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Framed Curves and Their Applications Based on a New Differential Equation

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ABSTRACT

Characterization is very important for non-regular curves in differential geometry. Recently, the concept of framed curve has been proposed to examine a non-regular curve. Framed curves are defined as smooth curves with a moving frame that can have singular points. In this paper, the differential equation is obtained by using distance squared functions for framed curves with for each $p, q \neq 0$ framed curvatures in Euclidean space. Next, we give the relationships between the differential equation and some special curves. Moreover we introduce a new equation of framed helices with the help of the differential equation. Moreover, these are supported by some examples and figures.

Keywords: *Framed curves, framed spherical curves, framed helices, framed rectifying curves, singular points*
AMS Subject Classification (2020): *Primary: 53A04; Secondary: 58K05, 58K30*

1. Introduction

The framed curve notion is defined as smooth curves that can have singular points by S. Honda and M. Takahashi but whose moving frame can be defined by creating a unit regular curve [1]. Furthermore, in particular, framed curves are important for general curve and surface theory as they are the generalized version of Legendre curves and linear independent conditions and regular curves. Moreover, Wang, Pei and Gao studied framed rectifying curves and define a new adapted frame for framed curves [2]. In addition, special framed curves such as framed normal curves in [3] and framed spherical curves in [4] were studied. See [1]- [6] for a detailed review of framed curves.

In [7], a differential equation is defined for each unit speed regular Frenet curves. With this equation, the characterizations of some special curves are examined in terms of their curvatures (such as helices with $\tau = c\kappa$ with a constant $c \neq 0$, rectifying curves with $\dfrac{\tau}{\kappa} = as + b$ with constants $a \neq 0, b$, etc.) [8]-[10]. However, these investigations are insufficient for non-regular curves. In this study, important characterizations are given for both regular and non-regular curves using framed curves.

In this paper, the differential equation is obtained by using distance squared functions for framed curves with for each $p, q \neq 0$ framed curvatures in Euclidean space. Next, we give the relationships between the differential equation and some special curves. Then, we give a new equation of framed helices with the help of the differential equation. Moreover, these are supported by some examples and figures.

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2. Framed curves in Euclidean space

Let $\gamma : I \to \mathbb{R}^3$ be a curve with singular points. In order to create framed curves, a special smooth manifold structure is defined as follows:

$$\Delta_\eta = \{\eta = (\eta_1, \eta_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \eta_i, \eta_j \rangle = \delta_{ij}, \quad i, j = 1, 2\}.$$

A unit vector is defined by $\nu = \eta_1 \times \eta_2$.

Definition 2.1 (**Framed curve**). $(\gamma, \eta) : I \to \mathbb{R}^3 \times \Delta_\eta$ is called a framed curve if $\langle \gamma'(s), \eta_i(s) \rangle = 0$ for all $s \in I$ and $i = 1, 2$ [1].

Definition 2.2 (**Framed base curve**). $\gamma : I \to \mathbb{R}^3$ is called a framed base curve if there exists $\eta : I \to \Delta_\eta$ such that (γ, η) is a framed curve [1].

Let $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_\eta$ be a framed curve and $\nu(s) = \eta_1(s) \times \eta_2(s)$. The derivative formula is given by

$$\begin{pmatrix} \nu'(s) \\ \eta_1'(s) \\ \eta_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \eta_1(s) \\ \eta_2(s) \end{pmatrix}, \tag{2.1}$$

where $l(s) = \langle \eta_1'(s), \eta_2(s) \rangle$, $m(s) = \langle \eta_1'(s), \nu(s) \rangle$ and $n(s) = \langle \eta_2'(s), \nu(s) \rangle$. Moreover, a smooth mapping $\alpha : I \to \mathbb{R}$ characterizing the singular points of the curve is given as

$$\gamma'(s) = \alpha(s)\nu(s).$$

In addition, s_0 is a singular point of the framed curve γ if and only if $\alpha(s_0) = 0$. Also, the existence and uniqueness theorems for these curvatures are given in [5]. Although the curve has singular points, if a unit regular vector ν can be defined, a moving frame of the curve can be constructed over this vector.

The generalized vectors are defined for framed curves [2]. In addition, rotated, reflected, etc. for framed curves in the literature such as many framed have been given. In this study, an adapted frame defined in [2] is used in terms of subject integrity. For the normal planes of $\gamma(s)$, which are spanned by $\eta_1(s)$ and $\eta_2(s)$. We define $(\eta_1(s), \eta_2(s)) \in \Delta_2$ by

$$\begin{pmatrix} \overline{\eta_1}(s) \\ \overline{\eta_2}(s) \end{pmatrix} = \begin{pmatrix} \cos\psi(s) & -\sin\psi(s) \\ \sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \eta_1(s) \\ \eta_2(s) \end{pmatrix},$$

where $\psi(s)$ is a smooth function. Consequently, $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$ is also a framed curve, and

$$\overline{\nu}(s) = \nu(s).$$

In order to obtain a frame similar to the Frenet-Serret frame of regular curves, $\psi : I \to \mathbb{R}$ is accepted as a function that satisfies the condition $m(s)\sin\psi(s) = -n(s)\cos\psi(s)$. Here, the notation for these curvatures is considered as follows:

$$m(s) = -p(s)\cos\psi(s), \quad n(s) = p(s)\sin\psi(s). \tag{2.2}$$

Hence, the derivative formulas of the generalized vectors of framed curves are calculated as follows:

$$\begin{aligned}
\nu'(s) &= -m(s)\eta_1(s) - n(s)\eta_2(s), \\
&= p(s)[\cos\psi(s))\eta_1(s) - \sin\psi(s)\eta_2(s)], \\
&= p(s)\overline{\eta_1}(s), \\
\overline{\eta_1}'(s) &= -(l(s) - \psi'(s))\sin\psi(s)\eta_1(s) + (l(s) - \psi'(s))\cos\psi(s)\eta_2(s) + (m(s)\cos\psi(s) - n(s)\sin\psi(s))\nu(s), \\
&= -p(s)\nu(s) + (l(s) - \psi'(s))\overline{\eta_2}(s), \\
\overline{\eta_2}'(s) &= -(l(s) - \psi'(s))\cos\psi(s)\eta_1(s) + (l(s) - \psi'(s))\sin\psi(s)\eta_2(s) + (m(s)\sin\psi(s) + n(s)\cos\psi(s))\nu(s), \\
&= -(l(s) - \psi'(s))\overline{\eta_1}(s).
\end{aligned}$$

Consequently, we have

$$\begin{pmatrix} \nu^{'}(s) \\ \overline{\eta_1^{'}}(s) \\ \overline{\eta_2^{'}}(s) \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & q(s) \\ 0 & -q(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\eta_1}(s) \\ \overline{\eta_2}(s) \end{pmatrix}.$$

The vectors $\nu(s), \overline{\eta_1}(s), \overline{\eta_2}(s)$ are called the generalized (tangent, principle normal, binormal) vectors of the framed curve, respectively, where

$$p(s) = \|\nu^{'}(s)\|,$$

and

$$q(s) = l(s) - \psi^{'}(s). \tag{2.3}$$

The functions $(p(s), q(s), \alpha(s))$ are called to as the framed curvature of $\gamma(s)$ [2]. Now let's introduce a new notation for $q(s)$ and show the equality of these two expressions:

$$q(s) = \frac{\langle \nu(s) \wedge \nu^{'}(s), \nu^{''}(s)\rangle}{(\|\nu'(s)\|)^2}.$$

According to equation 2.1, we have

$$\begin{aligned} \nu^{'}(s) &= -m(s)\eta_1(s) - n(s)\eta_2(s), \\ \nu^{''}(s) &= (-m^2(s) - n^2(s))\nu(s) + (-m^{'}(s) + l(s)n(s))\eta_1(s) + (-n^{'}(s) - l(s)m(s))\eta_2(s). \end{aligned}$$

Therefore, we have

$$\frac{\langle \nu(s) \wedge \nu^{'}(s), \nu^{''}(s)\rangle}{(\|\nu'(s)\|)^2} = \frac{l(s)m^2(s) + l(s)n^2(s) + n^{'}(s)m(s) - n(s)m^{'}(s)}{m^2(s) + n^2(s)}. \tag{2.4}$$

On the other hand, it was given as $q(s) = l(s) - \psi^{'}(s)$ in the equation 2.3. According to equation 2.2, we get

$$\psi(s) = \arctan\left(-\frac{n}{m}\right).$$

Consequently, we get

$$\begin{aligned} \psi^{'}(s) &= \frac{-n^{'}(s)m(s) + n(s)m^{'}(s)}{m^2(s) + n^2(s)}, \\ q(s) &= l(s) - \psi^{'}(s), \\ &= \frac{l(s)m^2(s) + l(s)n^2(s) + n^{'}(s)m(s) - n(s)m^{'}(s)}{m^2(s) + n^2(s)}. \end{aligned} \tag{2.5}$$

From here it is seen that the 2.4 and 2.5 equations are equal, that is,

$$q(s) = l(s) - \psi^{'}(s) = \frac{\langle \nu(s) \wedge \nu^{'}(s), \nu^{''}(s)\rangle}{(\|\nu'(s)\|)^2}.$$

Also, the following equations can be given for use in the following sections:

$$\begin{aligned} \langle \gamma(s), \nu(s)\rangle^{'} &= \alpha(s) + p(s)\langle \gamma(s), \overline{\eta_1}(s)\rangle, \\ \langle \gamma(s), \overline{\eta_1}(s)\rangle^{'} &= -p(s)\langle \gamma(s), \nu(s)\rangle + q(s)\langle \gamma(s), \overline{\eta_2}(s)\rangle, \\ \langle \gamma(s), \overline{\eta_2}(s)\rangle^{'} &= -q(s)\langle \gamma(s), \overline{\eta_1}(s)\rangle. \end{aligned} \tag{2.6}$$

3. A Differential Equation for Framed Curves in $\mathbb{R}^3$

In this section, we investigate a differential equation with the help of the distance squared function for framed curve where $p, q \neq 0$ for each $s \in I$ framed curvature. Let $\gamma : I \to \mathbb{R}^3$ be a framed curve with at least one non-singular point in Euclidean 3-space. The distance squared function satisfies d^2 of γ is given by $d^2(s) = \langle \gamma(s), \gamma(s)\rangle$. We give the key proposition in this section as follows:

Proposition 3.1. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve in $\mathbb{R}^3$ with framed curvatures $p, q \neq 0$ for every $s \in I$. Then, γ satisfies*

$$(yx)h''' + (y'x + 2yx')h'' + \left((yx')' + \frac{y}{x} + \frac{x}{y}\right) h' + \left(\frac{y}{x}\right)' h = (yx)\alpha'' + (y'x + 2yx')\alpha' + \left((yx')' + \frac{x}{y}\right) \alpha \tag{3.1}$$

where $x = p^{-1}$, $y = q^{-1}$, $\alpha h = dd'$ and $d^2 = \langle\gamma, \gamma\rangle$.

Proof. Assume that γ is a framed base curve. By differentiating $d^2 = \langle\gamma, \gamma\rangle$, we get

$$2dd^{'} = 2\alpha\langle\gamma, \nu\rangle,$$

and

$$h = \langle\gamma, \nu\rangle. \tag{3.2}$$

By taking the derivative of 3.2 and by using first equation in 2.6, we find

$$x(h' - \alpha) = \langle\gamma, \overline{\eta_1}\rangle. \tag{3.3}$$

After differentiating 3.3 and using 3.2, we get

$$x'(h' - \alpha) + x(h'' - \alpha') = -p\langle\gamma, \nu\rangle + q\langle\gamma, \overline{\eta_2}\rangle. \tag{3.4}$$

Therefore, by using equations 3.2 and 3.4, we find

$$yx'(h' - \alpha) + yx(h'' - \alpha') + \left(\frac{y}{x}\right) h = \langle\gamma, \overline{\eta_2}\rangle. \tag{3.5}$$

Finally, by differentiating 3.5 and applying 3.3 we have equation 3.1. □

Remark 3.1. i.) According to Proposition 3.1, if γ is a specially regular curve, since $h = \dfrac{dd^{'}}{\alpha}$ can be written, the differential equation is a 4-th order differential equation according to the distance function d. In particular, if $\alpha = 1$, since this curve unit speed regular curve, we get

$$(yx)h''' + (y'x + 2yx')h'' + \left((yx')' + \frac{y}{x} + \frac{x}{y}\right) h' + \left(\frac{y}{x}\right)' h = \left((yx')' + \frac{x}{y}\right),$$

where $x = p^{-1}$, $y = q^{-1}$, $h = dd'$ and $d^2 = \langle\gamma, \gamma\rangle$. This differential equation gives a differential equation for unit speed regular curves in [7].

ii.) However, the curve cannot be written as a non-regular curve (i.e if it has singular points) as $h = \dfrac{dd^{'}}{\alpha}$. Hence, differential equation 3.1 is a 3-th order differential equation with respect to h defined by the distance squared function.

Example 3.1. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve defined by

$$\gamma(s) = ((s - 3)\cos s - \sin s, -(s - 3)\sin s - \cos s, \frac{s^2}{2} - 3s). \tag{3.6}$$

(See Fig.1, Fig.2.) The curve γ has a singular point at $s = 3$, so that it is not a Frenet curve. On the other hand, γ is a framed base curve with the mapping $(\nu, \gamma) : \mathbb{R} \to S^2 \times \mathbb{R}$;

$$\nu(s) = \frac{1}{\sqrt{2}}(-\sin s, -\cos s, 1),$$

gives the generalized tangent vector and

$$\alpha(s) = \sqrt{2}(s - 3).$$

By a calculation, we get $p(s) = \dfrac{1}{\sqrt{2}}$ and $q(s) = \dfrac{1}{\sqrt{2}}$. On the other hand, we get the distance squared function

$$d^2(s) = \frac{s^4}{4} - 3s^3 + 10s^2 - 6s + 10.$$

Since $\alpha h = dd^{'}$, by a calculation, we get

$$h(s) = \frac{s^2 - 6s + 2}{2\sqrt{2}}.$$

Consequently, by simple calculations, we get

$$h^{'}(s) = \frac{s-3}{\sqrt{2}}, \quad h^{''}(s) = \frac{1}{\sqrt{2}}, \quad h^{'''}(s) = 0,$$

and

$$\alpha^{'}(s) = \sqrt{2}, \quad \alpha^{''}(s) = 0.$$

Consequently, the obtained values satisfy equation 3.1.

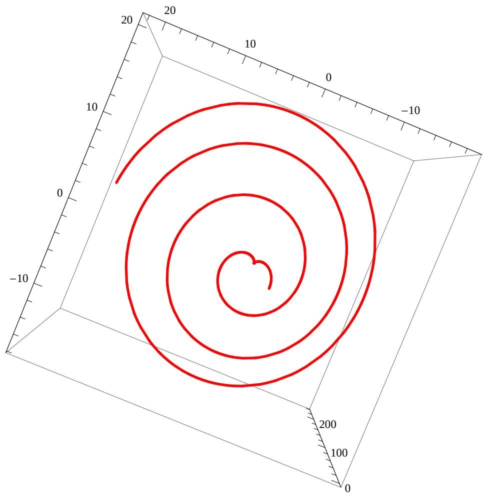

Figure 1. The curve with singular point at $t = 3 \in [0, 25]$ given in 3.6

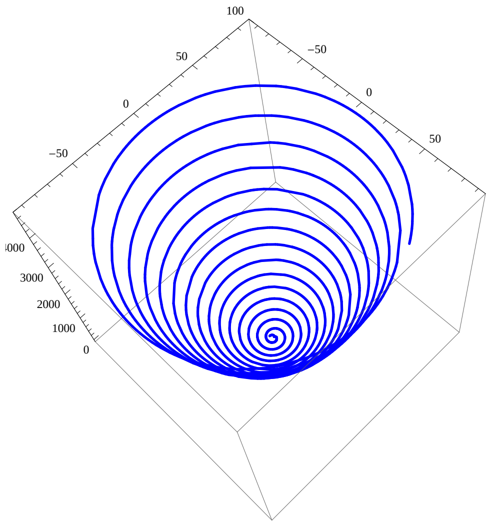

Figure 2. The curve for $t \in [0, 100]$ given in 3.6

4. Some Applications Framed Curves Based on Differential Equation (3.1)

In this section, some classifications related to framed spherical and framed rectifying curves are given with the help of differential equation 3.1. It is also seen that these classifications coincide with a characterization of framed spherical curves in [4] and a characterization of framed rectifying curves in [2].

Corollary 4.1 (Framed spherical curves). *Let γ be a framed curve with framed curvatures (p, q, α) in $\mathbb{R}^3$. Then $\gamma(s)$ is a framed spherical curve if and only if*

$$\left(y(\alpha x)'\right)' + \frac{\alpha x}{y} = 0, \tag{4.1}$$

for every $s \in I$ and $x = p^{-1}$, $y = q^{-1}$.

Proof. Suppose that γ be a framed spherical curve lying on a sphere with radius r and center origin. Then we have the distance squared function $d^2 = r^2$. Consequently, we have $\alpha h = dd' = 0$. Therefore, since $\alpha h = 0$ for each $s \in I$, it is either $\alpha = 0$ or $h = 0$. If $\alpha(s) = 0$ for all $s \in I$, then γ is a point [1]. Since γ is a framed spherical curve, $h = 0$ for every $s \in I$. Hence, the differential equation 3.1 reduces to

$$\left(y(\alpha x)'\right)' + \frac{\alpha x}{y} = 0.$$

Thus, γ is a framed spherical curve. The converse is trivial. □

Remark 4.1. If we consider C^∞ category, then the statement "functions $f(s)g(s) = 0$ for all $s \in I$, then $f(s) = 0$ for all $s \in I$ or $g(s) = 0$ for all $s \in I$" is wrong. Then there exist counter-examples, such as flat functions. Consequently, f and g must be analytic functions to provide this feature. Consequently, in Corollary 4.1, α and h are accepted as analytic functions. Also similarly, in Corollary 4.3 and Theorem 4.1, functions will be accepted as analytic functions.

Example 4.1. Let $\gamma : [0, 4\pi) \to S^2 \subset \mathbb{R}^3$ be a spherical cardioid is defined by

$$\gamma(s) = \frac{1}{3}\left(2\cos s - \cos 2s, 2\sin s - \sin 2s, 2\sqrt{2}\cos\frac{s}{2}\right).$$

(See Fig.3.) Then

$$\nu(s) = \frac{1}{3}\left(-2\sqrt{2}\cos\frac{s}{2} + 8\sqrt{2}\sin^2\frac{s}{2}\cos\frac{s}{2}, 2\sqrt{2}\sin\frac{s}{2} - 8\sqrt{2}\sin\frac{s}{2}\cos^2\frac{s}{2}, 1\right),$$

gives the generalized tangent vector and $\alpha(s) = -\sqrt{2}\sin\frac{s}{2}$. By a calculation, we get

$$\overline{\eta_1}(s) = \left(3\sin\frac{s}{2} - 4\sin^3\frac{s}{2}, 3\cos\frac{s}{2} - 4\cos^3\frac{s}{2}, 0\right),$$

$$\overline{\eta_2}(s) = \left(-3\cos\frac{s}{2} + 4\cos^3\frac{s}{2}, 3\sin\frac{s}{2} - 4\sin^3\frac{s}{2}, \frac{2\sqrt{2}}{3}\right).$$

Moreover, we find

$$x(s) = \frac{1}{p(s)} = \frac{1}{\sqrt{2}}, \quad y(s) = \frac{1}{q(s)} = 2.$$

Consequently, the equation 4.1 is provided.

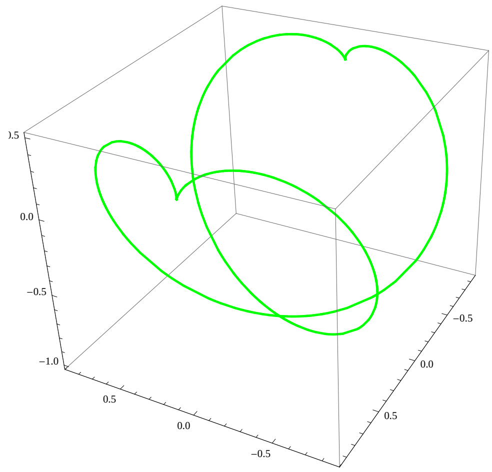

Figure 3. Spherical Cardioid

Definition 4.1. For the framed curve $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_\eta$, if the position vector of γ satisfies the equation $\gamma(s) = \delta(s)\nu(s) + \varepsilon(s)\overline{\eta_2}(s)$ for functions $\delta(s)$ and $\varepsilon(s)$, then γ is defined as a framed rectifying curve [2].

Corollary 4.2 (Framed rectifying curves). *A framed curve γ with $p, q \neq 0$ for every $s \in I$ is a framed rectifying curve if and only if it satisfies*

$$\left(\frac{p}{q}\right)' h + \left(\frac{p}{q}\right) h' = 0, \tag{4.2}$$

where $h = \langle \gamma, \nu \rangle$ for every $s \in I$.

Proof. Assume that γ is a framed rectifying curve. According to [2] the distance squared function is defined by

$$d^2(s) = \langle \gamma(s), \nu(s) \rangle^2 + c, \tag{4.3}$$

where c is constant and

$$\langle \gamma(s), \nu(s) \rangle' = \alpha(s). \tag{4.4}$$

Taking the derivative of the equation 4.3, we have $h(s) = \langle \gamma(s), \nu(s) \rangle$. According to equation 4.4, we get $h'(s) = \alpha(s)$. By using h, h', h'' and h''', differential equation 3.1 reduces to

$$\left(\frac{p}{q}\right)' h + \left(\frac{p}{q}\right) \alpha = 0.$$

Conversely, if 4.2 holds, then by integrating 4.2, we get $ph = c_1 q$ for a constant c_1. Then, since $h = \int \alpha(s)ds + c_2$ where constant c_2, we have $\dfrac{q}{p} = a_1 \int \alpha(s)ds + b_1$ where $a_1 = \dfrac{1}{c_1}$ and $b_1 = \dfrac{c_1}{c_2}$. Consequently, according to Theorem 5 in [2], γ is a framed rectifying curve. □

As a result of the differential equation 3.1, the characterization of framed rectifying curves and framed spherical curves are as follows:

Corollary 4.3. *Let $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve in $\mathbb{R}^3$ with framed curvatures $p, q \neq 0$ for every $s \in I$. Then*

$$\langle \gamma, \overline{\eta_1} \rangle^2 + \langle \gamma, \overline{\eta_2} \rangle^2 = c^2, \tag{4.5}$$

holds for a constant c if and only if either γ is a framed spherical curve or γ is a framed rectifying curve.

Proof. First assume that $\gamma(s)$ is a framed curve that satisfies condition 4.5. Therefore, the distance squared function of γ satisfies

$$d^2 = \langle \gamma, \nu \rangle^2 + c^2. \tag{4.6}$$

By differentiation in s of equation 4.6, we find

$$dd' = \langle \gamma, \nu \rangle \left(\alpha + p \langle \gamma, \overline{\eta_1} \rangle \right).$$

Therefore, by using the equations 3.2 and 3.3, we have

$$dd' = hh' = \alpha h.$$

Since α and h are considered as analytic functions, thus either $h = 0$ or $h' = \alpha$. Consequently, either γ is a framed spherical curve or it is framed rectifying curve. The converse is trivial. □

Theorem 4.1 (Framed helices). *A framed curve $(\gamma, \overline{\eta_1}, \overline{\eta_2}) : I \to \mathbb{R}^3 \times \Delta_2$ in $\mathbb{R}^3$ with framed curvatures $p, q \neq 0$ is a framed helix if and only if*

$$(x(h' - \alpha))' + \left(\frac{y}{x} + \frac{x}{y}\right) qh = \frac{x}{y^2}\left(\int \alpha(s)ds + b\right), \tag{4.7}$$

where $\alpha h = dd'$, $x = p^{-1}$ and $y = q^{-1}$.

Proof. Suppose that γ is a framed helix. Therefore, we have $\langle \nu, u\rangle = c$ for constant c and a fixed unit vector u [2]. Consequently, we have $\langle \overline{\eta_1}, u\rangle = 0$. Consequently, we have

$$u = c\nu + \sqrt{1 - c^2}\,\overline{\eta_2}. \tag{4.8}$$

By differentiating 4.8, we get

$$cp = q\sqrt{1 - c^2}. \tag{4.9}$$

Since $\langle \gamma, u\rangle' = \alpha c$, we get

$$\langle \gamma, u\rangle = c\int \alpha(s)ds + \overline{c}, \tag{4.10}$$

for some constant $\overline{c}$. By using 4.8 and 4.10, we get

$$\langle \gamma, \nu\rangle = \int \alpha(s)ds + \frac{\overline{c}}{c} - \frac{\sqrt{1 - c^2}}{c}\langle \gamma, \overline{\eta_2}\rangle.$$

By according to equation 3.2, we have

$$h = \int \alpha(s)ds + b - \frac{\sqrt{1 - c^2}}{c}\langle \gamma, \overline{\eta_2}\rangle, \quad b = \frac{\overline{c}}{c}. \tag{4.11}$$

By differentiating 4.11 and using equation 4.9, we get

$$x(h' - \alpha) = \langle \gamma, \overline{\eta_1}\rangle,$$

which again by differentiation leads to

$$x(h'' - \alpha') + x'(h' - \alpha) = q\langle \gamma, \overline{\eta_2}\rangle - ph.$$

By applying 4.9 and 4.11 we get

$$x(h'' - \alpha') + x'(h' - \alpha) = \frac{c\left(\int \alpha(s)ds + b\right) q}{\sqrt{1 - c^2}} - \left(\frac{\sqrt{1 - c^2}}{c} + \frac{c}{\sqrt{1 - c^2}}\right) qh.$$

Notice that from the equation 4.9, we can take

$$\frac{y}{x} = \frac{\sqrt{1 - c^2}}{c}, \quad \frac{x}{y} = \frac{c}{\sqrt{1 - c^2}}. \tag{4.12}$$

By according to equations 4.12, we have

$$(x(h' - \alpha))' + \left(\frac{y}{x} + \frac{x}{y}\right) qh = \frac{x}{y^2}\left(\int \alpha(s)ds + b\right).$$

Conversely, assume that γ is a framed curve that satisfies 4.7. By differentiating 4.7, we get

$$(yx)h^{'''} + \left(y^{'}x + 2yx^{'}\right)h^{''} + \left((yx^{'})^{'} + \frac{y}{x} + \frac{x}{y}\right)h^{'} + \left(\frac{y}{x} + \frac{x}{y}\right)^{'} h \tag{4.13}$$
$$= (yx)\alpha^{''} + (y^{'}x + 2yx^{'})\alpha^{'} + \left((yx^{'})^{'} + \frac{x}{y}\right)\alpha + \left(\frac{x}{y}\right)^{'}\left(\int \alpha(s)ds + b\right).$$

Comparing 4.13 with equation 3.1 in Proposition 3.1 gives

$$\left(\frac{x}{y}\right)^{'}\left(h - \int \alpha(s)ds + b\right) = 0. \tag{4.14}$$

If $h = \int \alpha(s)ds + b$, then 4.7 reduces to

$$\left(\frac{y}{x} + \frac{x}{y}\right)\left(\int \alpha(s)ds + b\right) - \left(\frac{x}{y}\right)\left(\int \alpha(s)ds + b\right) = 0.$$

Therefore, we have $\frac{y}{x}\left(\int \alpha(s)ds + b\right) = 0$. Since $y \neq 0$, we get $\left(\int \alpha(s)ds + b\right) = 0$. Hence we get $\left(\frac{x}{y}\right)^{'} = 0$ from 4.14, which implies that γ is a framed helix [2].

□

Example 4.2. (The astroid ([6]). Let $\gamma : [0, 2\pi) \to \mathbb{R}^3$ be a astroid is defined by

$$\gamma(s) = \left(\cos^3 s, \sin^3 s, \cos 2s\right).$$

(See Fig.4.) Then

$$d^2(s) = \cos^6 s + \sin^6 s + \cos^2 s,$$

gives the distance squared function. By a calculation, we get

$$h = \left(-\frac{3}{5}\cos^4 s + \frac{3}{5}\sin^4 s - \frac{4}{5}\cos 2s\right),$$

and

$$\alpha = 5\cos s \sin s.$$

Since $x = -\frac{5}{3}$ and $y = -\frac{5}{4}$, we get

$$(x(h' - \alpha))' + \left(\frac{y}{x} + \frac{x}{y}\right)qh = \frac{4}{3} - \frac{8}{3}\sin^2 s. \tag{4.15}$$

On the other hand, we have

$$\frac{x}{y^2}\left(\int \alpha(s)ds + b\right) = -\frac{8}{3}\sin^2 s + b. \tag{4.16}$$

According to equations 4.15 and 4.16, the astroid is a framed helix where $b = \frac{4}{3}$ and it satisfies equation 4.7.

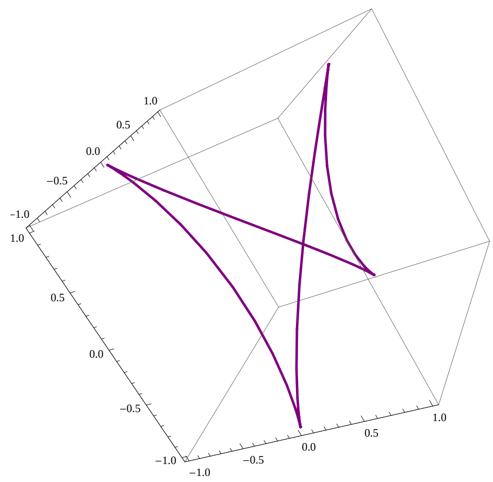

Figure 4. The astroid

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References

[1] Honda, S., Takahashi, M.: *Framed curves in the Euclidean space*. Advances in Geometry. 16(3), 265-276 (2016).
[2] Wang, Y., Pei, D., Gao, R.: *Generic properties of framed rectifying curves*. Mathematics. 7(1), 37 (2019).
[3] Doğan Yazıcı, B., Özkaldı Karakuş, S., Tosun, M.: *Framed normal curves in Euclidean space*. In: Tbilisi-Mathematics. Sciendo, (2020).
[4] Honda, S., Takahashi, M.: *Evolutes and focal surfaces of framed immersions in the Euclidean space*. Proceedings of the Royal Society of Edinburgh Sect. A. 150(1), 497-516 (2019).
[5] Fukunaga, T., Takahashi, M.: *Existence conditions of framed curves for smooth curves*. Journal of Geometry. 108, 763-774 (2017).
[6] Honda, S.: *Rectifying developable surfaces of framed base curves and framed helices*. Advanced Studies in Pure Mathematics. 78, 273-292 (2018).
[7] Deshmukh, S., Chen B.Y., Turki, N.B.: *A differential equations for Frenet curves in Euclidean 3-space and its applications*. Romanian Journal of Mathematics and Computer Science. 8(1), 1-6 (2018).
[8] Chen, B.Y.: *When does the position vector of a space curve always lie in its rectifying plane?*. The American Mathematical Monthly. 110, 147-152 (2003).
[9] Chen, B.Y., Dillen, F.: *Rectifying curves as centrodes and extremal curves*. Bulletin of the Institute of Mathematics Academia Sinica. 33, 77-90 (2005).
[10] Deshmukh, S., Chen B.Y., Alshammari S.H.: *On rectifying curves in Euclidean 3-space*. Turkish Journal of Mathematics. 42(2), 609-620 (2018).

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