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On the Classes of (n, m) Power (D, A)-Normal and (n,m) Power (D, A)-Quasinormal Operators in Semi-Hilbertian Space

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Keywords

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Abstract

The concept of (n, m) power D-normal operators on Hilbertian space is defined by Ould Ahmed Mahmoud Sid Ahmed and Ould Beinane Sid Ahmed in [1]. In this paper we introduce a new classes of operators on semi-Hilbertian space $(\mathcal{H}, \|.\|_A)$ called (n, m) power-(D, A)-normal denoted $[(n, m)DN]_A$ and (n, m) power-(D, A)-quasi-normal denoted $[(n, m)DQN]_A$ associated with a Drazin invertible operator using its Drazin inverse. Some properties of $[(n, m)DN]_A$ and $[(n, m)DQN]_A$ are investigated and some examples are also given. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (n, m) power-(D, A)-normal for some positive operator A and for some positive integers n and m if $(T^D)^n(T^\sharp)^m = (T^\sharp)^m(T^D)^n$.

1. Introduction

One of the most important subclasses of the algebra of all bounded linear operators acting on Hilbert space, the class of normal operators $(TT^* = T^*T)$ They have been the object of some intensive studies. The theory of these operators was investigated in [1] and [2] This class has been generalized, in some sense, to the larger sets of so-called quasinormal, Hyponormal, isometry, partial isometry, m-isometries operators on Hilbert spaces.

Recently, these classes of operators have been generalized by many authors when an Additional semi-inner product is considered, see([4], [5], [6], [7], [8], [9], [10]) and other papers. A bounded linear operator T on a complex Hilbert space is(n, m)power-D-normal if $(T^D)^n T^{*m} = T^{*m}(T^D)^n$.In the year 2019 the authors Ould Ahmed Mahmoud Sid Ahmed and Ould Beinane Sid Ahmed introduced the class of (n, m)power-D-normal operators and studied some proprietes of this class . For more details see [3].The purpose of this paper is to study the class of (n, m)power-(D, A)-normal and (n, m)power-(D, A)-quasi-normal in semi-Hilbertian spaces. The contents of the paper are the following. In section 1 we give notation and results about the concept of A-adjoint operators that will be useful in the sequel and the Drazin inverse of the operator. In Section 2 we introduce a new concept of normality of operators in semi-Hilbertian space $(\mathcal{H}, \langle .|.\rangle_A)$called (n, m) power-(D, A)-normal operator and we investigate various structural properties of this class of operators with some examples studied. Moreover, the product, direct sum, tensor product and the sum of finite numbers of this type are discussed. Also we study the relationship between this class and the other kinds of classes of operators in semi-Hilbertian spaces. In section three we define other new class called (n, m)power-(D, A)-quasi-normal and study some properties of this class.

We start by introducing some notations. Throughout this paper $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle .|.\rangle_A$, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on$\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+$is the cone of positive operators of $\mathcal{B}(\mathcal{H})$ defined as $\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) : \langle Tx|x\rangle \geq 0, \forall x \in \mathcal{H}\}$.

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For every $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$and $\overline{\mathcal{R}(T)}$stand for respectively the null space, range and closure of the range of T, its adjoint operator by T^*. The closed linear subspace $\mathcal{M}$ is called invariant subspace of T, if satisfying $T\mathcal{M} \subset \mathcal{M}$.In addition if $\mathcal{M}$ also is invariant subspace of T^*, then $\mathcal{M}$ is called a reducing subspace of T. We denote the orthogonal projection onto a closed linear subspace $\mathcal{M}$ by $P_{\mathcal{M}}$. Note that for $A \in \mathcal{B}(\mathcal{H})^+$, the functional

$\langle .|.\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $\langle u|v\rangle_A = \langle Au|v\rangle$ is a semi-inner product on $\mathcal{H}$. By $\|.\|_A$ we denote the semi-norm induced by $\langle .|.\rangle_A$ i.e $\|u\|_A = \langle u|u\rangle_A^{\frac{1}{2}} = \langle Au|u\rangle^{\frac{1}{2}}$. Observe that $\|u\|_A = 0$ if and only if $u \in \mathcal{N}(A)$, then $\|.\|_A$ is a norm if and only if A is an injective operator and the semi-normed space $(\mathcal{H}, \|.\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. The above semi-norm induces a semi-norm on the subspace $\mathcal{B}^A(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$.

$$\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) / \exists c > 0, \|Tu\|_A \leq c\|u\|_A, \forall x \in \mathcal{H}\}$$

Indeed, if $T \in \mathcal{B}^A(\mathcal{H})$, then $\| T \|_A = sup\left\{\frac{\|Tu\|_A}{\|u\|_A}, u \in \overline{\mathcal{R}(A)} \wedge u \neq 0\right\}$

Operator in $\mathcal{B}^A(\mathcal{H})$, is called A-bounded operator.

From now A denoted a positive operator on $\mathcal{H}$,that is $A \in \mathcal{B}(\mathcal{H})^+$.

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$, is called an A-adjoint operator of T if for every u , $v \in \mathcal{H}$, we have $\langle Tu|v\rangle$ = $\langle u|Sv\rangle$ that is $AS = T^*A$,if T is an A-adjoint of itself, then T is called an A-selfadjoint operator $AT = T^*A$,. Generally, the existence of an A-adjoint operator is not guaranteed .The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem [11].We have that

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \,/\, \mathcal{R}(T^*A) \subset \mathcal{R}(A)\}$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A-adjoint operator of T, namely the reduced solution of equation $AX = T^*A$, This operator is denoted by $T^\sharp$.Therefore ,

$T^\sharp = A^\dagger T^* A$ and $AT^\sharp = T^*A$, $\mathcal{R}(T^\sharp) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{N}(T^\sharp) = \mathcal{N}(T^*A)$

Note that in which $A^\dagger$ is the Moore-Penrose inverse of A. For more details see ([4],[5],[6]). In the next proposition we collect some properties of $T^\sharp$. and its relationship with the semi- norm $\|T\|_A$. For the proof see ([4],[5]) .

2. Main Results

Proposition 1.1 Let$T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold

1. $T^\sharp \in \mathcal{B}_A(\mathcal{H})$,$(T^\sharp)^\sharp = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $((T^\sharp)^\sharp)^\sharp = T^\sharp$
2. If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^\sharp = S^\sharp T^\sharp$
3. $TT^\sharp$ and $T^\sharp T$ are A-selfadjoint
4. $\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{\frac{1}{2}} = \|TT^\sharp\|_A^{\frac{1}{2}}$
5. $\|S\|_A = \|T^\sharp\|_A$ for every $S \in \mathcal{B}_A(\mathcal{H})$ which is an A-adjoint of T
6. If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$

In the following definition we collect the notions of some classes of operators.

Definition 1.1 Any operators $T \in \mathcal{B}_A(\mathcal{H})$. is called

1. A-normal if $TT^\sharp = T^\sharp T$

2. A-isometry if $T^\sharp T = P_{\overline{\mathcal{R}(A)}}$

3. A-unitary if $T^\sharp T = TT^\sharp = P_{\overline{\mathcal{R}(A)}}$

4.(n,m)power-A-normal if $T^n T^{*m} = T^{*m} T^n$

5. A-quasinormal if $TT^\sharp T = T^\sharp T^2$

The Drazin inverse in the setting of bounded linear operators on complex Banach spaces was investigated by Caradus[12] and King [13].The Drazin inverse has become a useful tool in a number of areas such that differential and difference equations, Markov chains, optimal control and iterative method ([14]). Recall([15]) that $T \in \mathcal{B}(\mathcal{H})$is Drazin Invertible if there exists an operator $T^D \in \mathcal{B}(\mathcal{H})$, such that
$[T^D, T] = T^D T - T T^D = 0$, $(T^D)^2 T = T^D$and $T^{p+1} T^D = T^p$ for some integer $p \geq 1$.
The operator T^D is then the Drazin inverse of T and p is the Drazin index of Tdenoted by ind (T). For $T \in \mathcal{B}_A(\mathcal{H})$ it was observed that the Drazin inverse $T^D$of T satisfies$\left(T^\sharp\right)^D = (T^D)^\sharp$ and $\left(T^k\right)^D = (T^D)^k$ for positive integer k. We denote by $\mathcal{B}(\mathcal{H})^D$the set of all Drazin invertible elements of $\mathcal{B}(\mathcal{H})$ and by $\mathcal{B}_A(\mathcal{H})^D$the set of all Drazin invertible elements of $\mathcal{B}_A(\mathcal{H})$. Very recently, the authors M. Dana and R. Yousfi in ([16]), has introduced the following classes of operators. Let $T \in \mathcal{B}(\mathcal{H})^D$, T is said to be

1. D-normal if $T^D T^* = T^* T^D$
2. D-quasinormal if $T^D T^* T = T^* T T^D$
3. n power D-normal if $(T^D)^n T^* = T^* (T^D)^n$

Lemma 1.1 ([12],[17]).Let $T, S \in \mathcal{B}(\mathcal{H})^D$. Then the following properties hold.
(1) TS is Drazin invertible if and only if $ST$is Drazin invertible. Moreover
$(TS)^D = T[(TS)^D]^2 S$ and $ind(TS) \leq ind(ST) + 1$
(2) If T is idempotent, then $T^D = T^\sharp = T$.
(3) If $TS = ST$,then $(TS)^D = S^D T^D = T^D S^D$,$T^D S = S T^D$and $T S^D = S^D T$.
(4) If $TS = ST = 0$, then $(T+S)^D = T^D + S^D$

2.1. (n, m) power-(D, A)-normal operators

Definition 2.1 Let $T \in \mathcal{B}_A(\mathcal{H})$ be Draizin invertible operator.We said that $T$is (n, m) power-(D, A)-normal operator for some positive integers n, m if

$$(T^D)^n \left(T^\sharp\right)^m = \left(T^\sharp\right)^m (T^D)^n$$

We denote the set of all (n, m) power-(D, A)-normal operators by $[(n, m)DN]_A$
Remark 2.1 We observed this results
1. if $n = m = 1$ then (1,1) power-(D, A)-normal is (D, A)-normal i.e $[(1,1)DN]_A = [DN]_A$
2. if $n = 1$ then $[(n, 1)DN]_A = [nDN]_A$
3. $T \in [(n, m)DN]_A \Leftrightarrow \left[(T^D)^n, \left(T^\sharp\right)^m\right]_A = 0$

Remark 2.2 Obviously that the following inclusions hold

1. $[(n, m)DN]_A \subset [(2n, m)DN]_A$
2. $[(n, m)DN]_A \subset [(n, 2m)DN]_A$
3. $[(n, m)DN]_A \subset [(2n, 2m)DN]_A$

Example 2.1 Let

$$T = \begin{pmatrix} -3 & 3 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Be operators acting on two dimensional Hilbert space space $\mathbb{C}^2$. A simple calculation shows that

$$A \geq 0 \ , \qquad T^D = \begin{pmatrix} \frac{-1}{3} & \frac{2}{9} \\ 0 & \frac{1}{3} \end{pmatrix} \quad and \quad T^\sharp = \begin{pmatrix} -3 & 0 \\ 6 & 3 \end{pmatrix}$$

It is easy to sheck that $(T^D)^3 \left(T^\sharp\right)^2 = \left(T^\sharp\right)^2 (T^D)^3$ then T is of class $[(3,2)DN]_A$

Example 2.2 Let

$$S=\begin{pmatrix}-1 & 0 & 1\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}\ \ and\ \ A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

A simple computation shows that

$$S^D=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\ \ and\ \ S^{\#}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ \frac{1}{2} & \frac{-1}{2} & 0\end{pmatrix}$$

It is easily to see that $T$is (n,m) power-(D,A)-normal operator for all positive integers $n\ and\ m$

The following examples shows that there exist a(n,m) power-(D,A)-normal operator but is not (n,m) power-A-normal operator

Example 2.3 Let

$$Z=\begin{pmatrix}3 & 6\\ 2 & 4\end{pmatrix}\quad \text{and}\quad A=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}.$$

A direct calculation shows that $(Z^D)^3(Z^{\#})^2=(Z^{\#})^2(Z^D)^3\ and\ Z^3(Z^{\#})^2\neq(Z^{\#})^2Z^3$, then Z is of class $[(3,2)DN]_A$ but is not of class$[(3,2)N]_A$.

It is well known that if $T\in\mathcal{B}_A(\mathcal{H})$ is A-normal, then T^n is A-normal.In the following theorem we extend this result to (n,m)power-(D,A)-normal operator it as follows

Theorem 2.1 Let $T\in\mathcal{B}_A(\mathcal{H})^D$,if T is (n,m)power-(D,A)-normal operator, then the following statements hold

1. T^k is(D,A)-normal operator where k is the least common multiple of n and m .i.e. $k=LCM(n,m)$
2. T^{nm} is (D,A)-normal operator

Proof **1.** Assume that T is a (n,m) power-(D,A)-normal operator that is

$$(T^D)^n(T^{\#})^m=(T^{\#})^m(T^D)^n$$

Let $k=pn$ and $k=qm$. By computation we get

$$\begin{aligned}(T^k)^D(T^k)^{\#}&=(T^D)^k(T^{\#})^k\\ &=(T^D)^{pn}(T^{\#})^{qm}\\ &=((T^D)^n)^p\left((T^{\#})^m\right)^q\\ &=\underbrace{(T^D)^n.(T^D)^n\cdots(T^D)^n}_{p\ times}\underbrace{(T^{\#})^m.(T^{\#})^m\cdots(T^{\#})^m}_{q\ times}\\ &=\underbrace{(T^{\#})^m.(T^{\#})^m\cdots(T^{\#})^m}_{q\ times}\underbrace{(T^D)^n.(T^D)^n\cdots(T^D)^n}_{p\ times}\\ &=(T^{\#})^{qm}(T^D)^{pn}\\ &=(T^{\#})^k(T^k)^D\\ &=(T^k)^{\#}(T^k)^D\end{aligned}$$

Then $T\in[DN]_A$

2. This statement is proved in the same way as in the statement (**1**)

Example 2.4 Let us consider the operators Z and A given in previous example $(\mathbf{2},\mathbf{3})$ we know that Z is (3,2) power-(D,A)-normal, then Z^6 is (D,A)-normal

The following proposition collects some of basic properties of (n,m)power-(D,A)-normal operators.

Proposition 2.2 Let $T\in\mathcal{B}_A(\mathcal{H})^D$.The following properties hold

1. If T is an (n,m)power-(D,A)-normal operator, then $T^D$is an (n,m)-A-normal

2. If T is an (n,n)power-(D,A)-normal operator, then $(T^D)^n$ is an A-normal operator
3. If T is an (n,m)power-(D,A)-normal operator, then $T^\sharp$ is an (n,m)-(D,A)-normal
4. If α is a real scalar and T is an (n,m)power-(D,A)-normal operator, then $(\alpha T)\in[(n,m)DN]_A$

Proof **1.** Assume that $T\in[(n,m)DN]_A$ that is $(T^D)^n\left(T^\sharp\right)^m=\left(T^\sharp\right)^m(T^D)^n$ then by lemma (1,1) it follows that $(T^D)^n\left(\left(T^\sharp\right)^m\right)^D=\left(\left(T^\sharp\right)^m\right)^D(T^D)^n$ then

$$(T^D)^n\left((T^D)^\sharp\right)^m=\left((T^D)^\sharp\right)^m(T^D)^n$$

Hence $T^D\in[(n,m)N]_A$

2. Assume that T is a (n,n) power-(D,A)-normal operator, then by statement (1) we have T^D is (n,n)-A-normal that is

$$(T^D)^n\left((T^D)^\sharp\right)^n=\left((T^D)^\sharp\right)^n(T^D)^n$$

Hence

$$(T^D)^n((T^D)^n)^\sharp=((T^D)^n)^\sharp(T^D)^n$$

Therefore $(T^D)^n$ is A-normal operator

3. Let $T\in[(n,m)DN]_A$ then

$$(T^D)^n\left(T^\sharp\right)^m=\left(T^\sharp\right)^m(T^D)^n\Rightarrow\left((T^D)^\sharp\right)^n\left(\left(T^\sharp\right)^\sharp\right)^m=\left(\left(T^\sharp\right)^\sharp\right)^m\left((T^D)^\sharp\right)^n$$
$$\Rightarrow\left(\left(T^\sharp\right)^D\right)^n\left(\left(T^\sharp\right)^\sharp\right)^m=\left(\left(T^\sharp\right)^\sharp\right)^m\left(\left(T^\sharp\right)^D\right)^n$$

Hence $T^\sharp\in[(n,m)DN]_A$

4. This statement obviously

The following discusses the conditions for product and sum of two (n,m)power-(D,A)-normal operators to be (n,m) power-(D,A)-normal.

Theorem 2.3 Let $T,S\in\mathcal{B}_A(\mathcal{H})^D$ are (n,m)power-(D,A)-normal operators such that $TS=ST$, $T^\sharp S^D=S^DT^\sharp$ and $S^\sharp T^D=T^DS^\sharp$,then the following statements hold.

1. TS is (n,m)power-(D,A)-normal

2. If $TS=0$ then $T+S$ is (n,m)power-(D,A)-normal

Proof. Assume that $T,S\in[(n,m)DN]_A$ and $T^\sharp S^D=S^DT^\sharp$ and $S^\sharp T^D=T^DS^\sharp$, it follows that $T^{\sharp m}(S^D)^n=(S^D)^nT^{\sharp m}$ and $S^{\sharp m}(T^D)^n=(T^D)^nS^{\sharp m}$ .So

1.
$$\begin{aligned}((TS)^D)^n.\left((TS)^\sharp\right)^m&=(T^DS^D)^n.\left(T^\sharp S^\sharp\right)^m\\&=(T^D)^n(S^D)^n.T^{\sharp m}S^{\sharp m}\\&=(T^D)^nT^{\sharp m}(S^D)^nS^{\sharp m}\\&=T^{\sharp m}(T^D)^nS^{\sharp m}(S^D)^n\\&=T^{\sharp m}S^{\sharp m}.(T^D)^n(S^D)^n\\&=\left((TS)^\sharp\right)^m.((TS)^D)^n\end{aligned}$$

Hence TS is (n,m)power-(D,A)-normal

2. Under the assumptions and from lemma (1.1) we get

$$\begin{aligned}((T+S)^D)^n\left((T+S)^\sharp\right)^m&=(T^D+S^D)^n\left(T^\sharp+S^\sharp\right)^m\\&=((T^D)^n+(S^D)^n)\left(T^{\sharp m}+S^{\sharp m}\right)\\&=(T^D)^nT^{\sharp m}+(T^D)^nS^{\sharp m}+(S^D)^nT^{\sharp m}+(S^D)^nS^{\sharp m}\\&=T^{\sharp m}(T^D)^n+S^{\sharp m}(T^D)^n+T^{\sharp m}(S^D)^n+S^{\sharp m}(S^D)^n\\&=\left(T^{\sharp m}+S^{\sharp m}\right)((T^D)^n+(S^D)^n)\\&=\left((T+S)^\sharp\right)^m((T+S)^D)^n\end{aligned}$$

Hence $(T+S)$ is an (n,m)power-(D,A)-normal

Proposition 2.4 Let $T, S \in \mathcal{B}_A(\mathcal{H})^D$ are commuting (D,A)-normal then TS is an (D,A)-normal operator if $T^D S^D S^\sharp = S^\sharp T^D S^D$ and $T^D S^D T^\sharp = T^\sharp T^D S^D$

Proof. Assume that $T,S \in [DN]_A$, $T^D S^D S^\sharp = S^\sharp T^D S^D$ and $T^D S^D T^\sharp = T^\sharp T^D S^D$ then

$$\begin{aligned}(TS)^D(TS)^\sharp &= T^D S^D S^\sharp T^\sharp \\ &= S^\sharp T^D S^D T^\sharp \\ &= S^\sharp T^\sharp T^D S^D \\ &= (TS)^\sharp (TS)^D\end{aligned}$$

Hence TS is an (D,A)-normal operator

Proposition 2.5 Let $T, S \in \mathcal{B}_A(\mathcal{H})^D$ are commuting $(n,1)$power-(D,A)-normal such that $T^D S^\sharp = S^\sharp T^D$, $T^\sharp S^D = S^D T^\sharp$ and $(T+S)^\sharp$ commutes with $\sum_{1\le p\le n-1}\binom{n}{p}(T^D)^p(S^D)^{n-p}$ then $(T+S)$ is an $(n,1)$ power-(D,A)-normal

Proof. Assume the conditions hold, then

$$\begin{aligned}(T+S)^{D^n}(T+S)^\sharp &= (T^D+S^D)^n (T+S)^\sharp \\ &= \left(\sum_{0\le p\le n}\binom{n}{p}(T^D)^p(S^D)^{n-p}\right)(T+S)^\sharp \\ &= \left(S^{D^n}+T^{D^n}+\sum_{1\le p\le n-1}\binom{n}{p}(T^D)^p(S^D)^{n-p}\right)(T+S)^\sharp \\ &= (S^{D^n}+T^{D^n})(T+S)^\sharp + \sum_{1\le p\le n-1}\binom{n}{p}(T^D)^p(S^D)^{n-p}.(T+S)^\sharp \\ &= S^{D^n}T^\sharp + S^{D^n}S^\sharp + T^{D^n}T^\sharp + T^{D^n}S^\sharp + (T+S)^\sharp \sum_{1\le p\le n-1}\binom{n}{p}(T^D)^p(S^D)^{n-p} \\ &= T^\sharp S^{D^n} + S^\sharp S^{D^n} + T^\sharp T^{D^n} + S^\sharp T^{D^n} + (T+S)^\sharp \sum_{1\le p\le n-1}\binom{n}{p}(T^D)^p(S^D)^{n-p} \\ &= (T+S)^\sharp (S^{D^n}+T^{D^n}) + (T+S)^\sharp \sum_{1\le p\le n-1}\binom{n}{p}(T^D)^p(S^D)^{n-p} \\ &= (T+S)^\sharp\left(\sum_{0\le p\le n}\binom{n}{p}(T^D)^p(S^D)^{n-p}\right) \\ &= (T+S)^\sharp (T+S)^{D^n}\end{aligned}$$

Therefore $(T+S)$ is an $(n,1)$ power-(D,A)-normal operator

Proposition 2.6 Let $T \in \mathcal{B}_A(\mathcal{H})^D$ be an (n,m)power-(D,A)-normal operator such that $\mathcal{N}(A)$is a reducing subspace of T, if $S = UTU^\sharp$where$U$is A-unitary operator,then S is (n,m) power-$(\mathrm{D,A})$-normal operator

Proof . Let T be an(n,m)power-(D,A)-normal operator and $S = UTU^\sharp$, easily we obtain $S^n = UT^nU^\sharp$, $S^D = UT^DU^\sharp$ and $S^\sharp = UT^\sharp U^\sharp$. Hence,

$$\begin{aligned}(S^D)^n(S^\sharp)^m &= (UT^DU^\sharp)^n(UT^\sharp U^\sharp)^m \\ &= U(T^D)^nU^\sharp.UT^{\sharp m}U^\sharp \\ &= U(T^D)^nP_{\overline{\mathcal{R}(A)}}T^{\sharp m}U^\sharp \\ &= U(T^D)^nT^{\sharp m}U^\sharp \\ &= UT^{\sharp m}(T^D)^nU^\sharp \\ &= (UT^{\sharp m}U^\sharp)(U(T^D)^nU^\sharp) \\ &= (S^\sharp)^m(S^D)^n\end{aligned}$$

Then $S \in [(n,m)DN]_A$

Proposition 2.7 Let $T \in \mathcal{B}_A(\mathcal{H})^D$, $X = (T^D)^n + T^{\sharp m}$, $Y = (T^D)^n - T^{\sharp m}$ and $Z = (T^D)^n.T^{\sharp m}$.Then the followig statements hold

1. T is an (n,m) power- $(\mathrm{D,A})$-normal operator if and only if $XY = YX$

2. If T is an (n,m) power- $(\mathrm{D,A})$-normal operator, then Z commutes with X and Y

3. T is of class $[(n,m)DN]_A$if and only if $(T^D)^n$ commutes with X

4. T is of class $[(n,m)DN]_A$if and only if $(T^D)^n$ commutes with Y

Proof .

1. $XY = YX \Leftrightarrow \left((T^D)^n + T^{\#m}\right)\left((T^D)^n - T^{\#m}\right) = \left((T^D)^n - T^{\#m}\right)\left((T^D)^n + T^{\#m}\right)$

$\Leftrightarrow (T^D)^{2n} - (T^D)^n T^{\#m} + T^{\#m}(T^D)^n - T^{\#2m} = (T^D)^{2n} + (T^D)^n T^{\#m} - T^{\#m}(T^D)^n - T^{\#2m}$

$\Leftrightarrow 2.T^{\#m}(T^D)^n = 2.(T^D)^n T^{\#m}$

$\Leftrightarrow T \in [(n,m)DN]_A$

The proof of statements $(2), (3)$ and (4) are straightforward.

In the following proposition, we study the relation between the two classes $[(2,m)DN]_A$ and $[(3,m)DN]_A$

Proposition 2.8 Let $T \in \mathcal{B}_A(\mathcal{H})$ be Draizin invertible operator such that T is of class $[(2,m)DN]_A$ and of class $[(3,m)DN]_A$ for some positive integre m, then T is of class $[(n,m)DN]_A$ for all positive integre $n \geq 4$.

Proof . We prove the assertion by using the mathematical induction. For $n = 4$, it a consequence of item (1) of remark (2.2).For $n = 5$ we prove it .Since $T \in [(2,m)DN]_A$ then $(T^D)^2\left(T^\#\right)^m = \left(T^\#\right)^m (T^D)^2$ (2.1)

Multiplying (2.1) to the left by $(T^D)^3$ we get $(T^D)^5\left(T^\#\right)^m = (T^D)^3\left(T^\#\right)^m(T^D)^2$.

Thus implies $(T^D)^5\left(T^\#\right)^m = \left(T^\#\right)^m(T^D)^5$. Now assume that the results is true for $n \geq 5$ that is $(T^D)^n\left(T^\#\right)^m = \left(T^\#\right)^m(T^D)^n$

Then

$$\begin{aligned}(T^D)^{n+1}\left(T^\#\right)^m &= T^D(T^D)^n\left(T^\#\right)^m\\ &= T^D\left(T^\#\right)^m(T^D)^n\\ &= T^D\left(T^\#\right)^m(T^D)^2(T^D)^{n-2}\\ &= (T^D)^3\left(T^\#\right)^m(T^D)^{n-2}\\ &= \left(T^\#\right)^m(T^D)^{n+1}\end{aligned}$$

This means that $T \in [(n+1,m)DN]_A$. The proof is complete.

Example 2.5 Let us consider the operators T and A given in Example $(\mathbf{2.1})$**.** A direct calculation shows that $T \in ([(2,2)DN]_A \cap [(3,2)DN]_A)$.

Therefore $T \in [(n,2)DN]_A$ for all $n \geq 4$.

The following Examples shows that the classes $[(n,m)DN]_A$ and $[(n+1,m)DN]_A$ are not the same

Example 2.6 Let us consider the matrix operators $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ acting on $\mathbb{C}^2$ .A simple calculation shows that $T^D = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $T^\# = \begin{pmatrix} 0 & 2 \\ \frac{-1}{2} & 1 \end{pmatrix}$

It is easily to check that $T \in [(3,2)DN]_A$ but $T \notin [(2,2)DN]_A$

Example 2.7 Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Then $S^D = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $S^\# = \begin{pmatrix} 0 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}$

A direct calculation shows that T is of class $[(2,3)DN]_A$ but $T \notin [(3,3)DN]_A$.

Proposition 2.9 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. if T is of class $[(n,m)DN]_A$ and of class $[(n+1,m)DN]_A$, then T is of class $[(n+2,m)DN]_A$ for some positive integers n and m. In particular T is of class $[(k,m)DN]_A$ for all $k \geq n$.

Proof . Since T is of class $[(n,m)DN]_A$ and of class $[(n+1,m)DN]_A$, it follows that $(T^D)^n T^{\#m} = T^{\#m}(T^D)^n$ and $(T^D)^{n+1}T^{\#m} = T^{\#m}(T^D)^{n+1}$ .So

$(T^D)^{n+2}T^{\#m} = (T^D)(T^D)^{n+1}T^{\#m}$

$= (T^D)T^{\sharp m}(T^D)^{n+1}$
$= (T^D)T^{\sharp m}(T^D)^n(T^D)$
$= (T^D)^{n+1}T^{\sharp m}(T^D)$
$= T^{\sharp m}(T^D)^{n+2}$

Hence T is of class $[(n+2,m)DN]_A$. By reapiting this process we can prove that T is of class $[(k,m)DN]_A$ for all $k \geq n$.

Proposition 2.10 Let $T \in \mathcal{B}_A(\mathcal{H})$ be Draizin invertible operator. If T is both of class $[(n,m)DN]_A$ and of class $[(n+1,m)DN]_A$ such that T^D is injective, then T is of class $[(1,m)DN]_A$.

Proof . Since T is of class $[(n,m)DN]_A$ and of class $[(n+1,m)DN]_A$, it follow that

$$(T^D)^n\big((T^D)T^{\sharp m} - T^{\sharp m}(T^D)\big) = 0$$

Since T^D is injective , then so is $(T^D)^n$ and we have $(T^D)T^{\sharp m} - T^{\sharp m}(T^D) = 0$, hence T is of class $[(1,m)DN]_A$.

In the following proposition, we study the relation between the two classes $[(n,2)DN]_A$ and $[(n,3)DN]_A$

Proposition 2.11 Let $T \in \mathcal{B}_A(\mathcal{H})$ be Draizin invertible operator such that T is of class $[(n,2)DN]_A$ and of class $[(,n,3)DN]_A$ for some positive integre n, then T is of class $[(n,m)DN]_A$ for all positive integre $m \geq 4$.

Proof . We omit the proof since the techniques are similar to those of the proof of Proposition $(\mathbf{2},\mathbf{8})$

Example 2.8 Let $T = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

The drazin invertible of T is $T^D = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{-1}{2} \end{pmatrix}$ and $T^\sharp = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix}$

By calculations we have that T is of class $[(2,2)DN]_A$ and $T \in [(2,3)DN]_A$. Therefore
$T \in [(2,m)DN]_A$ for all $m \geq 4$

The proof of the following proposition is very similar to the proof of proposition $(\mathbf{2},\mathbf{9})$**,** thus we omitted.

Proposition 2.12 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. if T is of class $[(n,m)DN]_A$ and of class$[(n,m+1)DN]_A$,then T is of class $[(n,m+2)DN]_A$ for some positive integers n and m. In particular T is of class $[(n,k)DN]_A$ for all $k \geq m$.

In the following proposition, we discuss conditions pertaining to an (n,m)power-(D,A)-normal operator to be an n power-(D,A)-normal operator.

Proposition 2.13 Let $T \in \mathcal{B}_A(\mathcal{H})$ be Draizin invertible operator. If T is both of class $[(n,m)DN]_A$ and of class $[(n,m+1)DN]_A$ such that $T^\sharp$ is injective, then T is of class $[(n,1)DN]_A = [nDN]_A$.

Proof . Since T is of class $[(n,m)DN]_A$ and of class $[(n,m+1)DN]_A$, it follow that

$$(T^D)^nT^{\sharp m+1} - T^{\sharp m+1}(T^D)^n = 0$$

i.e

$$T^{\sharp m}\big((T^D)^nT^\sharp - T^\sharp(T^D)^n\big) = 0$$

Since $T^\sharp$ is injective ,then so is $(T^\sharp)^m$ and we have $(T^D)^nT^\sharp - T^\sharp(T^D)^n = 0$, hence T is of class $[(n,1)DN]_A$ or equivalently T is of class $[nDN]_A$.

In [18] it was proved that if T is n-power normal which is a partial isometry, then T is $n+1$ power normal. In the following theorem we extend this result to (n,m) power-(D,A)-normal operator and (n,m) power-A-normal operator.

Theorem 2.14 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. be of class $[(n,m)DN]_A$. If $T^mT^{\sharp m}T^m = T^m$, then T is of class $[(n+m,m)DN]_A$.

Proof . Firstly, observe that since T is a Drazin invertible we have $T^DT = I$ and $(T^D)^2T = T^D$ from which it is easily to obtain that $(T^D)^{2k}T^k = (T^D)^K$ for $k \geq 1$

We have $$T^mT^{\sharp m}T^m = T^m \qquad (2,2)$$

Multiplying $(2,2)$ to the left by $(T^D)^{n+m}$ and to the right by $(T^D)^{2m}$ we get

$$(T^D)^{n+m}.T^mT^{\sharp m}T^m.(T^D)^{2m} = (T^D)^{n+m}.T^m.(T^D)^{2m}.$$

Then $$(T^D)^nT^{\sharp m}(T^D)^m = (T^D)^{n+2m} \qquad (2,3)$$

Multiplying $(2,2)$ to the left by $(T^D)^{2m}$ and to the right by $(T^D)^{n+m}$ we get

$$(T^D)^{2m}.T^mT^{\sharp m}T^m.(T^D)^{n+m} = (T^D)^{2m}.T^m.(T^D)^{n+m}$$

Therefore $$(T^D)^mT^{\sharp m}(T^D)^n = (T^D)^{n+2m} \qquad (2,4)$$

Combing $(2,3)$ and $(2,4)$ we get

$$(T^D)^nT^{\sharp m}(T^D)^m = (T^D)^mT^{\sharp m}(T^D)^n$$

By taking into account that T is of class $[(n,m)DN]_A$ we obtain

$$T^{\sharp m}(T^D)^{n+m} = (T^D)^{n+m}T^{\sharp m}$$

Thus T is of class $[(n,m)DN]_A$

Theorem 2.15 Let $T \in \mathcal{B}_A(\mathcal{H})$ be of class $[(n,m)N]_A$ for $n \geq m$. If $T^mT^{\sharp m}T^m = T^m$, then T is $(n+m,m)$ power A-normal.

Proof . Since $T^mT^{\sharp m}T^m = T^m$

Hence, we easily get $T^mT^{\sharp m}T^n = T^n$ and $T^nT^{\sharp m}T^m = T^n$

which means that $T^mT^{\sharp m}T^n = T^nT^{\sharp m}T^m$

Since T is (n,m) power A-normal, we get

$$T^{n+m}T^{\sharp m} = T^{\sharp m}T^{n+m}$$

Then T is $(n+m,m)$ power A-normal.

Proposition 2.16 Let $T \in \mathcal{B}_A(\mathcal{H})^D$, if T is (n,m) power-(D,A)-normal operator then so it T^k for every positive integer k

Proof . To prove that T^k is of class $[(n,m)DN]_A$,we have to prove that

$\left(\left(T^k\right)^D\right)^n.\left(\left(T^k\right)^\sharp\right)^m = \left(\left(T^k\right)^\sharp\right)^m\left(\left(T^k\right)^D\right)^n$ for all positive integer k

We prove the statement by using mathematical induction on k.Since T is (n,m) power-(D,A)-normal operator the result is true for $k=1$.Now we assume that the result is true for k,that is

$$\left(\left(T^k\right)^D\right)^n(T^k)^{\sharp m} = (T^k)^{\sharp m}\left(\left(T^k\right)^D\right)^n$$

and proved it for $k+1$.

$$\begin{aligned}
\left(\left(T^{k+1}\right)^D\right)^n(T^{k+1})^{\sharp m} &= (T^D)^n\left(\left(T^k\right)^D\right)^n(T^k)^{\sharp m}T^{\sharp m}\\
&= (T^D)^n(T^k)^{\sharp m}\left(\left(T^k\right)^D\right)^nT^{\sharp m}\\
&= T^{\sharp m}(T^k)^{\sharp m}\left(\left(T^k\right)^D\right)^n(T^D)^n \quad (\text{since } T \in [(n,m)DN]_A)\\
&= (T^{k+1})^{\sharp m}\left(\left(T^{k+1}\right)^D\right)^n)
\end{aligned}$$

Therefore T^{k+1} is of class$[(n,m)DN]_A$. We conclude that the statement of the proposition holds i.e T^k is of class$[(n,m)DN]_A$for all positive integer k.

Proposition 2.17 Let $T \in \mathcal{B}_A(\mathcal{H})^D$such that$T$ is idempotent then, T is (n,n)power-(D,A)-normal operator if and only if T is (D,A)-normal operator

Proof . **1.** Since T isidempotent, we have $T = T^2 = \cdots = T^n$, then

$$\begin{aligned}
T \in [(n,n)DN]_A &\Leftrightarrow (T^D)^nT^{\sharp n} = T^{\sharp n}(T^D)^n\\
&\Leftrightarrow T^DT^\sharp = T^\sharp T^D\\
&\Leftrightarrow T \in [DN]_A
\end{aligned}$$

And the proof is complete.

In the following theorem we will prove the stability of the class of (n,m) power-(D,A)-normal operators under the direct sum and tensor product.

Theorem 2.18 Let $T_1, T_2 \cdots T_k$ are (n,m) power-(D,A)-normal in $\mathcal{B}_A(\mathcal{H})^D$,then
1. $T_1 \oplus T_2 \oplus \cdots \oplus T_k$ is of class $[(n,m)DN]_{A\oplus A\oplus\cdots\oplus A}$
2. $T_1 \otimes T_2 \otimes \cdots \otimes T_k$ is of class $[(n,m)DN]_{A\otimes A\otimes\cdots\otimes A}$
Proof . Assume that each T_d for $d = 1,\cdots,k$ is (n,m) power-(D,A)-normal, then
$((T_d)^D)^n T_d^{\sharp m} = T_d^{\sharp m}((T_d)^D)^n$ and we have.
1.

$$\begin{aligned}
&((T_1 \oplus T_2 \oplus \cdots \oplus T_k)^D)^n (T_1 \oplus T_2 \oplus \cdots \oplus T_k)^{\sharp m}\\
&= (((T_1)^D)^n \oplus ((T_2)^D)^n \oplus \cdots \oplus ((T_k)^D)^n)\big(T_1^{\sharp m} \oplus T_2^{\sharp m} \oplus \cdots \oplus T_k^{\sharp m}\big)\\
&= ((T_1)^D)^n . T_1^{\sharp m} \oplus ((T_2)^D)^n . T_2^{\sharp m} \oplus \cdots \oplus ((T_k)^D)^n . T_k^{\sharp m}\\
&= T_1^{\sharp m}((T_1)^D)^n \oplus T_2^{\sharp m}((T_2)^D)^n \oplus \cdots \oplus T_k^{\sharp m}((T_k)^D)^n\\
&= \big(T_1^{\sharp m} \oplus T_2^{\sharp m} \oplus \cdots \oplus T_k^{\sharp m}\big)(((T_1)^D)^n \oplus ((T_2)^D)^n \oplus \cdots \oplus ((T_k)^D)^n)\\
&= (T_1 \oplus T_2 \oplus \cdots \oplus T_k)^{\sharp m}((T_1 \oplus T_2 \oplus \cdots \oplus T_k)^D)^n
\end{aligned}$$

Thus $T_1 \oplus T_2 \oplus \cdots \oplus T_d$ is of class$[(n,m)DN]_{A\oplus A\oplus\cdots\oplus A}$
2.

$$\begin{aligned}
&((T_1 \otimes T_2 \otimes \cdots \otimes T_d)^D)^n (T_1 \otimes T_2 \otimes \cdots \otimes T_d)^{\sharp m}\\
&= (((T_1)^D)^n \otimes ((T_2)^D)^n \otimes \cdots \otimes ((T_d)^D)^n)\big(T_1^{\sharp m} \otimes T_2^{\sharp m} \otimes \cdots \otimes T_d^{\sharp m}\big)\\
&= ((T_1)^D)^n . T_1^{\sharp m} \otimes ((T_2)^D)^n . T_2^{\sharp m} \otimes \cdots \otimes ((T_d)^D)^n . T_d^{\sharp m}\\
&= T_1^{\sharp m} . ((T_1)^D)^n \otimes T_2^{\sharp m} . ((T_2)^D)^n \otimes \cdots \otimes T_d^{\sharp m} . ((T_d)^D)^n\\
&= (T_1 \otimes T_2 \otimes \cdots \otimes T_d)^{\sharp m}((T_1 \otimes T_2 \otimes \cdots \otimes T_d)^D)^n
\end{aligned}$$

Hence $T_1 \otimes T_2 \otimes \cdots \otimes T_d$ is of class $[(n,m)DN]_{A\otimes A\otimes\cdots\otimes A}$

Proposition 2.19 Let $T, S \in \mathcal{B}_A(\mathcal{H})^D$ are an (n,m) power-(D,A)-normal such that $[T^D, S^\sharp] = [S^D, T^\sharp] = 0$, then $(TS \otimes T)$,($TS \otimes S$), $(ST \otimes T)$ and $(ST \otimes S)$ in $\mathcal{B}_{A\otimes A}(\mathcal{H})^D$ are (n,m) power-$(D, A\otimes A)$-normal operators.
Proof . Assume that T, S are an (n,m) power-(D,A)-normal operator. From the conditions we have $T^D S^\sharp = S^\sharp T^D$ and $S^D T^\sharp = T^\sharp S^D$, then easily we get $(T^D)^n S^{\sharp m} = S^{\sharp m}(T^D)^n$.So

$$\begin{aligned}
((TS \otimes T)^D)^n (TS \otimes T)^{\sharp m} &= (((TS)^D)^n \otimes (T^D)^n)\big((TS)^{\sharp m} \otimes T^{\sharp m}\big)\\
&= \big(((TS)^D)^n . (TS)^{\sharp m} \otimes (T^D)^n . T^{\sharp m})\big)\\
&= \big((S^D)^n (T^D)^n . S^{\sharp m} T^{\sharp m} \otimes (T^D)^n . T^{\sharp m}\big)\\
&= \big((S^D)^n S^{\sharp m} (T^D)^n T^{\sharp m} \otimes T^{\sharp m}(T^D)^n\big)\\
&= \big(S^{\sharp m}(S^D)^n T^{\sharp m}(T^D)^n \otimes T^{\sharp m}(T^D)^n\big)\\
&= \big(S^{\sharp m} T^{\sharp m}(S^D)^n (T^D)^n \otimes T^{\sharp m}(T^D)^n\big)\\
&= ((TS)^{\sharp m} . ((TS)^D)^n \otimes T^{\sharp m}(T^D)^n)\\
&= \big((TS)^{\sharp m} \otimes T^{\sharp m}\big)(((TS)^D)^n \otimes (T^D)^n)\\
&= (TS \otimes T)^{\sharp m}((TS \otimes T)^D)^n
\end{aligned}$$

Hence $(TS \otimes T)$ of class$[(n,m)DN]_{A\otimes A}$. In the same way, we may deduce the (n,m) power-(D,A)-normality of $(TS \otimes S)$, $(ST \otimes T)$ and $(ST \otimes S)$

3. (n,m) power-(D,A)-quasi-normal operators

Definition 3.1 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. We said that T is (n,m) power-(D,A)-quasi-normal operator if

$$(T^D)^n . T^{\sharp m} T = T^{\sharp m} T . (T^D)^n$$

for some positive integers n, m.This class ofoperators will be denote by $[(n,m)DQN]_A$
Remark 3.1 We make the following observations;

1. Clearly if $n = m = 1$, then (1,1) power-(D, A)-quasi-normal is precisely -(D, A)-quasi-normal i.e $[(1,1)DQN]_A = [DQN]_A$.
2. $[(n, 1)DQN]_A$ is the class of n power-(D, A)-quasi-normal i.e $[(n, 1)DQN]_A = [nDQN]_A$.
3. Every(n, m) power-(D, A)-normal is (n, m) power-(D, A)-quasi-normal i.e.
$[(n, m)DN]_A \subset [(n, m)DQN]_A$.
4. $[(n, m)DQN]_A \subset [(2n, m)DQN]_A$
5.$T \in [(n, m)DQN]_A \Leftrightarrow [(T^D)^n, T^{\sharp m}T] = 0$
6. If $T^D = T$, then $[(n, m)QDN]_A = [(n, m)QN]_A$

Example 3.1 Let $T = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

The drazin invertible of T is $T^D = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{-1}{2} \end{pmatrix}$ and $T^\sharp = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix}$

A simple calculation shows that T is of class $[(2,2)DQN]_A$
Example 3.2 Let

$$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad and \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A simple computation shows that

$$T^D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = T \quad and \quad T^\sharp = \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A Direct computation shows that $(T^D)^2 . T^{\sharp 3}T = T^{\sharp 3}T . (T^D)^2$. This implies that T isof class$[(2,3)QDN]_A$

Proposition 3.1 Let $T \in \mathcal{B}_A(\mathcal{H})^D$, $B = (T^D)^n + T^{\sharp m}T$ and $C = (T^D)^n - T^{\sharp m}T$, Then the followig statements hold
1. T is of class$[(n, m)DQN]_A$if and only if B commutes with C
2. If T is of class$[(n, m)DQN]_A$, then $(T^D)^n . T^{\sharp m}T$ commutes with B and C
3. T is of class $[(n, m)DQN]_A$if and only if $(T^D)^n$ commutes with B and C
Proof .
1. $BC = CB \Leftrightarrow \big((T^D)^n + T^{\sharp m}T\big)\big((T^D)^n - T^{\sharp m}T\big) = \big((T^D)^n - T^{\sharp m}T\big)\big((T^D)^n + T^{\sharp m}T\big)$

$$\Leftrightarrow (T^D)^{2n} - (T^D)^n T^{\sharp m}T + T^{\sharp m}T(T^D)^n - (T^{\sharp m}T)^2 = (T^D)^{2n} + (T^D)^n T^{\sharp m}T - T^{\sharp m}T(T^D)^n - (T^{\sharp m}T)^2$$
$$\Leftrightarrow 2. T^{\sharp m}T(T^D)^n = 2. (T^D)^n T^{\sharp m}T$$
$$\Leftrightarrow T \in [(n, m)QDN]_A$$

2. By definition $(3,1)$ we have that

$$(T^D)^n T^{\sharp m}T\big((T^D)^n \pm T^{\sharp m}T\big) = \big((T^D)^n \pm T^{\sharp m}T\big)(T^D)^n T^{\sharp m}T$$

3. Easily we obtain this statement.

Proposition 3.2 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. if T is of class $[(n, m)DQN]_A$ and of class $[(n + 1, m)DQN]_A$, then T is of class $[(n + 2, m)DQN]_A$,i.e.

$$([(n, m)DQN]_A \cap [(n + 1, m)DQN]_A) \subset [(n + 2, m)DQN]_A$$

Proof . Since T is of class $[(n, m)DQN]_A$ and of class $[(n + 1, m)DQN]_A$, it follows that
$(T^D)^n T^{\sharp m}T = T^{\sharp m}T(T^D)^n$ and $(T^D)^{n+1} T^{\sharp m}T = T^{\sharp m}T(T^D)^{n+1}$ .So

$$\begin{aligned}(T^D)^{n+2} T^{\sharp m}T &= (T^D)(T^D)^{n+1} T^{\sharp m}T \\ &= (T^D)T^{\sharp m}T(T^D)^{n+1}\end{aligned}$$

$= (T^D)T^{\sharp m}T(T^D)^n(T^D)$
$= (T^D)^{n+1}T^{\sharp m}T(T^D)$
$= T^{\sharp m}T(T^D)^{n+2}$

Hence T is of class $[(n+2,m)DQN]_A$.
In [19] it was proved that if T is of class $[nQN]$ such that T^m is a partial isometry, then T is of class $[(n+1)QN]$. We extend this result to the classes of $[(n,m)QN]_A$ and $[(n,m)DQN]_A$

Theorem 3.4 Let $T \in \mathcal{B}_A(\mathcal{H})$ be of class $[(n,m)QN]_A$ for $n \geq m$ .If $T^mT^{\sharp m}T^m = T^m$, then T is $(n+m,m)$ power-A-quasi-normal.

Proof . Under the assumption we have $T^mT^{\sharp m}T^m = T^m$ or equivalently

$$T^m(T^{\sharp m}T)T^{m-1} = T^m \quad (3,1)$$

Multiplying (3,1) to the left by T^{n-m} and to the right by T we get

$$T^n(T^{\sharp m}T)T^m = T^{n+1} \quad (3,2)$$

Multiplying (3,1) to the right by T^{n-m+1} we get

$$T^m(T^{\sharp m}T)T^n = T^{n+1} \quad (3,3)$$

Combining (3,2) and (3,3) we get

$$T^n(T^{\sharp m}T)T^m = T^m(T^{\sharp m}T)T^n$$

Using the fact that $T \in [(n,m)QN]_A$ we get

$$(T^{\sharp m}T)T^{n+m} = T^{n+m}(T^{\sharp m}T)$$

Therefore T is $(n+m,m)$ power-A-quasi-normal.

Theorem 3.5 Let $T \in \mathcal{B}_A(\mathcal{H})^D$ be an (n,m)power (D,A)-quasi-normal for some integres n and m with $n \geq m$. If T^m is A-partial isometry, then T is of class $[(n+m,m)DQN]_A$.

Proof . Firstly, observe that since T is a Drazin invertible we have $T^DT = I$ and $(T^D)^2T = T^D$ from which it is easily to obtain that $(T^D)^{2k-1}T^{k-1} = (T^D)^k$ and $(T^D)^{2k}T^k = (T^D)^k$ for $k \geq 1$
Since T^m is A-partial isometry, then

$$T^mT^{\sharp m}T^m = T^m$$

Or equivalently

$$T^m(T^{\sharp m}T)T^{m-1} = T^m \quad (3,4)$$

Multiplying (3,4) to the left by $(T^D)^{n+m}$ and to the right by $(T^D)^{2m-1}$ we get

$$(T^D)^{n+m}T^m(T^{\sharp m}T)T^{m-1}(T^D)^{2m-1} = (T^D)^{2m-1}(T^D)^{n+m}T^m$$

i.e.

$$(T^D)^n(T^{\sharp m}T)(T^D)^m = (T^D)^{n+2m-1} \quad (3,5)$$

Multiplying (3,4) to the left by $(T^D)^{2m}$ and to the right by $(T^D)^{n+m-1}$ we get

$$(T^D)^{2m}T^m(T^{\sharp m}T)T^{m-1}(T^D)^{n+m-1} = (T^D)^{2m}T^m(T^D)^{n+m-1}$$

i.e

$$(T^D)^m(T^{\sharp m}T)(T^D)^n = (T^D)^{n+2m-1} \quad (3,6)$$

In view of (3,5) and (3,6) we obtain

$$(T^D)^n(T^{\sharp m}T)(T^D)^m = (T^D)^m(T^{\sharp m}T)(T^D)^n$$

By taking in to account that T is of class $[(n,m)DQN]_A$.we obtain

$$(T^{\sharp m}T)(T^D)^{n+m} = (T^D)^{n+m}(T^{\sharp m}T)$$

Thus, T is of class $[(n+m,m)DQN]_A$

The class of $[(n,m)QDN]_A$ has the following properties.

Proposition 3.6 Let $T \in \mathcal{B}_A(\mathcal{H})^D$ be an (n,m) power-$(D,$A$)$-quasinormal such that $\mathcal{N}(A)$ is a reducing subspace for T, if S is A-unitary equivalent operator to T,then S is (n,m) power-$(D,$A$)$- normal operator

Proof. Assume that T is an (n,m) power-(D,A)-normal operator. Since S is A-unitary equivalent to T, then there existe A-unitary operator $U \in \mathcal{B}_A(\mathcal{H})$ such that $S = UTU^\sharp$, easily we obtain $S^n = UT^nU^\sharp$,

$S^{\#} = UT^{\#}U^{\#}$ and $S^{D} = UT^{D}U^{\#}$. Hence,

$$\begin{aligned}(S^D)^n S^{\#m} S &= \big(U(T^D)^n U^{\#}\big)\big(UT^{\#m}U^{\#}\big)\big(UT^{\#m}U^{\#}\big)\\
&= U(T^D)^n U^{\#} U T^{\#m} U^{\#} U T U^{\#}\\
&= U(T^D)^n P_{\overline{\mathcal{R}(A)}} T^{\#m} P_{\overline{\mathcal{R}(A)}} T U^{\#}\\
&= U(T^D)^n T^{\#m} T U^{\#}\\
&= U T^{\#m} T (T^D)^n U^{\#}\\
&= (UT^{\#m}U^{\#})\big(UTU^{\#}\big)\big(U(T^D)^n U^{\#},\big)\\
&= S^{\#m} S (S^D)^n\end{aligned}$$

Hence, $(S^D)^n S^{\#m} S = S^{\#m} S (S^D)^n$, then $S \in [(n,m)DQN]_A$

The following examples show that a (n,m) power-(D,A)-normal need not be a $(n+1,m)$ power-(D,A)-normal and vice versa.

Example 3.3 Let us consider the matrix opertors $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Acting on $\mathbb{C}^2$. By simple calculations, it follows that T is of class $[(3,2)QDN]_A$ but not of class $[(2,2)QDN]_A$.

Example 3.4 Let $S = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

It easily to check that S is of class $[(2,2)DQN]_A$ but not of class$[(3,2)DQN]_A$.

Proposition 3.7 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. If T is both of class $[(n,m)DQN]_A$ and of class $[(n+1,m)DQN]_A$ such that T^D is injective, then T is of class $[(1,m)QDN]_A$.

Proof . Since T is of class $[(n,m)DQN]_A$ and of class $[(n+1,m)DQN]_A$, it follow that

$$(T^D)^{n+1} T^{\#m} T - T^{\#m} T (T^D)^{n+1} = 0$$

i.e.

$$(T^D)^n (T^D T^{\#m} T - T^{\#m} T T^D) = 0$$

If T^D is injective, then so is $(T^D)^n$ and we have $T^D T^{\#m} T - T^{\#m} T T^D = 0$, whence T is of class $[(1,m)QDN]_A$.

Proposition 3.8 Let $T \in \mathcal{B}_A(\mathcal{H})^D$. such that T is of class $[(2,m)DN]_A$ and of class $[(3,m)DN]_A$ for some positive integre m, then T is of class $[(n,m)DN]_A$ for all positive integre $n \geq 4$.

Proof . We prove the assertion by using the mathematical induction. For $n = 4$, it a consequence of statement(4) of remark (3.1).

We prove this for $n = 5$. Since $T \in [(2,m)QDN]_A$, then

$$(T^D)^2 T^{\#m} T = T^{\#m} T (T^D)^2 \qquad (3.7)$$

Multiplying (3.7) to the left by $(T^D)^3$ we get $(T^D)^5 \big(T^{\#}\big)^m T = (T^D)^3 \big(T^{\#}\big)^m T (T^D)^2$.

Thus implies $(T^D)^5 \big(T^{\#}\big)^m T = \big(T^{\#}\big)^m T (T^D)^5$.

Now assume that the results is true for $n \geq 5$ that is

$$(T^D)^n \big(T^{\#}\big)^m T = \big(T^{\#}\big)^m T (T^D)^n$$

Then

$$\begin{aligned}(T^D)^{n+1} \big(T^{\#}\big)^m T &= T^D (T^D)^n \big(T^{\#}\big)^m T\\
&= T^D \big(T^{\#}\big)^m T (T^D)^n\\
&= T^D \big(T^{\#}\big)^m T (T^D)^2 (T^D)^{n-2}\\
&= (T^D)^3 \big(T^{\#}\big)^m T (T^D)^{n-2}\\
&= \big(T^{\#}\big)^m T (T^D)^{n+1}\end{aligned}$$

Thus $T \in [(n+1,m)DN]_A$. The proof is complete.

Proposition 3.9 Let $T \in \mathcal{B}_A(\mathcal{H})^D$ be an $(n, m+1)$ power-(D,A)-quasinormal. The following properties hold
1. If $T \in [(n,m)DN]_A$ and $T^\sharp$ is injective, then T is of class $[nDQN]_A$.
2. If T is A-normal and $T^{\sharp m}T$ is injective, then T is of class $[nDN]_A$.
Proof .
1. Since T is of class $[(n,m)DN]_A$ and of class $[(n,m+1)DQN]_A$, it follow that

$$(T^D)^n T^{\sharp m+1}T - T^{\sharp m+1}T(T^D)^n = 0$$

i.e. $(T^D)^n T^{\sharp m}T^\sharp T - T^{\sharp m}T^\sharp T(T^D)^n = 0$
Hence $T^{\sharp m}((T^D)^n T^\sharp T - T^\sharp T(T^D)^n) = 0$
If $T^\sharp$is injective, then so is $T^{\sharp m}$ and we have $(T^D)^n T^\sharp T - T^\sharp T(T^D)^n = 0$, whence T is of class $[(n,1)DQN]_A = [nDQN]_A$.
2. Assume T is A-normal, then $T^\sharp T = TT^\sharp$, and since T is of class $[(n,m+1)DQN]_A$, we have $(T^D)^n T^{\sharp m+1}T - T^{\sharp m+1}T(T^D)^n = 0$
Then $(T^D)^n T^{\sharp m}T^\sharp T - T^{\sharp m}T^\sharp T(T^D)^n = 0$
Hence $(T^D)^n T^{\sharp m}TT^\sharp - T^{\sharp m}TT^\sharp(T^D)^n = 0$
Hence $T^{\sharp m}T((T^D)^n T^\sharp - T^\sharp(T^D)^n) = 0$
$T^{\sharp m}T$ is injective, then $(T^D)^n T^\sharp - T^\sharp(T^D)^n = 0$, thus T is of class $[(n,1)DN]_A = [nDN]_A$.

Theorem 3.10 Let $T \in \mathcal{B}_A(\mathcal{H})^D$of class $[(n,m)DQN]_A$, if T is $(1,m)$ power A-normal, then T^D is of class $[(n,m)QN]_A$.
Proof . Since T is $(1,m)$ power A-normal,then$T^{\sharp m}T = TT^{\sharp m}$,we have

$$(T^D)^n T^{\sharp m}T - T^{\sharp m}T(T^D)^n = 0$$

By lemma (1,1) we get

$$(T^D)^n \left(T^{\sharp m}T\right)^D - \left(T^{\sharp m}T\right)^D (T^D)^n = 0$$

Then

$$(T^D)^n (T^D)^{\sharp m}(T)^D - (T^D)^{\sharp m}(T)^D(T^D)^n = 0$$

Hence T^D is of class $[(n,m)QN]_A$

Proposition 3.11 Let $T \in \mathcal{B}_A(\mathcal{H})^D$ such that $\mathcal{N}(A)$ is an invariant subspace under the action of T , then we have the following properties
1. If T is of class $[2DQN]_A$ and T is an $(A,2)$ isometry, then T^2 is of class $[nDQN]_A$ for all $n \geq 2$
2. If T is of class $[2DQN]_A$ and T is an $(A,2)$ isometry, then T is an A-isometry
3. If T is of class $[2DQN]_A$ and T is A-normal, then T^2 is of class $[DQN]_A$
Proof . By analogous arguments as in the proof of (proposition (2,3)in [19]), we show the Statements(1) and (2)
3. Assume $T \in ([2DQN]_A \cap [N]_A)$, then we have $(T^D)^2 T^\sharp T = T^\sharp T(T^D)^2$ and $T^\sharp T = TT^\sharp$
Now

$$\begin{aligned}
(T^2)^D (T^2)^\sharp T^2 &= (T^D)^2 T^{\sharp 2}T^2 \\
&= (T^D)^2 T^\sharp (T^\sharp T)T \\
&= (T^D)^2 T^\sharp (TT^\sharp)T \\
&= (T^D)^2 (T^\sharp T)T^\sharp T \\
&= (T^\sharp T)T^\sharp T(T^D)^2 \\
&= T^\sharp (TT^\sharp)T(T^D)^2 \\
&= (T^2)^\sharp T^2 (T^D)^2
\end{aligned}$$

Therefore $T^2 \in [(1,1)DQN]_A = [DQN]_A$

Tensor Product and Direct sum of (n,m)Power-(D,A)-quasi-normal Operators in Semi-Hilbertian Spaces

Theorem 3.12 Let $(T_d)_{1\leq d\leq k}\in(\mathcal{B}_A(\mathcal{H})^D)^k$ such that each T_d is (n,m) power-$(D,$A)-quasinormal, then

1. $T_1\oplus T_2\oplus\cdots\oplus T_k$ is (n,m) power-$(D,A\oplus A\oplus\cdots\oplus A)$-quasinormal

2. $T_1\otimes T_2\otimes\cdots\otimes T_k$ is of class $[(n,m)DQN]_{A\otimes A\otimes\cdots\otimes A}$

Proof . Since $T_d\in[(n,m)DQN]_A$, then $((T_d)^D)^n T_d^{\sharp m}T_d=T_d^{\sharp m}T_d((T_d)^D)^n$ and we have.

1.

$$((T_1\oplus T_2\oplus\cdots\oplus T_k)^D)^n(T_1\oplus T_2\oplus\cdots\oplus T_k)^{\sharp m}(T_1\oplus T_2\oplus\cdots\oplus T_k)$$
$$=(((T_1)^D)^n\oplus\cdots\oplus((T_k)^D)^n)\left(T_1^{\sharp m}\oplus T_2^{\sharp m}\oplus\cdots\oplus T_k^{\sharp m}\right)(T_1\oplus T_2\oplus\cdots\oplus T_k)$$
$$=((T_1)^D)^n.T_1^{\sharp m}T_1\oplus((T_2)^D)^n.T_2^{\sharp m}T_2\oplus\cdots\oplus((T_k)^D)^n.T_k^{\sharp m}T_k$$
$$=T_1^{\sharp m}T_1.((T_1)^D)^n\oplus T_2^{\sharp m}T_2.((T_2)^D)^n\oplus\cdots\oplus T_k^{\sharp m}T_k.((T_k)^D)^n$$
$$=\left(T_1^{\sharp m}\oplus\cdots\oplus T_k^{\sharp m}\right)(T_1\oplus T_2\oplus\cdots\oplus T_k)(((T_1)^D)^n\oplus((T_2)^D)^n\oplus\cdots\oplus((T_k)^D)^n)$$
$$=(T_1\oplus T_2\oplus\cdots\oplus T_k)^{\sharp m}(T_1\oplus T_2\oplus\cdots\oplus T_k)((T_1\oplus T_2\oplus\cdots\oplus T_k)^D)^n$$

Thus $T_1\oplus T_2\oplus\cdots\oplus T_k$ is of class$[(n,m)DQN]_{A\oplus A\oplus\cdots\oplus A}$

2. The proof of this statement is formally the same as the proof of statement(1)

4. Conclusions

In this article we have worked on two new classes of operators $(\boldsymbol{n},\boldsymbol{m})$ power-$(\boldsymbol{D},\boldsymbol{A})$-normal and $(\boldsymbol{n},\boldsymbol{m})$ power $(\boldsymbol{D},\boldsymbol{A})$-quasi-normal in Semi-Hilbertian Space, we have given the definition of these classes and found the conditions so that the sum and the product of two $(\boldsymbol{n},\boldsymbol{m})$ power-$(\boldsymbol{D},\boldsymbol{A})$-normal and $(\boldsymbol{n},\boldsymbol{m})$ power-$(\boldsymbol{D},\boldsymbol{A})$ quasi-normal to be $(\boldsymbol{n},\boldsymbol{m})$ power-$(\boldsymbol{D},\boldsymbol{A})$-normal and $(\boldsymbol{n},\boldsymbol{m})$ power-$(\boldsymbol{D},\boldsymbol{A})$-quasi-normal respectively and we have proved that these classes are stable by the tensor product and the direct sum.

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Declaration of competing interest

The authors declare that there is no known conflict of interest.

Authorship contribution statement

The authors' contribution to the manuscript is as follows

Djilali Bekai: Conceptualization and Methodology
Abdelkader Benali: Writing, Reviewing and Supervision
Ali Hakem: Data Preparation, Reviewing and Editing

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