \gamma_p(B) = \begin{bmatrix} B_1 & -\sqrt{|p|}B_2 \\ \sqrt{|p|}B_2 & B_1 \end{bmatrix} \end{aligned}$$

is a linear isomorphism where $M^{\Omega}_{2m\times 2n}(\mathbb{R}) = \left\{ \begin{bmatrix} E & -\sqrt{|p|}F \\ \sqrt{|p|}F & E \end{bmatrix} : E, F \in M_{m\times n}(\mathbb{R}) \right\}$ [11].

3. Exact Solutions of Linear Elliptic Biquaternion Equations

In this section, we get 8×8 real matrix representations of elliptic biquaternions derived from left Hamilton operator. Then, we use these representations to give a method investigating the exact solutions of linear elliptic biquaternion equations. We must emphasize that the computations in this section are performed by using the MATLAB package Symbolic Math Toolbox. The aim of using this Toolbox is to ensure the exact solution. Note that a similar matrix representation derived from right Hamilton operator and a similar method obtained by means of this representation were given in the study [12]. We will follow similar steps and approaches in this study.

We continue to take into consideration the aforementioned arbitrary elliptic biquaternion $U = U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \in H\mathbb{C}_p$. Then, the matrix $H^{-}(U) = H^{-}(U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k})$ can be written as

$$H^{-}(U) = \begin{bmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{bmatrix} + I \begin{bmatrix} u_0^* & -u_1^* & -u_2^* & -u_3^* \\ u_1^* & u_0^* & u_3^* & -u_2^* \\ u_2^* & -u_3^* & u_0^* & u_1^* \\ u_3^* & u_2^* & -u_1^* & u_0^* \end{bmatrix} = H^{-}(U)^{\#} + IH^{-}(U)',$$

where $U_i = u_i + Iu_i^* \in \mathbb{C}_p$, $0 \le i \le 3$. As can be seen easily, the matrices

$$H^{-}(U)^{\#} = \begin{bmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{bmatrix}$$

and

$$H^{-}(U)' = \begin{bmatrix} u_0^* & -u_1^* & -u_2^* & -u_3^* \\ u_1^* & u_0^* & u_3^* & -u_2^* \\ u_2^* & -u_3^* & u_0^* & u_1^* \\ u_3^* & u_2^* & -u_1^* & u_0^* \end{bmatrix}$$

are real matrices. When $H^{-}(U)^{\#}$ and $H^{-}(U)'$ are considered in (2), 8×8 real matrix representation of U is found as follows:

$$\gamma_p\left(H^{-}(U)\right) = \begin{bmatrix} H^{-}(U)^{\#} & -\sqrt{|p|}H^{-}(U)' \\ \sqrt{|p|}H^{-}(U)' & H^{-}(U)^{\#} \end{bmatrix}. \tag{3}$$

For convenience, we will show the matrix representation of U with $(U)_{p\gamma}$ instead of $\gamma_p(H^{-}(U))$ in the rest of the study. In this case we can rewrite (3) as follows:

$$(U)_{p\gamma}=\begin{bmatrix} u_0 & -u_1 & -u_2 & -u_3 & -u_0^*\sqrt{|p|} & u_1^*\sqrt{|p|} & u_2^*\sqrt{|p|} & u_3^*\sqrt{|p|} \\ u_1 & u_0 & u_3 & -u_2 & -u_1^*\sqrt{|p|} & -u_0^*\sqrt{|p|} & -u_3^*\sqrt{|p|} & u_2^*\sqrt{|p|} \\ u_2 & -u_3 & u_0 & u_1 & -u_2^*\sqrt{|p|} & u_3^*\sqrt{|p|} & -u_0^*\sqrt{|p|} & -u_1^*\sqrt{|p|} \\ u_3 & u_2 & -u_1 & u_0 & -u_3^*\sqrt{|p|} & -u_2^*\sqrt{|p|} & u_1^*\sqrt{|p|} & -u_0^*\sqrt{|p|} \\ u_0^*\sqrt{|p|} & -u_1^*\sqrt{|p|} & -u_2^*\sqrt{|p|} & -u_3^*\sqrt{|p|} & u_0 & -u_1 & -u_2 & -u_3 \\ u_1^*\sqrt{|p|} & u_0^*\sqrt{|p|} & u_3^*\sqrt{|p|} & -u_2^*\sqrt{|p|} & u_1 & u_0 & u_3 & -u_2 \\ u_2^*\sqrt{|p|} & -u_3^*\sqrt{|p|} & u_0^*\sqrt{|p|} & u_1^*\sqrt{|p|} & u_2 & -u_3 & u_0 & u_1 \\ u_3^*\sqrt{|p|} & u_2^*\sqrt{|p|} & -u_1^*\sqrt{|p|} & u_0^*\sqrt{|p|} & u_3 & u_2 & -u_1 & u_0 \end{bmatrix}. \tag{4}$$

As mentioned earlier, a similar 8×8 real matrix representation, which is derived from the right Hamilton operator, can be found in the study [12].

Proposition 1. *Let $U,V\in H\mathbb{C}_p$ be any elliptic biquaternions. In this case*

1. $U=V\Leftrightarrow (U)_{p\gamma}=(V)_{p\gamma}$,
2. $(U+V)_{p\gamma}=(U)_{p\gamma}+(V)_{p\gamma}$, $(UV)_{p\gamma}=(V)_{p\gamma}(U)_{p\gamma}$,
3. *If U is invertible, then $(U)_{p\gamma}$ is invertible and* $\left(U^{-1}\right)_{p\gamma}=\left(U_{p\gamma}\right)^{-1}$,
4. $(U)_{p\gamma}=T^{-1}(U)_{p\gamma}T$ *where* $T=\begin{bmatrix} 0 & -I_4 \\ I_4 & 0 \end{bmatrix}$.

Proof. Here we will prove the third item. The other items can be proved easily.

(3) It is not difficult to see the following equality

$$(1)_{p\gamma}=I_8.$$

Taking into consideration the inverse property, the equality

$$UU^{-1}=U^{-1}U=1$$

can be written. In this case, we get

$$\left(U^{-1}\right)_{p\gamma}(U)_{p\gamma}=\left(UU^{-1}\right)_{p\gamma}=(1)_{p\gamma}=I_8$$

and

$$(U)_{p\gamma}\left(U^{-1}\right)_{p\gamma}=\left(U^{-1}U\right)_{p\gamma}=(1)_{p\gamma}=I_8$$

by the help of first two properties in this proposition. Consequentially, the equality $\left(U^{-1}\right)_{p\gamma}=\left(U_{p\gamma}\right)^{-1}$ holds. ■

Remark 2. *Let the general linear elliptic biquaternion equation*

$$U_1XV_1+\ldots+U_kXV_k=W \tag{5}$$

be given where $W,U_i,V_i\in H\mathbb{C}_p$, $1\le i\le k$ are known elliptic biquaternions and $X\in H\mathbb{C}_p$ is an unknown elliptic biquaternion. Then, one can easily define the real representation of (5) as follows:

$$(V_1)_{p\gamma}Y(U_1)_{p\gamma}+\ldots+(V_k)_{p\gamma}Y(U_k)_{p\gamma}=(W)_{p\gamma}. \tag{6}$$

Also, it is clear that (5) is equivalent to the following

$$(V_1)_{p\gamma}(X)_{p\gamma}(U_1)_{p\gamma}+\ldots+(V_k)_{p\gamma}(X)_{p\gamma}(U_k)_{p\gamma}=(W)_{p\gamma}$$

from the first and second items of the Proposition 1.

Theorem 3. *Let $U_1, U_2, ..., U_k \in H\mathbb{C}_p$, $V_1, V_2, ..., V_k \in H\mathbb{C}_p$ and $W \in H\mathbb{C}_p$ be given. Then (5) has a solution $X \in H\mathbb{C}_p$ if and only if (6) has a solution $Y \in M_{8\times 8}(\mathbb{R})$. Additionally, if the matrix $Y = (y_{ij}) \in M_{8\times 8}(\mathbb{R})$ is a solution of (6), then the elliptic biquaternion*

$$X = (x_0 + Ix_0^*) + (x_1 + Ix_1^*)\mathbf{i} + (x_2 + Ix_2^*)\mathbf{j} + (x_3 + Ix_3^*)\mathbf{k} \in H\mathbb{C}_p \tag{7}$$

is a solution of (5) where

$$\begin{aligned}
x_0 &= \frac{1}{8}(y_{11} + y_{22} + y_{33} + y_{44} + y_{55} + y_{66} + y_{77} + y_{88})\\
x_0^* &= \frac{1}{8\sqrt{|p|}}(y_{51} - y_{15} + y_{62} - y_{26} + y_{73} - y_{37} + y_{84} - y_{48})\\
x_1 &= \frac{1}{8}(y_{21} - y_{12} + y_{65} - y_{56} + y_{34} - y_{43} + y_{78} - y_{87})\\
x_1^* &= \frac{1}{8\sqrt{|p|}}(y_{16} - y_{52} + y_{61} - y_{25} + y_{74} - y_{38} + y_{47} - y_{83})\\
x_2 &= \frac{1}{8}(y_{31} - y_{13} + y_{75} - y_{57} + y_{42} - y_{24} + y_{86} - y_{68})\\
x_2^* &= \frac{1}{8\sqrt{|p|}}(y_{17} - y_{53} + y_{71} - y_{35} + y_{28} - y_{64} + y_{82} - y_{46})\\
x_3 &= \frac{1}{8}(y_{41} - y_{14} + y_{85} - y_{58} + y_{23} - y_{32} + y_{67} - y_{76})\\
x_3^* &= \frac{1}{8\sqrt{|p|}}(y_{18} - y_{54} + y_{81} - y_{45} + y_{36} - y_{72} + y_{63} - y_{27}).
\end{aligned} \tag{8}$$

Proof. Considering Remark 2, we can conclude that an elliptic biquaternion $X' \in H\mathbb{C}_p$ satisfies (5) if and only if $(X')_{p\gamma} \in M_{8\times 8}(\mathbb{R})$ satisfies (6). In that case, the proof remains to reveal that if the matrix

$$Y = (y_{ij}) \in M_{8\times 8}(\mathbb{R}) \tag{9}$$

is a solution of (6), then the elliptic biquaternion in (7) is a solution of (5). When Y is a solution of (6), from the fourth item of Proposition 1, the equation

$$(V_1)_{p\gamma}\left(T^{-1}YT\right)(U_1)_{p\gamma} + ... + (V_k)_{p\gamma}\left(T^{-1}YT\right)(U_k)_{p\gamma} = (W)_{p\gamma}$$

can be written. This shows that $T^{-1}YT$ is also a solution of (6). Then, the matrix

$$Y' = \frac{1}{2}\left(Y + T^{-1}YT\right) = \frac{1}{2}\begin{bmatrix}
y_{11}+y_{55} & y_{12}+y_{56} & y_{13}+y_{57} & y_{14}+y_{58} & y_{15}-y_{51} & y_{16}-y_{52} & y_{17}-y_{53} & y_{18}-y_{54}\\
y_{21}+y_{65} & y_{22}+y_{66} & y_{23}+y_{67} & y_{24}+y_{68} & y_{25}-y_{61} & y_{26}-y_{62} & y_{27}-y_{63} & y_{28}-y_{64}\\
y_{31}+y_{75} & y_{32}+y_{76} & y_{33}+y_{77} & y_{34}+y_{78} & y_{35}-y_{71} & y_{36}-y_{72} & y_{37}-y_{73} & y_{38}-y_{74}\\
y_{41}+y_{85} & y_{42}+y_{86} & y_{43}+y_{87} & y_{44}+y_{88} & y_{45}-y_{81} & y_{46}-y_{82} & y_{47}-y_{83} & y_{48}-y_{84}\\
y_{51}-y_{15} & y_{52}-y_{16} & y_{53}-y_{17} & y_{54}-y_{18} & y_{11}+y_{55} & y_{12}+y_{56} & y_{13}+y_{57} & y_{14}+y_{58}\\
y_{61}-y_{25} & y_{62}-y_{26} & y_{63}-y_{27} & y_{64}-y_{28} & y_{21}+y_{65} & y_{22}+y_{66} & y_{23}+y_{67} & y_{24}+y_{68}\\
y_{71}-y_{35} & y_{72}-y_{36} & y_{73}-y_{37} & y_{74}-y_{38} & y_{31}+y_{75} & y_{32}+y_{76} & y_{33}+y_{77} & y_{34}+y_{78}\\
y_{81}-y_{45} & y_{82}-y_{46} & y_{83}-y_{47} & y_{84}-y_{48} & y_{41}+y_{85} & y_{42}+y_{86} & y_{43}+y_{87} & y_{44}+y_{88}
\end{bmatrix} \tag{10}$$

satisfies the equation (6) as it is well known from the matrix theory. In other words, Y' is another solution of (6). In regard to (10) and (4), let us construct the elliptic biquaternion

$$X = (x_0 + Ix_0^*) + (x_1 + Ix_1^*)\mathbf{i} + (x_2 + Ix_2^*)\mathbf{j} + (x_3 + Ix_3^*)\mathbf{k} \in H\mathbb{C}_p$$

which satisfies $(X)_{p\gamma} = Y'$. Then, we achieve x_i, x_i^*, $0 \le i \le 3$ as in (8). In a similar way followed at the beginning of the proof, we conclude that aforesaid $X \in H\mathbb{C}_p$ is a solution of (5). This completes the proof. ∎

Theorem 3 can be seen as a numerical method for solving linear elliptic biquaternion equations. As it is clear this method is derived from the left Hamilton operator and it does not preserve the order of matrices in the multiplication case. A similar method, which is derived from the right Hamilton operator and preserves the order of matrices in the multiplication case, can be found in the study [12].

Example 4. *Let us try to solve the equation*

$$[(2-I)+(1+2I)\mathbf{j}]X+X[(1+I)+(3-2I)\mathbf{k}]=(-6+4I)+(5-5I)\mathbf{i}+(21+11I)\mathbf{j}+(-19+I)\mathbf{k}$$

over the space $H\mathbb{C}_{-4}$. We can rewrite this equation as in the following:

$$((2-I)+(1+2I)\mathbf{j})(X)(1)+(1)(X)((1+I)+(3-2I)\mathbf{k})=((-6+4I)+(5-5I)\mathbf{i}+(21+11I)\mathbf{j}+(-19+I)\mathbf{k}).$$

By keeping the equality $(1)_{p\gamma}=I_8$, Remark 2 and Theorem 3 in mind, we get real representation of the last equation as

$$Y\begin{bmatrix} 2 & 0 & -1 & 0 & 2 & 0 & 4 & 0\\ 0 & 2 & 0 & -1 & 0 & 2 & 0 & 4\\ 1 & 0 & 2 & 0 & -4 & 0 & 2 & 0\\ 0 & 1 & 0 & 2 & 0 & -4 & 0 & 2\\ -2 & 0 & -4 & 0 & 2 & 0 & -1 & 0\\ 0 & -2 & 0 & -4 & 0 & 2 & 0 & -1\\ 4 & 0 & -2 & 0 & 1 & 0 & 2 & 0\\ 0 & 4 & 0 & -2 & 0 & 1 & 0 & 2 \end{bmatrix}+\begin{bmatrix} 1 & 0 & 0 & -3 & -2 & 0 & 0 & -4\\ 0 & 1 & 3 & 0 & 0 & -2 & 4 & 0\\ 0 & -3 & 1 & 0 & 0 & -4 & -2 & 0\\ 3 & 0 & 0 & 1 & 4 & 0 & 0 & -2\\ 2 & 0 & 0 & 4 & 1 & 0 & 0 & -3\\ 0 & 2 & -4 & 0 & 0 & 1 & 3 & 0\\ 0 & 4 & 2 & 0 & 0 & -3 & 1 & 0\\ -4 & 0 & 0 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}Y$$

$$=\begin{bmatrix} -6 & -5 & -21 & 19 & -8 & -10 & 22 & 2\\ 5 & -6 & -19 & -21 & 10 & -8 & -2 & 22\\ 21 & 19 & -6 & 5 & -22 & 2 & -8 & 10\\ -19 & 21 & -5 & -6 & -2 & -22 & -10 & -8\\ 8 & 10 & -22 & -2 & -6 & -5 & -21 & 19\\ -10 & 8 & 2 & -22 & 5 & -6 & -19 & -21\\ 22 & -2 & 8 & -10 & 21 & 19 & -6 & 5\\ 2 & 22 & 10 & 8 & -19 & 21 & -5 & -6 \end{bmatrix}.$$

This equation yields

$$Y=\begin{bmatrix} 0 & -1 & 0 & -2 & 0 & -6 & 0 & 0\\ 1 & 0 & 2 & 0 & 6 & 0 & 0 & 0\\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 6\\ 2 & 0 & -1 & 0 & 0 & 0 & -6 & 0\\ 0 & 6 & 0 & 0 & 0 & -1 & 0 & -2\\ -6 & 0 & 0 & 0 & 1 & 0 & 2 & 0\\ 0 & 0 & 0 & -6 & 0 & -2 & 0 & 1\\ 0 & 0 & 6 & 0 & 2 & 0 & -1 & 0 \end{bmatrix}.$$

Consequently, we get

$$X=(1-3I)\mathbf{i}+2\mathbf{k}\in H\mathbb{C}_{-4}$$

from (7) and (8).

Finally, we give an algorithm for solving the problems related to Theorem 3.

Algorithm

1. Input $U_i, V_i, W\in H\mathbb{C}_p$.
2. Form $(U_i)_{p\gamma}$, $(V_i)_{p\gamma}$, $W_{p\gamma}$ by considering (4) $(1\leq i\leq k)$.
3. Compute $Y=(Y_{ij})\in M_{8\times 8}(\mathbb{R})$ satisfying (6).
4. Calculate x_i, x_i^* by considering (8) $(0\leq i\leq 3)$.
5. Output $X=(x_0+Ix_0^*)+(x_1+Ix_1^*)\mathbf{i}+(x_2+Ix_2^*)\mathbf{j}+(x_3+Ix_3^*)\mathbf{k}\in H\mathbb{C}_p$.

4. Conclusion

In this study, a new real matrix representation of an elliptic biquaternion in $H\mathbb{C}_p$, which can be needed to discuss many topics on elliptic biquaternions, is obtained. By the help of this representation, a new method for investigating the exact solutions of linear elliptic biquaternion equations is developed.

$AX \pm B = 0$, $AX \pm XB = C$ and $X \pm AXB = C$ are some examples for the linear elliptic biquaternion equations and they are well known in the literature. Since our method is a general method, it solves these linear equations and all of the others if they have a solution. Existing or not existing of the solution is determined based on the real representation in virtue of Theorem 3.

Since elliptic biquaternions are generalized form of complex quaternions and so real quaternions, we expect that the results obtained in this study will be used as valuable tools in the various areas of science.

A natural question is to investigate the solutions of quadratic elliptic biquaternion equations. We leave that as a future project.

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