



An Application to the Existence of Solutions of the Integral Equations

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ABSTRACT. Integral equations provide mathematical models of many important problems in the physical sciences and engineering. This paper treats one class of such equations, concentrating on methods involving the use of classical fixed point theorem. The study of integral equations in connection with nonlinear equations has a long history, during which a variety of approaches has emerged. Here, we effectively use a strategy that derives key properties of the solvability of integral equations from previously established results in Hölder spaces. Moreover, our approach leads to solvability of the Fredholm integral equations.

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1. INTRODUCTION

Fredholm integral equations show up thoroughly in many scientific areas like approximation theory, computational mathematics, physical mathematics and contact problems in the theory of elasticity. A short time since, several authors have comprehensively studied the integral equations and the solution of the Fredholm integral equation with the fixed point approach [1–10].

For example, J. Caballero, M. Darwish and K. Sadarangani et al. [3] study the following equation;

$$x(t) = p(t) + x(t) \int_0^1 k(t, \tau)x(\tau)d\tau. \quad (1.1)$$

This article concerns the entity of solutions of the a quadratic integral equation of Fredholm type,

$$x(t) = (T_1x)(t) + (T_2x)(t) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau, \quad t \in I = [0, 1], \quad (1.2)$$

where k, ρ are given functions, T_1, T_2 are given operators satisfying conditions specified later and x is unknown function. Notice that equation (1.1) in study [3] is a particular case of (1.2), for $(T_1x)(t) = p(t)$ and $(T_2x)(t) = x(t)$.

2. PRELIMINARIES

Let $[a, b]$ be a closed interval in \mathbb{R} , by $C[a, b]$ we indicate the space of continuous functions defined on $[a, b]$ equipped with the supremum norm, i.e.,

$$\|x\|_\infty = \sup \{|x(t)| : t \in [a, b]\}$$

for $x \in C[a, b]$. For a fixed α with $0 < \alpha \leq 1$, by $H_\alpha[a, b]$ we will indicate the spaces of the real functions x defined on $[a, b]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that

$$|x(t) - x(s)| \leq H_x^\alpha |t - s|^\alpha \tag{2.1}$$

for all $t, s \in [a, b]$. It is well proved that $H_\alpha[a, b]$ is a linear subspaces of $C[a, b]$. Also, for $x \in H_\alpha[a, b]$, by H_x^α we will indicate the least possible stable for which inequality (2.1) is satisfied. Rather, we put

$$H_x^\alpha = \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b] \text{ and } t \neq s \right\}. \tag{2.2}$$

The space $H_\alpha[a, b]$ with $0 < \alpha \leq 1$ may be equipped with the norm

$$\|x\|_\alpha = |x(a)| + H_x^\alpha$$

for $x \in H_\alpha[a, b]$. Here, H_x^α is defined by (2.2). In [1], the authors demonstrated that $(H_\alpha[a, b], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space.

Lemma 2.1 ([1]). *For $0 < \alpha \leq 1$ and $x \in H_\alpha[a, b]$, we have:*

$$\|x\|_\infty \leq \max(1, (b - a)^\alpha) \|x\|_\alpha.$$

In particular, the inequality $\|x\|_\infty \leq \|x\|_\alpha$ is satisfied for $a = 0$ and $b = 1$.

Lemma 2.2 ([1]). *For $0 < \alpha < \beta \leq 1$, we have*

$$H_\beta[a, b] \subset H_\alpha[a, b] \subset C[a, b].$$

Furthermore, for $x \in H_\beta[a, b]$, we have:

$$\|x\|_\alpha \leq \max(1, (b - a)^{\beta - \alpha}) \|x\|_\beta.$$

Particularly, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\beta$ is satisfied for $a = 0$ and $b = 1$.

Lemma 2.3 ([3]). *Let's assume that $0 < \alpha < \beta \leq 1$ and E is a bounded subset in $H_\beta[a, b]$, then E is a relatively compact subset in $H_\alpha[a, b]$.*

Lemma 2.4 ([3]). *Assume that $0 < \alpha < \beta \leq 1$ and by B_r^β we indicate the ball centered at θ and radius r in the space $H_\beta[a, b]$, i.e., $B_r^\beta = \{x \in H_\beta[a, b] : \|x\|_\beta \leq r\}$. Then B_r^β is a closed subset of $H_\alpha[a, b]$.*

Corollary 2.5 ([3]). *Assume that $0 < \alpha < \beta \leq 1$ and $B_r^\beta = \{x \in H_\beta[a, b] : \|x\|_\beta \leq r\}$. Then B_r^β is a compact subset in the space $H_\alpha[a, b]$.*

Theorem 2.6 (Schauder's fixed point theorem [10]). *Let E be a nonempty and convex subset of a Banach space $(X, \|\cdot\|)$ and let $T : E \rightarrow E$ be a continuity mapping. Then T has at least one fixed point in E .*

3. MAIN RESULT

Theorem 3.1. *Assume that the following conditions (i) – (iv) are satisfied:*

- (i) *The operators $T_1, T_2 : H_\beta[0, 1] \rightarrow H_\beta[0, 1]$ are continuous on $H_\beta[0, 1]$ with respect to the norm $\|\cdot\|_\alpha$. Also, T_1 and T_2 hold the inequalities*

$$\|T_1 x\|_\beta \leq f_1(\|x\|_\beta) \text{ and } \|T_2 x\|_\beta \leq f_2(\|x\|_\beta)$$

for any $x \in H_\beta[0, 1]$, where α and β are the fixed constants satisfying $0 < \alpha < \beta \leq 1$ and the functions $f_1, f_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nondecreasing on \mathbb{R}_+ .

- (ii) *$k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is a continuous function such that there exists a constant $k_\beta > 0$ satisfying*

$$|k(t, \tau) - k(s, \tau)| \leq k_\beta |t - s|^\beta,$$

for any $t, s, \tau \in [0, 1]$.

(iii) $\rho : [0, 1] \rightarrow [0, 1]$ is measurable function.

(iv) There exists a positive solution r_0 of the inequality

$$f_1(r) + (2K + k_\beta)r f_2(r) \leq r,$$

where the constant K is defined by

$$\sup \left\{ \int_0^1 |k(t, \tau)| d\tau : t \in [0, 1] \right\} \leq K.$$

Then the equation (1.2) has at least one solution $x = x(t)$ belonging to space $H_\alpha[0, 1]$.

Proof. Let us consider $x \in H_\beta[0, 1]$ and the operator F defined on the space $H_\beta[0, 1]$ by the formula:

$$(F x)(t) = (T_1 x)(t) + (T_2 x)(t) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau,$$

for $t \in [0, 1]$. Then for arbitrarily fixed $t, s \in [0, 1], (t \neq s)$, in view of our assumptions we get

$$\begin{aligned} (F x)(t) - (F x)(s) &= (T_1 x)(t) + (T_2 x)(t) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau \\ &\quad - (T_1 x)(s) - (T_2 x)(s) \int_0^1 k(s, \tau)x(\rho(\tau))d\tau \\ &= (T_1 x)(t) - (T_1 x)(s) + (T_2 x)(t) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau \\ &\quad - (T_2 x)(s) \int_0^1 k(s, \tau)x(\rho(\tau)) \\ &\quad + (T_2 x)(s) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau \\ &\quad - (T_2 x)(s) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau \\ &= (T_1 x)(t) - (T_1 x)(s) + ((T_2 x)(t) - (T_2 x)(s)) \int_0^1 k(t, \tau)x(\rho(\tau))d\tau \\ &\quad + (T_2 x)(s) \int_0^1 (k(t, \tau) - k(s, \tau))x(\rho(\tau))d\tau \end{aligned}$$

and

$$\begin{aligned} \frac{|(F x)(t) - (F x)(s)|}{|t - s|^\beta} &\leq \frac{|(T_1 x)(t) - (T_1 x)(s)|}{|t - s|^\beta} + \frac{|(T_2 x)(t) - (T_2 x)(s)|}{|t - s|^\beta} \int_0^1 |k(t, \tau)| |x(\rho(\tau))| d\tau \\ &\quad + \frac{|(T_2 x)(s)|}{|t - s|^\beta} \int_0^1 |k(t, \tau) - k(s, \tau)| |x(\rho(\tau))| d\tau \\ &\leq H_{T_1 x}^\beta + \|x\|_\infty \|T_2 x\|_\beta \int_0^1 |k(t, \tau)| d\tau \\ &\quad + \|x\|_\infty \|T_2 x\|_\infty \int_0^1 \frac{|k(t, \tau) - k(s, \tau)|}{|t - s|^\beta} d\tau \\ &\leq H_{T_1 x}^\beta + \|x\|_\beta \|T_2 x\|_\beta K + \|x\|_\beta \|T_2 x\|_\beta \int_0^1 k_\beta \frac{|t - s|^\beta}{|t - s|^\beta} d\tau \\ &\leq H_{T_1 x}^\beta + \|x\|_\beta f_2(\|x\|_\beta) K + \|x\|_\beta f_2(\|x\|_\beta) k_\beta \\ &= H_{T_1 x}^\beta + (K + k_\beta) \|x\|_\beta f_2(\|x\|_\beta). \end{aligned} \tag{3.1}$$

This demonstrates that the operator F maps $H_\beta[0, 1]$ into itself.

Besides, for any $x \in H_\beta[0, 1]$, we get

$$\begin{aligned}
 |(Fx)(0)| &\leq |(T_1x)(0)| + |(T_2x)(0)| \int_0^1 |k(0, \tau)| |x(\rho(\tau))| d\tau \\
 &\leq |(T_1x)(0)| + \|T_2x\|_\infty \|x\|_\infty K \\
 &\leq |(T_1x)(0)| + \|T_2x\|_\beta \|x\|_\beta K \\
 &\leq |(T_1x)(0)| + \|x\|_\beta f_2(\|x\|_\beta) K.
 \end{aligned} \tag{3.2}$$

By the inequalities by (3.1) and (3.2), we derive that

$$\begin{aligned}
 \|Fx\|_\beta &\leq \|T_1x\|_\beta + (2K + k_\beta) \|x\|_\beta f_2(\|x\|_\beta) \\
 &\leq f_1(\|x\|_\beta) + (2K + k_\beta) \|x\|_\beta f_2(\|x\|_\beta).
 \end{aligned} \tag{3.3}$$

Since positive number r_0 is the solution of the inequality given in hypothesis (iv), from (3.3), we conclude that the inequality

$$\|Fx\|_\beta \leq f_1(r_0) + (2K + k_\beta) r_0 f_2(r_0) \leq r_0 \tag{3.4}$$

holds. As a results, it follows from (3.4) that F transforms the ball

$$B_{r_0}^\beta = \{x \in H_\beta[0, 1] : \|x\|_\beta \leq r_0\}$$

into itself. That is, $F : B_{r_0}^\beta \rightarrow B_{r_0}^\beta$. Thus, we have that the set $B_{r_0}^\beta$ is relatively compact in $H_\alpha[0, 1]$ for any $0 < \alpha < \beta \leq 1$. Furthermore, $B_{r_0}^\beta$ is a compact subset in $H_\alpha[0, 1]$.

We will show that the operator F is continuous on $B_{r_0}^\beta$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$. Let $y \in B_{r_0}^\beta$ be an arbitrary point in $B_{r_0}^\beta$. Then, we get

$$\begin{aligned}
 (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) &= (T_1x)(t) + (T_2x)(t) \int_0^1 k(t, \tau) x(\rho(\tau)) d\tau \\
 &\quad - (T_1y)(t) - (T_2y)(t) \int_0^1 k(t, \tau) y(\rho(\tau)) d\tau \\
 &\quad - (T_1x)(s) - (T_2x)(s) \int_0^1 k(s, \tau) x(\rho(\tau)) d\tau \\
 &\quad + (T_1y)(s) + (T_2y)(s) \int_0^1 k(s, \tau) y(\rho(\tau)) d\tau
 \end{aligned} \tag{3.5}$$

for any $x \in B_{r_0}^\beta$ and $t, s \in [0, 1]$. Let us take

$$(T_3x)(\tau) = x(\rho(\tau)) \text{ and } (T_3y)(\tau) = y(\rho(\tau))$$

for the sake of shortness, then the equality (3.5) can be rewritten as:

$$\begin{aligned}
 (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) &= (T_1x)(t) - (T_1y)(t) - ((T_1x)(s) - (T_1y)(s)) \\
 &\quad + (T_2x)(t) \int_0^1 k(t, \tau) (T_3x)(\tau) d\tau - (T_2y)(t) \int_0^1 k(t, \tau) (T_3x)(\tau) d\tau \\
 &\quad + (T_2y)(t) \int_0^1 k(t, \tau) (T_3x)(\tau) d\tau - (T_2y)(t) \int_0^1 k(t, \tau) (T_3y)(\tau) d\tau \\
 &\quad - (T_2x)(s) \int_0^1 k(s, \tau) (T_3x)(\tau) d\tau + (T_2y)(s) \int_0^1 k(s, \tau) (T_3x)(\tau) d\tau \\
 &\quad - (T_2y)(s) \int_0^1 k(s, \tau) (T_3x)(\tau) d\tau + (T_2y)(s) \int_0^1 k(s, \tau) (T_3y)(\tau) d\tau.
 \end{aligned} \tag{3.6}$$

By (3.6), we have

$$\begin{aligned}
 (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) &= (T_1x)(t) - (T_1y)(t) - ((T_1x)(s) - (T_1y)(s)) \\
 &\quad + ((T_2x)(t) - (T_2y)(t)) \int_0^1 k(t, \tau)(T_3x)(\tau) d\tau \\
 &\quad + (T_2y)(t) \int_0^1 k(t, \tau)((T_3x)(\tau) - (T_3y)(\tau)) d\tau \\
 &\quad - ((T_2x)(s) - (T_2y)(s)) \int_0^1 k(s, \tau)(T_3x)(\tau) d\tau \\
 &\quad - (T_2y)(s) \int_0^1 k(s, \tau)((T_3x)(\tau) - (T_3y)(\tau)) d\tau. \tag{3.7}
 \end{aligned}$$

(3.7) yields the following equality:

$$\begin{aligned}
 ((Fx)(t) - (Fy)(t)) - ((Fx)(s) - (Fy)(s)) &= (T_1x)(t) - (T_1y)(t) - ((T_1x)(s) - (T_1y)(s)) \\
 &\quad + [((T_2x)(t) - (T_2y)(t)) - ((T_2x)(s) - (T_2y)(s))] \int_0^1 k(t, \tau)(T_3x)(\tau) d\tau \\
 &\quad + ((T_2x)(s) - (T_2y)(s)) \int_0^1 (k(t, \tau) - k(s, \tau))(T_3x)(\tau) d\tau \\
 &\quad + ((T_2y)(t) - (T_2y)(s)) \int_0^1 k(t, \tau)((T_3x)(\tau) - (T_3y)(\tau)) d\tau \\
 &\quad + (T_2y)(s) \int_0^1 (k(t, \tau) - k(s, \tau))((T_3x)(\tau) - (T_3y)(\tau)) d\tau. \tag{3.8}
 \end{aligned}$$

Since $|(T_3x)(\tau)| \leq \|x\|_\infty$ and $|(T_3x)(\tau) - (T_3y)(\tau)| \leq \|x - y\|_\infty$ for all $x, y \in B_{r_0}^\beta$ and $\tau \in [0, 1]$, taking into account (3.8) and hypotheses, we can write:

$$\begin{aligned}
 \frac{|(Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s))|}{|t - s|^\alpha} &\leq \frac{|(T_1x)(t) - (T_1y)(t) - ((T_1x)(s) - (T_1y)(s))|}{|t - s|^\alpha} \\
 &\quad + \frac{|(T_2x)(t) - (T_2y)(t) - ((T_2x)(s) - (T_2y)(s))|}{|t - s|^\alpha} \int_0^1 |k(t, \tau)|(T_3x)(\tau) d\tau \\
 &\quad + \frac{|(T_2x)(s) - (T_2y)(s)|}{|t - s|^\alpha} \int_0^1 |k(t, \tau) - k(s, \tau)|(T_3x)(\tau) d\tau \\
 &\quad + \frac{|(T_2y)(t) - (T_2y)(s)|}{|t - s|^\alpha} \int_0^1 |k(t, \tau)|(T_3x)(\tau) - (T_3y)(\tau) d\tau \\
 &\quad + \frac{|(T_2y)(s)|}{|t - s|^\alpha} \int_0^1 |k(t, \tau) - k(s, \tau)|(T_3x)(\tau) - (T_3y)(\tau) d\tau \\
 &\leq \|T_1x - T_1y\|_\alpha + \|T_2x - T_2y\|_\alpha \|x\|_\infty K + \|T_2x - T_2y\|_\infty \|x\|_\infty \int_0^1 k_\beta |t - s|^{\beta - \alpha} d\tau \\
 &\quad + \|T_2y\|_\alpha \|x - y\|_\infty K + \|T_2y\|_\infty \|x - y\|_\infty \int_0^1 k_\beta |t - s|^{\beta - \alpha} d\tau \\
 &\leq \|T_1x - T_1y\|_\alpha + K \|T_2x - T_2y\|_\alpha \|x\|_\alpha + k_\beta \|T_2x - T_2y\|_\alpha \|x\|_\alpha \\
 &\quad + K \|T_2y\|_\alpha \|x - y\|_\alpha + k_\beta \|T_2y\|_\alpha \|x - y\|_\alpha \\
 &= \|T_1x - T_1y\|_\alpha + (K + k_\beta) \|T_2x - T_2y\|_\alpha \|x\|_\alpha + (K + k_\beta) \|T_2y\|_\alpha \|x - y\|_\alpha \tag{3.9}
 \end{aligned}$$

for all $t, s \in [0, 1]$ with $t \neq s$. Besides, for $x, y \in \mathcal{B}_{r_0}^\beta$, we obtain following equality:

$$\begin{aligned}
(Fx)(0) - (Fy)(0) &= (T_1x)(0) + (T_2x)(0) \int_0^1 k(0, \tau)(T_3x)(\tau) d\tau \\
&\quad - (T_1y)(0) - (T_2y)(0) \int_0^1 k(0, \tau)(T_3y)(\tau) d\tau \\
&= (T_1x)(0) - (T_1y)(0) + (T_2x)(0) \int_0^1 k(0, \tau)(T_3x)(\tau) d\tau \\
&\quad - (T_2y)(0) \int_0^1 k(0, \tau)(T_3x)(\tau) d\tau + (T_2y)(0) \int_0^1 k(0, \tau)(T_3x)(\tau) d\tau \\
&\quad - (T_2y)(0) \int_0^1 k(0, \tau)(T_3y)(\tau) d\tau \\
&= (T_1x)(0) - (T_1y)(0) + ((T_2x)(0) - (T_2y)(0)) \int_0^1 k(0, \tau)(T_3x)(\tau) d\tau \\
&\quad + (T_2y)(0) \int_0^1 k(0, \tau)((T_3x)(\tau) - (T_3y)(\tau)) d\tau.
\end{aligned} \tag{3.10}$$

By (3.10), we get that

$$\begin{aligned}
|(Fx)(0) - (Fy)(0)| &\leq |(T_1x)(0) - (T_1y)(0)| + |(T_2x)(0) - (T_2y)(0)| |k(0, \tau)| \int_0^1 |(T_3x)(\tau)| d\tau \\
&\quad + |(T_2y)(0)| |k(0, \tau)| \int_0^1 |(T_3x)(\tau) - (T_3y)(\tau)| d\tau \\
&\leq \|T_1x - T_1y\|_\infty + \|T_2x - T_2y\|_\infty K \|x\|_\infty + \|T_2y\|_\infty K \|x - y\|_\infty \\
&\leq \|T_1x - T_1y\|_\alpha + \|T_2x - T_2y\|_\alpha K \|x\|_\alpha + \|T_2y\|_\alpha K \|x - y\|_\alpha.
\end{aligned} \tag{3.11}$$

From (3.9) and (3.11), we have that

$$\begin{aligned}
\|Fx - Fy\|_\alpha &= |(Fx - Fy)(0)| + H_{Fx-Fy}^\alpha \\
&= |(Fx)(0) - (Fy)(0)| \\
&\quad + \sup \left\{ \frac{|(Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s))|}{|t - s|^\alpha} : t, s \in [0, 1] \text{ and } t \neq s \right\} \\
&\leq 2 \|T_1x - T_1y\|_\alpha + (2K + k_\beta) \|T_2x - T_2y\|_\alpha \|x\|_\alpha \\
&\quad + (2K + k_\beta) \|T_2y\|_\alpha \|x - y\|_\alpha \\
&\leq 2 \|T_1x - T_1y\|_\alpha + (2K + k_\beta) \|T_2x - T_2y\|_\alpha \|x\|_\beta \\
&\quad + (2K + k_\beta) \|T_2y\|_\beta \|x - y\|_\alpha \\
&\leq 2 \|T_1x - T_1y\|_\alpha + (2K + k_\beta) \|T_2x - T_2y\|_\alpha \|x\|_\beta \\
&\quad + (2K + k_\beta) f_2(\|y\|_\beta) \|x - y\|_\alpha.
\end{aligned} \tag{3.12}$$

Moreover, since $\|x\|_\beta \leq r_0$ and $\|y\|_\beta \leq r_0$, we derive from (3.12) that the following inequality holds:

$$\begin{aligned}
\|Fx - Fy\|_\alpha &\leq 2 \|T_1x - T_1y\|_\alpha + (2K + k_\beta) r_0 \|T_2x - T_2y\|_\alpha \\
&\quad + (2K + k_\beta) f_2(r_0) \|x - y\|_\alpha.
\end{aligned} \tag{3.13}$$

Since the operators $T_1, T_2 : H_\beta[0, 1] \rightarrow H_\beta[0, 1]$ are continuous on $H_\beta[0, 1]$ with respect to the norm $\|\cdot\|_\alpha$, they are also continuous at the point $y \in \mathcal{B}_{r_0}^\beta$. Let us take an arbitrary $\varepsilon > 0$, then there exists the number δ satisfying

$$0 < \delta < \frac{\varepsilon}{3(2K + k_\beta) f_2(r_0)}$$

such that the inequalities

$$\|T_1x - T_1y\|_\alpha < \frac{\varepsilon}{6}$$

and

$$\|T_2x - T_2y\|_\alpha < \frac{\varepsilon}{3(2K + k_\beta)r_0}$$

hold for all $x \in B_{r_0}^\beta$, where $\|x - y\|_\alpha < \delta$. Then, taking into account (3.13), we derive the following inequality:

$$\|Fx - Fy\|_\alpha < \varepsilon$$

for all $x \in B_{r_0}^\beta$ with $\|x - y\|_\alpha < \delta$. As a results, we infer that the operator F is continuous at the point $y \in B_{r_0}^\beta$. Because y was chosen arbitrarily, we deduce that F is continuous on $B_{r_0}^\beta$ with respect to the norm $\|\cdot\|_\alpha$. As $B_{r_0}^\beta$ is compact in $H_\alpha[0, 1]$, from the classical Schauder fixed point theorem, we get the desired result. \square

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this article.

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