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On Parseval Identity of *q*-Sturm-Liouville Problem with Transmission Conditions on Semi Axis

Nida PALAMUT KOŞAR*

Gaziantep University, Nizip Education Faculty, Mathematics and Science Education Department, Gaziantep
(ORCID: 0000-0003-2421-7872)

Abstract
The paper is concerned with the existence of a spectral function for the singular *q*-Sturm-Liouville problem with transmission conditions. Furthermore, the Parseval identity and the expansion formula in the eigenfunctions is established.

Keywords: *q*-Sturm-Liouville operator, Parseval identity, spectral function, eigenfunction expansion.

Yarı Eksende Transfer Koşullu *q*-Sturm-Liouville Probleminin Parseval Özdeşiliği Üzerine

Öz
Makale, transfer koşullu tekil *q*-Sturm-Liouville problemi için bir spektral fonksiyonun varlığı ile ilgilidir. Ayrıca, özfonksiyonlarda genişleme formülü ve Parseval eşitliği oluşturulmuştur.

Anahtar kelimeler: *q*-Sturm-Liouville operatorü, Parseval eşitliği, spectral funksiyon, özfonksiyon genişlemesi.

1. Introduction

The growth and applications of the *q*-calculus which is known as the classical calculus without limits has been of great interest recently. It was seen that it has an important role in different fields of mathematics such as mathematical physics, calculus of variations, statistical mechanic and the theory of quantum. There are lots of details about *q*-calculus for interested researchers Ernst [1], Kac and Cheung [2]. Furthermore, for a general introduction to the *q*-calculus very useful studies have been done such as Allahverdiev and Tuna [3-5], Annaby and Mansour [6, 7] and Jackson [8, 9].

It is well known that eigenfunction expansion problems are important for solving varies problems and there are lots of technics for obtained (see Allahverdiev and Tuna [3], Levitan and Sagsjan [10], Titchmarsh [11], Annaby and Mansour [12]). On the other hand, many researchers have focus on certain generalizations of Sturm-Liouville problems.

Annaby et al. [13] also studied the eigenfunction expansion for a certain *q*-Sturm-Liouville problems by using Titchmarch's technique and defined some concepts for deriving eigenfunction expansion problems. Mamedov et al. [14] gave sampling theory associated with *q*-Sturm-Liouville operator with discontinuity conditions.

Sturm-Liouville problems with transmission conditions have been investigated by many authors such as Allahverdiev and Tuna [5], Mukhtarov and Tunç [15], Mukhtarov and Yakubov [16] et al. gave asymptotic formulas for eigenvalues and the corresponding eigenfunction for these problems. Furthermore, inverse nodal problem for polynomial pencil of Sturm-Liouville operator was studied by Goktas et al. [17], and scattering properties of eigenparameter was given by Bairamov et al. [18]. It was proved that the existence of a spectral function for singular *q*-Sturm-Liouville operators on semi

*Corresponding author: npkosar@gmail.com
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unbounded interval by Allahverdiev and Tuna [4] and also they obtained Parseval identity on expansion formula.

In this study, we consider q-Sturm-Liouville expression as follows:

$$\ell(y) := -\frac{1}{q} D_{q^{-1}} D_q y(\zeta) + u(\zeta) y(\zeta), \qquad \zeta \in J := [0,c) \cup (c, q^{-n}), n \in \mathbb{N}.$$

Here we denote $J_1 := [0,c)$, $J_2 := (c, q^{-n})$ and so $J := J_1 \cup J_2$. Suppose that the points 0, c, q^{-n} are q-regular for the differential expression ℓ. u is real, Lebesque measurable function in J and $u \in L_q^1(J_k)$, $k = 1, 2$. Recall that c is a q-regular point of the function u which belongs to $L_q^1[c-\varepsilon, c+\varepsilon]$ for some $\varepsilon > 0$.

The rest of the study is arranged as follows. In section 2, we give some preliminaries for q-calculus. In section 3, we investigate the existence of a spectral function, Parseval identity has been obtained and expansion formula with eigenfunctions for a singular q-Sturm-Liouville problems with transmission conditions.

2. Preliminaries

We begin with some preliminary facts and notations for quantum calculus (see Kac and Cheung [2], Annaby and Mansour [6]). Our main tools are dealt with q-derivative and q-integral. Let q be any fixed constant with $q \in (0,1)$, $A \subset \mathbb{R}$, and $a \in A$. Also a q-difference equation is an equation which includes an equation which includes q-derivatives of a function defined on A. The q-difference operator is denoted by D_q , the Jackson q-derivative of a function $\varphi: A \to \mathbb{C}$ is defined by

$$D_q \varphi(a) := \frac{\varphi(aq) - \varphi(a)}{aq - a}, \quad \forall a \in A \setminus \{0\}.$$

We say that the point 0 in A is the Jackson q-derivative at zero if the limit

$$D_q \varphi(0) := \frac{\varphi(aq) - \varphi(0)}{aq^n}, \quad \forall a \in A$$

exists and belongs to $\mathbb{C}$. Here, note that the value of the limit is independent of a (see Jackson [8]). The Jackson q-integral is given by

$$\int_0^a \varphi(\zeta) d_q \zeta = a(1-q) \sum_{k=0}^{\infty} q^k \varphi(aq^k), \quad (a \in A),$$

where the series is convergent (see Jackson [9]). Additionally, the following result is satisfied

$$\int_a^b \varphi(\zeta) d_q \zeta = \int_0^b \varphi(\zeta) d_q \zeta - \int_0^a \varphi(\zeta) d_q \zeta, \quad \forall a, b \in A.$$

The Jackson q-integration of φ on $[0, \infty)$ is defined by Hahn [19] by the formula

$$\int_0^\infty \varphi(\zeta) d_q \zeta = a(1-q) \sum_{k=-\infty}^{\infty} q^k \varphi(aq^k), \quad (a \in A),$$

provided that some converges absolutely. A function φ is q-regular at the point zero if the limit

$$\lim_{n \to \infty} \varphi(aq^n) = \varphi(0), \quad (a \in A \cup \{0\})$$

exists. Through the study the functions will be acknowledged q-regular at zero. If φ and ψ are q-regular at zero, then the following equality holds

$$\int_0^a \psi(\zeta) D_q \varphi(\zeta) d_q\zeta + \int_0^a \varphi(q\zeta) D_q \psi(\zeta) d_q\zeta = \varphi(a)\psi(a) - \varphi(0)\psi(0).$$

The separable Hilbert space is $L_q^2(0,\infty) := \{\varphi \big| \ \int_0^\infty |\varphi(\zeta)|^2 \, d_q\zeta < \infty, \ \varphi: [0,\infty] \to \mathbb{C}\}$ with the norm by

$$\|\varphi\| := \left(\int_0^\infty |\varphi(\zeta)|^2 \, d_q\zeta\right)^{\frac{1}{2}} < \infty,$$

and given with the inner product as

$$\langle \varphi, \psi \rangle := \int_0^\infty \varphi(\zeta)\overline{\psi(\zeta)} \, d_q\zeta, \qquad \varphi, \psi \in L_q^2(0,\infty)$$

(see Annaby et al. [13]).

We call the q-Wronskian of φ, ψ functions on A if

$$W_q[\varphi,\psi](a) = \varphi(a) D_q \psi(a) - \psi(a) D_q \varphi(a) \tag{2.1}$$

exists.

3. Results

In this section, we begin with the q-Sturm-Liouville equations as follows

$$\ell(y) := -\tfrac{1}{q} D_{q^{-1}} D_q y(\zeta) + u(\zeta) y(\zeta) := \lambda y, \qquad \zeta \in J \tag{3.1}$$

with the boundary conditions

$$y(0) cos\alpha + D_{q^{-1}} y(0) sin\alpha = 0, \tag{3.2}$$

$$y(q^{-n}) cos\alpha + D_{q^{-1}} y(q^{-n}) sin\alpha = 0, \quad (\alpha, \beta \in \mathbb{R}, n \in \mathbb{N}) \tag{3.3}$$

and transmission conditions:

$$y(c+) - \gamma_1 y(c-) - \gamma_2 D_q y(c-) = 0 \tag{3.4}$$

$$D_q y(c+) - \gamma_3 y(c-) - \gamma_4 D_q y(c-) = 0, \tag{3.5}$$

where λ is a complex eigenparameter and the potential function $u \in L_q^1(J)$ and notice that it guarantees $y(c\mp)$ and $D_q y(c\mp)$ in (3.4) - (3.5) make sense; here we assume that

$$\gamma = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{vmatrix} > 0 \tag{3.6}$$

Furthermore, the class $H_q := L_q^2(J_1) \oplus L_q^2(J_2)$ is introduced as Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{H_q} := \int_0^c \varphi_1 \overline{\psi_1} d_q\zeta + \frac{1}{\gamma} \int_c^{q^{-n}} \varphi_2 \overline{\psi_2} d_q\zeta$$

where

$$\varphi(\zeta) := \begin{cases} \varphi_1(\zeta) & ;\zeta \in J_1 \\ \varphi_2(\zeta) & ;\zeta \in J_2 \end{cases}, \qquad \psi(\zeta) := \begin{cases} \psi_1(\zeta), & ;\zeta \in J_1 \\ \psi_2(\zeta) & ;\zeta \in J_2 \end{cases}.$$

It can be easily obtained by direct manipulation from Annaby [20], (pg. 217). A compact resolvent of the regular self-adjoint boundary value problem (3.1) -(3.3), (3.4) -(3.5) with transmission was proved by the same method like Dehghani and Akbarfam [21] and Wang et al. [22] and they also showed that it has a completely discrete spectrum.

Let us define the eigenvalues of this problem with $\lambda_{m;q^{-n}}$ $(m \in \mathbb{N})$ and

$$\phi_{m;q^{-n}}(\zeta) = \begin{cases} \phi^{(1)}_{m;q^{-n}}(\zeta) & ;\zeta \in J_1 \\ \phi^{(2)}_{m;q^{-n}}(\zeta), & ;\zeta \in J_2 \end{cases},$$

$\phi_{m;q^{-n}}(\zeta) := \phi(\zeta, \lambda_{m;q^{-n}})$ the corresponding real valued eigenfunctions which satisfy conditions (3.2) -(3.5). If $\varphi \in H_q$ is a real valued function with

$$\varphi(\zeta) := \begin{cases} \varphi_1(\zeta) & ;\zeta \in J_1 \\ \varphi_2(\zeta) & ;\zeta \in J_2 \end{cases},$$

then

$$\begin{aligned} \|\varphi\|^2_{H_q} &= \int_0^c \big(\varphi_1(\zeta)\big)^2 d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-n}} \big(\varphi_2(\zeta)\big)^2 d_q\zeta \\ &= \sum_{m=1}^{\infty} \frac{1}{\alpha^2_{m;q^{-n}}} \left\{ \int_0^c \varphi_1(\zeta)\phi^{(1)}_{m;q^{-n}}(\zeta) d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-n}} \varphi_2(\zeta)\phi^{(2)}_{m;q^{-n}}(\zeta) d_q\zeta \right\} \end{aligned} \tag{3.7}$$

where

$$\alpha^2_{m;q^{-n}} = \int_0^c \left(\phi^{(1)}_{m;q^{-n}}(\zeta)\right)^2 d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-n}} \left(\phi^{(2)}_{m;q^{-n}}(\zeta)\right)^2 d_q\zeta$$

is obtained. Here the equality (3.7) is called the Parseval identity (see Allahverdiev and Tuna [5]).

Now we give a monotone increasing step function on $\mathbb{R}$,

$$\sigma_{q^{-n}}(\lambda) = \begin{cases} -\displaystyle\sum_{\lambda < \lambda_{m;q^{-n}} < 0} \frac{1}{\alpha^2_{m;q^{-n}}} & ;\lambda < 0 \\ \displaystyle\sum_{0 < \lambda_{m;q^{-n}} < \lambda} \frac{1}{\alpha^2_{m;q^{-n}}} & ;\lambda \geq 0. \end{cases} \tag{3.8}$$

Then we can write (3.7) as;

$$\int_0^c \big(\varphi_1(\zeta)\big)^2 d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-n}} \big(\varphi_2(\zeta)\big)^2 d_q\zeta = \int_{-\infty}^{\infty} \Phi^2(\lambda) d\sigma_{q^{-n}}(\lambda) \tag{3.9}$$

where

$$\Phi(\lambda)=\int_0^c \varphi_1(\zeta)\phi^{(1)}_{m;q^{-n}}(\zeta)d_q\zeta+\frac{1}{\gamma}\int_c^{q^{-n}}\varphi_2(\zeta)\phi^{(2)}_{m;q^{-n}}(\zeta)d_q\zeta.$$

We will obtain the Parseval identity for (3.1) -(3.5) from (3.9) by letting $q^{-n}\to\infty$.

The function φ is bounded variation on interval $[a,b]$ if and only if there exists a positive constant M such that

$$\sum_{k=1}^{n}|\varphi(\zeta_k)-\varphi(\zeta_{k-1})|\le M$$

for all finite partitions $\mathbb{P}=\{\zeta_0,\zeta_1,\dots,\zeta_n\}$ of $[a,b]$.

If $\varphi:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then the total variation of φ on $[a,b]$ is defined to be

$$V_a^b(\varphi):=sup\sum_{k=1}^{n}|\varphi(\zeta_k)-\varphi(\zeta_{k-1})|,$$

where we take the supremum over all partitions of $[a,b]$ (see Allahverdiev and Tuna [5]).

Lemma 3.1: For an arbitrary M, the formula

$$V_{-M}^{M}\left(\sigma_{q^{-n}}(\lambda)\right)=\sum_{-M<\lambda_{m,q^{-n}}<M}\frac{1}{\alpha^2_{m,q^{-n}}}=\sigma_{q^{-n}}(M)-\sigma_{q^{-n}}(-M)<\Upsilon \tag{3.10}$$

holds for a positive constant $\Upsilon=\Upsilon(M)$.

Proof: Firstly assume that $sin\alpha\neq 0$. Since $\phi(\zeta,\lambda)$ is continuous on domain $[0,c]\times[-M,M]$ with the condition $\phi(0,\lambda)=sin\alpha$, there exists a positive number k such that

$$\left(\frac{1}{k}\int_0^k \phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta\right)^2>\frac{sin^2\alpha}{2} \tag{3.11}$$

Let us define

$$\varphi_k(\zeta):=\begin{cases}\dfrac{1}{k} & ;0\le\zeta<k\\ 0 & ;\zeta\ge k.\end{cases}$$

From (3.9) and (3.11) we get

$$\begin{aligned}\int_0^k \varphi_k^2(\zeta)d_q\zeta=\frac{1}{k}&=\int_{-\infty}^{+\infty}\left(\frac{1}{k}\int_0^k\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)\,d_q\zeta\right)^2 d\sigma_{q^{-n}}(\lambda)\\ &\ge\int_{-M}^{M}\left(\frac{1}{k}\int_0^k\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)\,d_q\zeta\right)^2 d\sigma_{q^{-n}}(\lambda)>\frac{1}{2}sin^2\alpha\int_{-M}^{M}d\sigma_{q^{-n}}(\lambda)\\ &=\frac{1}{2}sin^2\alpha\{\sigma_{q^{-n}}(M)-\sigma_{q^{-n}}(-M)\},\end{aligned}$$

so it gives us the inequality (3.10).

If $sin\alpha = 0$, we give a formula for the function $\varphi_k(\zeta)$ by

$$\varphi_k(\zeta) := \begin{cases} \frac{1}{k^2} & ; 0 \leq \zeta < k \\ 0 & ; \zeta \geq k \end{cases}$$

Thus we obtain (3.10) by applying the Parseval identity.

Let us now mention the following well known Helly's first and second theorems, for more details see Kolmogorov and Fomin [23].

Firstly, recall Helly's first theorem that given a uniformly bounded sequence $\{\psi_n\}$ of monotone increasing real functions on $[a, b]$, there exists a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ converging to a monotone increasing real function ψ on $[a, b]$.

Secondly, given a sequence $\{\psi_n\}$ of monotone increasing real functions on $[a, b]$, converging to a monotone increasing real function ψ, then for every continuous function φ on $[a, b]$ we have

$$\lim_{n\to\infty} \int_a^b \varphi(\lambda) d\psi_n(\lambda) = \int_a^b \varphi(\lambda) d\psi(\lambda).$$

We introduce the Hilbert space $H := L_q^2(J_1) \oplus L_q^2(J_3)$, $(J_1 = [0, c),\ J_3 = (c,\infty))$ with the inner product

$$\langle \varphi, \psi \rangle_H := \int_0^c \varphi_1 \overline{\psi_1} d_q\zeta + \frac{1}{\gamma}\int_c^\infty \varphi_2 \overline{\psi_2} d_q\zeta$$

where

$$\varphi(\zeta) := \begin{cases} \varphi_1(\zeta) & ; \zeta \in J_1 \\ \varphi_2(\zeta) & ; \zeta \in J_3 \end{cases}, \qquad \psi(\zeta) := \begin{cases} \psi_1(\zeta), & ; \zeta \in J_1 \\ \psi_2(\zeta) & ; \zeta \in J_3 \end{cases}.$$

We assume that let σ is any non-decreasing function for $-\infty < \lambda < \infty$. Let us define all measurable real functions of Hilbert space by $L_\sigma^2(\mathbb{R})$ which holds

$$\int_{-\infty}^{\infty} \varphi^2(\lambda) d\sigma(\lambda) < \infty,$$

with the inner product

$$\langle \varphi, \psi \rangle_\sigma := \int_{-\infty}^{\infty} \varphi(\lambda)\overline{\psi(\lambda)}\, d\sigma(\lambda).$$

The fundamental result of the study is given as follows.

Theorem 3.2. The non-decreasing function $\sigma(\lambda)$ on $-\infty < \lambda < \infty$ for the q-Sturm-Liouville problem (2.1)-(2.3) satisfies the following properties:

(i) If

$$\varphi(\zeta) := \begin{cases} \varphi_1(\zeta) & ; \zeta \in J_1 \\ \varphi_2(\zeta) & ; \zeta \in J_3 \end{cases}$$

is a real valued function and φ belongs to H, then there is a function $\Phi \in L_\sigma^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left\{ \Phi(\lambda) - \int_0^c \varphi_1(\zeta)\phi_{m;q^{-n}}^{(1)}(\zeta,\lambda) d_q\zeta - \frac{1}{\gamma}\int_c^n \varphi_2(\zeta)\phi_{m;q^{-n}}^{(2)}(\zeta,\lambda) d_q\zeta \right\} d\sigma(\lambda) = 0 \qquad (3.12)$$

and the Parseval identity

$$\|\varphi\|_H^2 = \int_0^c (\varphi_1(\zeta))^2 d_q\zeta + \frac{1}{\gamma}\int_c^\infty (\varphi_2(\zeta))^2 d_q\zeta = \int_{-\infty}^\infty \Phi^2(\lambda)d\sigma(\lambda) \tag{3.13}$$

(ii) The integral $\int_{-\infty}^{\infty}\Phi(\lambda)\,\phi(\zeta,\lambda_{m;q^{-n}})d\sigma(\lambda)$ converges to φ in H; that is,

(iii)

$$\lim_{n\to\infty}\left\{\int_0^c\left(\varphi_1(\zeta)-\int_{-n}^n\Phi(\lambda)\,\phi^{(1)}(\zeta,\lambda_{m;q^{-n}})d\sigma(\lambda)\right)^2 d_q\zeta + \frac{1}{\gamma}\int_c^\infty\left(\varphi_2(\zeta)-\int_{-n}^n\Phi(\lambda)\,\phi^{(2)}(\zeta,\lambda_{m;q^{-n}})d\sigma(\lambda)\right)^2 d_q\zeta\right\}=0.$$

It should be known that the function σ is said to be a spectral function for the boundary-value problem (3.1)-(3.5).

Proof. We may assume that

$$\varphi_\xi(\zeta) := \begin{cases}\varphi_{1;\xi}(\zeta) & ;\zeta\in[0,c)\\ \varphi_{2;\xi}(\zeta) & ;\zeta\in(c,q^{-\xi}]\end{cases}$$

satisfies three conditions as follows:

(a) Let $\varphi_\xi(\zeta)$ be identically zero outside the set $[0,c)\cup(c,q^{-\xi}]$ with $q^{-\xi}<q^{-n}$.
(b) Let $\varphi_\xi(\zeta)$ and $D_q\varphi_\xi(\zeta)$ be q-regular functions at c.
(c) Let $\varphi_\xi(\zeta)$ satisfy the boundary conditions (3.1)-(3.5).

Applying the Parseval identity (3.9) to the function $\varphi_\xi(\zeta)$ we obtain;

$$\int_0^c (\varphi_{1;\xi}(\zeta))^2 d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-\xi}} (\varphi_{2;\xi}(\zeta))^2 d_q\zeta = \int_{-\infty}^\infty \Phi_\xi^2(\lambda)d\sigma(\lambda) \tag{3.14}$$

where

$$\Phi_\xi(\lambda) = \int_0^c \varphi_{1;\xi}(\zeta)\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-\xi}} \varphi_{2;\xi}(\zeta)\phi^{(2)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta. \tag{3.15}$$

Because of $\phi(t,\lambda_{m;q^{-n}})$ holds (3.1), it is clear that

$$\phi_{m;q^{-n}}(\zeta,\lambda)=\frac{1}{\lambda}\left[-\frac{1}{q}D_{q^{-1}}D_q\phi_{m;q^{-n}}(\zeta,\lambda)+u(\zeta)\phi_{m;q^{-n}}(\zeta,\lambda)\right].$$

From (3.14), we get

$$\Phi_\xi(\lambda)=\frac{1}{\lambda}\int_0^c\varphi_{1;\xi}(\zeta)\left[-\frac{1}{q}D_{q^{-1}}D_q\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)+u(\zeta)\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)\right]d_q\zeta + \frac{1}{\lambda\gamma}\int_c^{q^{-\xi}}\varphi_{2;\xi}(\zeta)\left[-\frac{1}{q}D_{q^{-1}}D_q\phi^{(2)}_{m;q^{-n}}(\zeta,\lambda)+u(\zeta)\phi^{(2)}_{m;q^{-n}}(\zeta,\lambda)\right]d_q\zeta.$$

Since $\varphi_\xi(\zeta)$ is identically zero in a neighborhood of the point q^{-n} and both $\varphi_\xi(\zeta)$ and $\phi_{m;q^{-n}}(\zeta,\lambda)$ satisfy the boundary conditions (3.1)-(3.3), taking by q-integration by parts we obtain;

$$\Phi_{\xi}(\lambda)=\frac{1}{\lambda}\int_0^c \phi_{m;q^{-n}}^{(1)}(\zeta,\lambda)\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{1;\xi}(\zeta)+u(\zeta)\varphi_{1;\xi}(\zeta)\right]d_q\zeta + \frac{1}{\lambda\gamma}\int_c^{q^{-n}} \phi_{m;q^{-n}}^{(2)}(\zeta,\lambda)\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{2;\xi}(\zeta)+u(t)\varphi_{2;\xi}(\zeta)\right]d_q\zeta.$$

For any finite $M>0$, from (3.9) we have

$$\begin{aligned}\int_{|\lambda|>M}\Phi_{\xi}^2(\lambda)d\sigma_{q^{-n}}(\lambda) &\le \frac{1}{M^2}\int_{|\lambda|>M}\left\{\int_0^c \phi_{m;q^{-n}}^{(1)}(\zeta,\lambda)\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{1;\xi}(\zeta)+u(\zeta)\varphi_{1;\xi}(\zeta)\right]d_q\zeta\right.\\
&\left.+\frac{1}{\gamma}\int_c^{q^{-n}} \phi_{m;q^{-n}}^{(2)}(\zeta,\lambda)\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{2;\xi}(\zeta)+u(\zeta)\varphi_{2;\xi}(\zeta)\right]d_q\zeta\right\}^2 d\sigma_{q^{-n}}(\lambda)\\
&\le \frac{1}{M^2}\int_{-\infty}^{\infty}\left\{\int_0^c \phi_{m;q^{-n}}^{(1)}(\zeta,\lambda)\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{1;\xi}(\zeta)+u(\zeta)\varphi_{1;\xi}(\zeta)\right]d_q\zeta\right.\\
&\left.+\frac{1}{\gamma}\int_c^{q^{-n}} \phi_{m;q^{-n}}^{(2)}(\zeta,\lambda)\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{2;\xi}(\zeta)+u(\zeta)\varphi_{2;\xi}(\zeta)\right]d_q\zeta\right\}^2 d\sigma_{q^{-n}}(\lambda)\\
&=\frac{1}{M^2}\int_0^c\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{1;\xi}(\zeta)+u(\zeta)\varphi_{1;\xi}(\zeta)\right]^2 d_q\zeta\\
&+\frac{1}{M^2}\frac{1}{\gamma}\int_c^{q^{-n}}\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{2;\xi}(\zeta)+u(\zeta)\varphi_{2;\xi}(\zeta)\right]^2 d_q\zeta.\end{aligned}$$

From (3.14), we obtain that

$$\begin{aligned}&\left|\int_0^c\left(\varphi_{1;\xi}(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{q^{-\xi}}\left(\varphi_{2;\xi}(\zeta)\right)^2 d_q\zeta-\int_{-M}^{M}\Phi_{\xi}^2(\lambda)d\sigma_{q^{-n}}(\lambda)\right|\\
&<\frac{1}{M^2}\int_0^c\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{1;\xi}(\zeta)+u(\zeta)\varphi_{1;\xi}(\zeta)\right]^2 d_q\zeta\\
&+\frac{1}{M^2}\frac{1}{\gamma}\int_c^{q^{-\xi}}\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{2;\xi}(\zeta)+u(\zeta)\varphi_{2;\xi}(\zeta)\right]^2 d_q\zeta \qquad (3.16)\end{aligned}$$

We know that the set $\{\sigma_{q^{-n}}(\lambda)\}$ is bounded from Lemma 3.1. A sequence $\{\psi_k\}$, $(\psi_k\to\infty)$ such that the function $\sigma_{q^{-n};\psi_k}(\lambda)$ converges to a monotone function $\sigma(\lambda)$ can be found from Helly's first and second theorems. By taking limit with respect to $\{\psi_k\}$ in (3.16) we get;

$$\begin{aligned}&\left|\int_0^c\left(\varphi_{1;\xi}(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{q^{-\xi}}\left(\varphi_{2;\xi}(\zeta)\right)^2 d_q\zeta-\int_{-M}^{M}\Phi_{\xi}^2(\lambda)d\sigma(\lambda)\right|\\
&<\frac{1}{M^2}\int_0^c\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{1;\xi}(\zeta)+u(\zeta)\varphi_{1;\xi}(\zeta)\right]^2 d_q\zeta\\
&+\frac{1}{M^2}\frac{1}{\gamma}\int_c^{q^{-\xi}}\left[-\frac{1}{q}D_{q^{-1}}D_q\varphi_{2;\xi}(\zeta)+u(\zeta)\varphi_{2;\xi}(\zeta)\right]^2 d_q\zeta\end{aligned}$$

Therefore, letting $M\to\infty$ we get,

$$\int_0^c\left(\varphi_{1;\xi}(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{q^{-\xi}}\left(\varphi_{2;\xi}(\zeta)\right)^2 d_q\zeta=\int_{-\infty}^{+\infty}\Phi_{\xi}^2(\lambda)d\sigma(\lambda).$$

Let φ be any real function on H. It is known that there is a sequence of function $\{\varphi_\xi(\zeta)\}$ satisfying the condition (3.1)-(3.5) and such that;

$$\lim_{\xi\to\infty}\left\{\int_0^c \left(\varphi_1(\zeta)-\varphi_{1;\xi}(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{q^{-\xi}} \left(\varphi_2(\zeta)-\varphi_{2;\xi}(\zeta)\right)^2 d_q\zeta\right\}=0$$

Let;

$$\Phi_\xi(\lambda)=\int_0^{+\infty} \varphi_{1;\xi}(\zeta)\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta+\frac{1}{\gamma}\int_c^{+\infty}\varphi_{2;\xi}(\zeta)\phi^{(2)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta.$$

Then from this we can get;

$$\int_0^c \left(\varphi_{1;\xi}(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{+\infty}\left(\varphi_{2;\xi}(\zeta)\right)^2 d_q\zeta=\int_{-\infty}^{+\infty}\Phi_\xi^2(\lambda)d\sigma(\lambda).$$

Since

$$\int_0^c \left(\varphi_{1;\xi_1}(\zeta)-\varphi_{1;\xi_2}(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{+\infty}\left(\varphi_{2;\xi_1}(\zeta)-\varphi_{2;\xi_2}(\zeta)\right)^2 d_q\zeta\to 0$$

as $\xi_1,\xi_2\to\infty$, we have

$$\int_{-\infty}^{+\infty}\left(\Phi_{\xi_1}(\lambda)-\Phi_{\xi_2}(\lambda)\right)^2 d\sigma(\lambda)\to 0,\qquad (\xi_1,\xi_2\to\infty,).$$

Accordingly there is a limit function Φ such that

$$\int_0^c \left(\varphi_1(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{+\infty}\left(\varphi_2(\zeta)\right)^2 d_q\zeta=\int_{-\infty}^{+\infty}\Phi^2(\lambda)d\sigma(\lambda),$$

holds by the completeness of the space $L^2_\sigma(\mathbb{R})$.
Our next aim is to see that;

$$K_\xi(\lambda)=\int_0^c \varphi_1(\zeta)\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta+\frac{1}{\gamma}\int_c^{q^{-\xi}}\varphi_2(\zeta)\phi^{(2)}_{m;q^{-n}}(\zeta,\lambda)d_q\zeta$$

converges to Φ as $\xi\to\infty$ in the space $L^2_\sigma(\mathbb{R})$. Let ψ be an another real valued function in H. By the same way; let $\Psi(\lambda)$ be related with ψ. It is clear that for;

$$\psi(\zeta):=\begin{cases}\psi_1(\zeta) & ;\zeta\in[0,c)\cup[c,q^{-\xi}]\\ \psi_2(\zeta) & ;\zeta\in(q^{-\xi},+\infty)\end{cases}$$

we have;

$$\int_0^c \left(\varphi_1(\zeta)-\psi_1(\zeta)\right)^2 d_q\zeta+\frac{1}{\gamma}\int_c^{+\infty}\left(\varphi_2(\zeta)-\psi_2(\zeta)\right)^2 d_q\zeta=\int_{-\infty}^{+\infty}\{\Phi(\lambda)-\Psi(\lambda)\}^2 d\sigma(\lambda).$$

Let us define

$$\psi(\zeta)=\begin{cases}\varphi(\zeta) & ;\zeta\in[0,c)\cup[c,q^{-\xi}]\\ 0 & ;\zeta\in(q^{-\xi},+\infty)\end{cases}$$

then

$$\int_{-\infty}^{+\infty} \{\Phi(\lambda)- K_{\xi}(\lambda)\}^2 d\sigma(\lambda) = \frac{1}{\gamma}\int_{q^{-\xi}}^{+\infty} (\varphi_2(\zeta))^2 d_q\zeta \to 0$$

as $\xi \to \infty$ which proves that K_{ξ} converges to Φ in $L^2_{\sigma}(\mathbb{R})$. This gives us (i).

Now let us give the prove of (ii). Assume that the functions $\Phi(\lambda)$, $\Psi(\lambda)$ are Fourier transforms of $\varphi,\ \psi \in H$ (see [10]). Then $\Phi \mp \Psi$ are transforms of $\varphi \mp \psi$. In this manner, by (3.13) we have;

$$\int_0^c (\varphi_1(\zeta)+\psi_1(\zeta))^2 d_q\zeta + \frac{1}{\gamma}\int_c^{+\infty} (\varphi_2(\zeta)+\psi_2(\zeta))^2 d_q\zeta = \int_{-\infty}^{+\infty} \{\Phi(\lambda)+ \Psi(\lambda)\}^2 d\sigma(\lambda),$$

$$\int_0^c (\varphi_1(\zeta)-\psi_1(\zeta))^2 d_q\zeta + \frac{1}{\gamma}\int_c^{+\infty} (\varphi_2(\zeta)-\psi_2(\zeta))^2 d_q\zeta = \int_{-\infty}^{+\infty} \{\Phi(\lambda)- \Psi(\lambda)\}^2 d\sigma(\lambda).$$

Let us subtract the resulting two equation side by side, we have;

$$\int_0^c \varphi_1(\zeta)\psi_1(\zeta) d_q\zeta + \frac{1}{\gamma}\int_c^{+\infty} \varphi_2(\zeta)\psi_2(\zeta) d_q\zeta = \int_{-\infty}^{+\infty} \Phi(\lambda)\Psi(\lambda) d\sigma(\lambda) \tag{3.17}$$

which is may be called the generalized Parseval identity. Set;

$$\varphi_{\tau}^{(j)}(\zeta) = \int_{-\tau}^{+\tau} \Phi(\lambda)\, \phi^{(j)}_{m;q^{-n}}(\zeta,\lambda) d\sigma(\lambda), \qquad j=1,2$$

where Φ is the function determined in (3.12). Let $\psi \in H$ be a real function which equals zero outside the set $[0,c) \cup [c,q^{-\mu}]$. So we can get;

$$\begin{aligned}
\int_0^c \varphi_{\tau}^{(1)}(\zeta)\psi_1(\zeta) d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-\mu}} \varphi_{\tau}^{(2)}(\zeta)\psi_2(\zeta) d_q\zeta &= \int_0^c \left\{\int_{-\tau}^{\tau} \Phi(\lambda)\phi^{(1)}_{m;q^{-n}}(\zeta,\lambda) d\sigma(\lambda)\right\}\psi_1(\zeta) d_q\zeta \\
&+ \frac{1}{\gamma}\int_c^{q^{-\mu}} \left\{\int_{-\tau}^{\tau} \Phi(\lambda)\phi^{(2)}_{m;q^{-n}}(\zeta,\lambda) d\sigma(\lambda)\right\}\psi_2(\zeta) d_q\zeta \\
&= \int_{-\tau}^{\tau} \Phi(\lambda)\left\{\int_0^c \phi^{(1)}_{m;q^{-n}}(\zeta,\lambda)\psi_1(\zeta) d_q\zeta + \frac{1}{\gamma}\int_c^{q^{-\mu}} \phi^{(2)}_{m;q^{-n}}(\zeta,\lambda)\psi_2(\zeta) d_q\zeta\right\} d\sigma(\lambda) \\
&= \int_{-\tau}^{+\tau} \Phi(\lambda)\,\Psi(\lambda) d\sigma(\lambda).
\end{aligned}$$

Subtracting (3.17) and (3.18), we have

$$\int_0^c \left(\varphi_1(\zeta)-\varphi_{\tau}^{(1)}(\zeta)\right)\psi_1(\zeta) d_q\zeta + \frac{1}{\gamma}\int_c^{+\infty} \left(\varphi_2(\zeta)-\varphi_{\tau}^{(2)}(\zeta)\right)\psi_2(\zeta) d_q\zeta = \int_{|\lambda|>\tau} \Phi(\lambda)\Psi(\lambda) d\sigma(\lambda).$$

From Cauchy-Schwartz inequality, we get

$$\left(\int_0^c \left(\varphi_1(\zeta)-\varphi_\tau^{(1)}(\zeta)\right)\psi_1(\zeta)d_q\zeta+\frac{1}{\gamma}\int_c^{+\infty}\left(\varphi_2(\zeta)-\varphi_\tau^{(2)}(\zeta)\right)\psi_2(\zeta)d_q\zeta\right)^2 \leq \int_{|\lambda|>\tau}\Phi^2(\lambda)d\sigma(\lambda)\int_{|\lambda|>\tau}\Psi^2(\lambda)d\sigma(\lambda).$$

If we carry out this inequality to the function;

$$\psi(\zeta)=\begin{cases}\varphi_\tau(\zeta)-\varphi(\zeta) & ;\zeta\in[0,c)\cup[c,q^{-\mu}]\\ 0 & ;\zeta\in(q^{-\mu},+\infty),\end{cases}$$

we get

$$\int_0^c \left(\varphi_1(\zeta)-\varphi_\tau^{(1)}(\zeta)\right)^2 d_q\zeta+\int_c^{q^{-\mu}}\left(\varphi_2(\zeta)-\varphi_\tau^{(2)}(\zeta)\right)^2 d_q\zeta \leq \int_{|\lambda|>\tau}\Phi^2(\lambda)d\sigma(\lambda).$$

Since the above inequality is not depend of μ , the result is achieved by letting $\tau\to\infty$. □

4. Conclusion

In this study, we investigate the existence of a spectral function for the singular q-Sturm-Liouville problem with transmission conditions. We prove the Parseval identity with the help of the inner product in class $H_q := L_q^2(J_1)\oplus L_q^2(J_2)$ as Hilbert space. We also give the expansion formula in the eigenfunctions.

Author's Contribution

All contributions belong to the author in this paper.

Statement of Conflicts of Interest

No potential conflict of interest was reported by author.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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