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A STUDY ON GCR-LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. We study GCR-lightlike submanifolds of semi-Riemannian product manifolds. We give some equivalent conditions for the integrability of various distributions of GCR-lightlike submanifolds of semi-Riemannian product manifolds and investigate the geometry of leaves of distributions.

1. Introduction

The theory of lightlike submanifolds is an important research topic in differential geometry due to its application in mathematical physics, especially in the general relativity. Since the normal vector bundle intersects with the tangent bundle, contrary to classical theory of submanifolds, the theory of lightlike (degenerate) submanifolds becomes more interesting and remarkably different from the theory of non-degenerate submanifolds. The geometry of lightlike submanifolds was initiated by Kupeli [16]. After it was developed by Duggal and Bejancu [3] and Duggal and Şahin [7]. Many authors studied the lightlike submanifolds in various manifolds, for example [2], [11], [18] and [20].

Duggal and Bejancu introduced CR-lightlike submanifolds of indefinite Kaehler manifolds [3]. Similar to CR-lightlike submanifolds, semi-invariant lightlike submanifolds of semi-Riemannian product manifolds were introduced by Atçeken and Kılıç in [1]. Since CR-lightlike submanifolds exclude the complex and totally real submanifolds, therefore, in [4], Duggal and Şahin introduced screen CR-lightlike submanifolds, which contains complex and screen real subcases. The SCR-lightlike submanifolds, analogously, Screen semi-invariant lightlike submanifolds, of semi-Riemannian product manifolds were introduced by Khursheed, Thakur and Advin [10] and Kılıç, Şahin and Keleş [12], respectively. But there is no inclusion relation between CR and SCR submanifolds, so Duggal and Şahin introduced a new class called GCR-lightlike submanifolds of indefinite Kaehler manifolds which is an umbrella for all these types of submanifolds [5]. Kumar, Kumar, Nagaich studied GCR-lightlike submanifolds of a semi-Riemannian product manifold [15]. These

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types of submanifolds have been studied in various manifolds by many authors [6, 8, 9, 13, 14, 17].

In this paper, we study GCR-lightlike submanifolds of semi-Riemannian product manifolds. We give some equivalent conditions for the integrability of various distributions of GCR-lightlike submanifolds of semi-Riemannian product manifolds and investigate the geometry of leaves of distributions.

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $\tilde{M}$, where g is the induced metric of $\tilde{g}$ on M. If $\tilde{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\tilde{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup \left\{ u \in T_x\tilde{M} : \tilde{g}(u, v) = 0, \forall v \in T_xM, x \in M \right\} \tag{2.1}$$

is a degenerate $n-$dimensional subspace of $T_x\tilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, called radical distribution on M of rank $r > 0$ then the submanifold M of $\tilde{M}$ is called an $r-$lightlike submanifold.

Let $S\,(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S\,(TM) \perp Rad\,(TM) \tag{2.2}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr\,(TM)$ and $ltr\,(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\tilde{M}_{|M}$ and $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr\,(TM) = ltr\,(TM) \perp S(TM^{\perp}), \tag{2.3}$$

$$T\tilde{M}\mid_M = TM \oplus tr\,(TM) = \{Rad(TM) \oplus ltr\,(TM)\} \perp S\,(TM) \perp S(TM^{\perp}). \tag{2.4}$$

Theorem 2.1. [3] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Suppose U is a coordinate neighbourhood of M and $\{\xi_i\}$, $i \in \{1, .., r\}$ is a basis of $\Gamma(Rad\left(TM\right)_{|U})$. Then, there exist a complementary vector subbundle $ltr\,(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}_{|U}$ and a basis $\{N_i\}$, $i \in \{1, .., r\}$ of $\Gamma(ltr\,(TM)_{|U})$ such that*

$$\tilde{g}\,(N_i, \xi_j) = \delta_{ij}, \quad \tilde{g}\,(N_i, N_j) = 0 \tag{2.5}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\tilde{M}$ is
Case 1: $r-$lightlike if $r < min\,\{m, n\}$,
Case 2: Coisotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$; $S\,(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$; $S\,(TM) = \{0\} = S(TM^{\perp})$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Then, using (2.4), the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.6}$$

$$\tilde{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr\,(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr\,(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr\,(TM)$, respectively. According to (2.3), considering the projection morphisms L and S of $tr\,(TM)$ on $ltr\,(TM)$ and $S(TM^{\perp})$, respectively, (2.6) and (2.7) become

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \tag{2.9}$$

$$\tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^l_X N, D^l(X,W) \in \Gamma(ltr\,(TM))$ and $\nabla^s_X W, D^s(X,N) \in \Gamma(S(TM^{\perp}))$. Then, by using (2.8)-(2.10) and taking into account that $\tilde{\nabla}$ is a metric connection we obtain

$$\tilde{g}(h^s(X,Y),W) + \tilde{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.11}$$

$$\tilde{g}(D^s(X,N),W) = \tilde{g}(A_W X, N). \tag{2.12}$$

Let Q be a projection of TM on $S(TM)$. Then, using (2.2) we can write

$$\nabla_X QY = \nabla^*_X QY + h^*(X,QY), \tag{2.13}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X QY, A^*_\xi X\}$ and $\{h^*(X,QY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S\,(TM))$ and $\Gamma(Rad\,(TM))$, respectively.

Using the equations given above, we derive

$$\tilde{g}(h^l(X,QY),\xi) = g(A^*_\xi X, QY), \tag{2.15}$$

$$\tilde{g}(h^*(X,QY),N) = g(A_N X, QY), \tag{2.16}$$

$$\tilde{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.17}$$

Generally, the induced connection ∇ on M is not metric connection. Since $\tilde{\nabla}$ is a metric connection, from (2.8), we obtain

$$(\nabla_X g)(Y,Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(h^l(X,Z),Y). \tag{2.18}$$

But, ∇^* is a metric connection on $S(TM)$.

3. Semi-Riemannian product manifolds

Let (M_1, g_1) and (M_2, g_2) be two (m_1+1) and (m_2+1) dimensional Lorentzian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ be the projections which are given by $\pi(x,y) = x$ and $\sigma(x,y) = y$ for any $(x,y) \in M_1 \times M_2$. We denote the product manifold by $\tilde{M} = M_1 \times M_2$, where

$$\tilde{g}(X,Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y),$$

for any $X, Y \in \Gamma(T\tilde{M})$ and $*$ means tangent mapping. Then we have $\pi_*^2 = \pi$, $\sigma_*^2 = \sigma$, $\pi_*\sigma_* = \sigma_*\pi_* = 0$ and $\pi_* + \sigma_* = I$, where I is identity transformation. Thus

$(\tilde{M}, \tilde{g})$ is a $(m_1+m_2)-$dimensional semi-Riemannian manifold with constant index (q_1+q_2). The semi-Riemannian product manifold $\tilde{M} = M_1 \times M_2$ is characterized by M_1 and M_2 which are totally geodesic submanifolds of $\tilde{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that

$$F^2 = I \tag{3.1}$$

and

$$\tilde{g}(FX, Y) = \tilde{g}(X, FY) \tag{3.2}$$

for any $X, Y \in \Gamma(T\tilde{M})$. Then it can be seen that

$$(\tilde{\nabla}_X F)Y = 0, \tag{3.3}$$

for any $X, Y \in \Gamma(T\tilde{M})$, that is, F is parallel with respect to $\tilde{\nabla}$ [19].

4. Generalized Cauchy-Riemann (GCR)-Lightlike Submanifolds

Definition 4.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then we say that M is a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2,\ F(D_1) = D_1,\ F(D_2) \subset S(TM). \tag{4.1}$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM) = \{F(D_2) \oplus D'\} \perp D_0,\ F(D_0) = D_0,\ F(D^{'}) = L_1 \perp L_2, \tag{4.2}$$

where D_0 is a non-degenerate distribution on M, L_1 and L_2 are vector subbundles of $ltr\,(TM)$ and $S(TM^{\perp})$, respectively.

Thus we have the following decomposition:

$$TM = D \oplus D',\ D = Rad(TM) \perp D_0 \perp F(D_2). \tag{4.3}$$

We say that M is a proper GCR-lightlike submanifold of a semi-Riemannian product manifold if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $L_2 \neq \{0\}$.

Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Thus, for any $X \in \Gamma(TM)$ we derive

$$FX = fX + wX, \tag{4.4}$$

where fX and wX are tangential and transversal parts of FX.

For $V \in \Gamma(tr(TM))$ we write

$$FV = BV + CV, \tag{4.5}$$

where BV and CV are tangential and transversal parts of FV.

Theorem 4.2. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then the distribution D_0 is integrable iff*

(i) $\tilde{g}(h^*(X, Y), N) = \tilde{g}(h^*(Y, X), N)$,

(ii) $\tilde{g}(h^*(X, FY), N') = \tilde{g}(h^*(Y, FX), N')$,

(iii) $\tilde{g}(h^s(X, FY), W) = \tilde{g}(h^s(Y, FX), W)$,

(iv) $g(\nabla^*_X Y, F\xi) = g(\nabla^*_Y X, F\xi)$,

for any $X, Y \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $N' \in \Gamma(L_1)$ *and* $W \in \Gamma(L_2)$.

Proof. Using the definition of GCR-lightlike submanifold, the distribution D_0 is integrable iff

$$\tilde{g}([X,Y],N) = \tilde{g}([X,Y],FN') = \tilde{g}([X,Y],FW) = \tilde{g}([X,Y],F\xi) = 0.$$

for any $X, Y \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $N' \in \Gamma(L_1)$ and $W \in \Gamma(L_2)$. Then using (2.8) and (2.13) we obtain

$$\tilde{g}([X,Y],N) = \tilde{g}(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, N) = \tilde{g}(h^*(X,Y) - h^*(Y,X), N), \tag{4.6}$$

$$\begin{aligned} g([X,Y],FN') &= g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, FN') = \tilde{g}(\tilde{\nabla}_X FY - \tilde{\nabla}_Y FX, N') \\ &= \tilde{g}(h^*(X,FY) - h^*(Y,FX), N'), \end{aligned} \tag{4.7}$$

$$\begin{aligned} g([X,Y],FW) &= g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, FW) = \tilde{g}(\tilde{\nabla}_X FY - \tilde{\nabla}_Y FX, W) \\ &= \tilde{g}(h^s(X,FY) - h^s(Y,FX), W), \end{aligned} \tag{4.8}$$

$$g([X,Y],F\xi) = g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, F\xi) = g(\nabla^*_X Y, F\xi) - g(\nabla^*_Y X, F\xi). \tag{4.9}$$

Thus from (4.6)-(4.9) the result follows. □

From Theorem 4.2 we obtain the following corollary.

Corollary 4.3. Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then the distribution D_0 is integrable iff
(i) $g(A_N X, Y) = g(A_N Y, X)$,
(ii) $g(A_{N'} X, FY) = g(A_{N'} Y, FX)$,
(iii) $g(A_W X, FY) = g(A_W Y, FX)$,
(iv) $\tilde{g}(h^l(X,FY),\xi) = \tilde{g}(h^l(Y,FX),\xi)$,
for any $X, Y \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $N' \in \Gamma(L_1)$ and $W \in \Gamma(L_2)$.

Theorem 4.4. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then $Rad(TM)$ is integrable iff*
(i) $\tilde{g}(h^l(\xi,Z),\xi') = \tilde{g}(h^l(\xi',Z),\xi)$,
(ii) $\tilde{g}(h^l(\xi,F\xi''),\xi') = \tilde{g}(h^l(\xi',F\xi''),\xi)$,
(iii) $\tilde{g}(h^*(\xi,F\xi'),N) = \tilde{g}(h^*(\xi',F\xi),N)$,
(iv) $\tilde{g}(h^s(\xi,F\xi'),W) = \tilde{g}(h^s(\xi',F\xi),W)$,
for any $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, $\xi, \xi' \in \Gamma(Rad(TM))$, $N \in \Gamma(L_1)$, $W \in \Gamma(L_2)$.

Proof. Using the definition of GCR-lightlike submanifold, $Rad(TM)$ is integrable iff

$$\tilde{g}([\xi,\xi'],Z) = \tilde{g}([\xi,\xi'],F\xi'') = \tilde{g}([\xi,\xi'],FN) = \tilde{g}([\xi,\xi'],FW) = 0,$$

for any $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, $\xi, \xi' \in \Gamma(Rad(TM))$, $N \in \Gamma(L_1)$, $W \in \Gamma(L_2)$. Then taking into account that $\tilde{\nabla}$ is a metric connection and using (2.8) and (2.13) we get

$$\begin{aligned} g([\xi,\xi'],Z) &= g(\tilde{\nabla}_\xi \xi', Z) - g(\tilde{\nabla}_{\xi'} \xi, Z) = -g(\xi', \tilde{\nabla}_\xi Z) + g(\xi, \tilde{\nabla}_{\xi'} Z) \\ &= -\tilde{g}(h^l(\xi,Z),\xi') + \tilde{g}(h^l(\xi',Z),\xi), \end{aligned} \tag{4.10}$$

$$\begin{aligned} g([\xi,\xi'],F\xi'') &= g(\tilde{\nabla}_\xi \xi', F\xi'') - g(\tilde{\nabla}_{\xi'} \xi, F\xi'') = -\tilde{g}(\xi', \tilde{\nabla}_\xi F\xi'') + \tilde{g}(\xi, \tilde{\nabla}_{\xi'} F\xi'') \\ &= -\tilde{g}(h^l(\xi,F\xi''),\xi') + \tilde{g}(h^l(\xi',F\xi''),\xi), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \tilde{g}([\xi,\xi'],FN) &= g(\tilde{\nabla}_\xi \xi', FN) - g(\tilde{\nabla}_{\xi'} \xi, FN) = g(\tilde{\nabla}_\xi F\xi', N) - g(\tilde{\nabla}_{\xi'} F\xi, N) \\ &= \tilde{g}(h^*(\xi,F\xi'),N) - \tilde{g}(h^*(\xi',F\xi),N), \end{aligned} \tag{4.12}$$

$$g([\xi,\xi'],FW) = g(\tilde{\nabla}_\xi \xi',FW) - g(\tilde{\nabla}_{\xi'}\xi,FW) = \tilde{g}(\tilde{\nabla}_\xi F\xi',W) - \tilde{g}(\tilde{\nabla}_{\xi'}F\xi,W)$$
$$= \tilde{g}(h^s(\xi,F\xi'),W) - \tilde{g}(h^s(\xi',F\xi),W). \tag{4.13}$$

Hence proof is complete. □

From Theorem 4.4 we get the following corollary.

Corollary 4.5. Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M},\tilde{g},F)$. Then $Rad(TM)$ is integrable iff
(i) $g(A^*_{\xi'}\xi,Z) = g(A^*_\xi \xi',Z)$,
(ii) $g(A^*_{\xi'}\xi,F\xi'') = g(A^*_\xi \xi',F\xi'')$,
(iii) $g(A_N\xi,F\xi') = g(A_N\xi',F\xi)$,
(iv) $g(A_W\xi,F\xi') = g(A_W\xi',F\xi)$,
for any $Z\in\Gamma(D_0)$, $\xi''\in\Gamma(D_2)$, $\xi,\xi'\in\Gamma(Rad(TM))$, $N\in\Gamma(L_1)$, $W\in\Gamma(L_2)$.

Theorem 4.6. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M},\tilde{g},F)$. Then, each leaf of radical distribution is totally geodesic on M iff*
(i) $A^*_{\xi'}\xi\in\Gamma(F(D_2)\perp F(L_1))$,
(ii) $\tilde{g}(h^*(\xi,FN),\xi') = 0$,
(iii) $\tilde{g}(h^s(\xi,F\xi'),W) = 0$,
for any $Z\in\Gamma(D_0)$, $\xi,\xi'\in\Gamma(Rad(TM))$, $N\in\Gamma(L_1)$, $W\in\Gamma(L_2)$.

Proof. Using the definition of GCR-lightlike submanifold, each leaf of radical distribution is totally geodesic iff

$$g(\nabla_\xi \xi',Z) = g(\nabla_\xi \xi',F\xi'') = g(\nabla_\xi \xi',FN) = g(\nabla_\xi \xi',FW) = 0,$$

for any $Z\in\Gamma(D_0)$, $\xi''\in\Gamma(D_2)$, $\xi,\xi'\in\Gamma(Rad(TM))$, $N\in\Gamma(L_1)$, $W\in\Gamma(L_2)$. Then taking into account that $\tilde{\nabla}$ is a metric connection and using (2.8) and (2.13), (2.14), we derive

$$g(\nabla_\xi \xi',Z) = g(\tilde{\nabla}_\xi \xi',Z) = -g(A^*_{\xi'}\xi,Z), \tag{4.14}$$

$$g(\nabla_\xi \xi',F\xi'') = g(\tilde{\nabla}_\xi \xi',F\xi'') = -g(A^*_{\xi'}\xi,F\xi''), \tag{4.15}$$

$$g(\nabla_\xi \xi',FN) = g(\tilde{\nabla}_\xi \xi',FN) = -\tilde{g}(\xi',\tilde{\nabla}_\xi FN) = -\tilde{g}(h^*(\xi,FN),\xi'), \tag{4.16}$$

$$g(\nabla_\xi \xi',FW) = g(\tilde{\nabla}_\xi \xi',FW) = \tilde{g}(\tilde{\nabla}_\xi F\xi',W) = \tilde{g}(h^s(\xi,F\xi'),W). \tag{4.17}$$

Hence from (4.14)-(4.17) the assertion follows. □

Theorem 4.7. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M},\tilde{g},F)$. Then the distribution D_1 is integrable iff*
(i) $\nabla^{*t}_X FY - \nabla^{*t}_Y FX\in\Gamma(D_1)$,
(ii) $A^*_{FX}Y = A^*_{FY}X$,
(iii) $Bh(X,FY) = Bh(Y,FX)$,
for any $X,Y\in\Gamma(D_1)$.

Proof. For any $X,Y\in\Gamma(Rad(TM))$ using (3.3), we have $\tilde{\nabla}_X FY = F\tilde{\nabla}_X Y$, applying F to both sides and then using (2.6), (2.14) and (3.1), we obtain

$$\nabla_X Y + h(X,Y) = F(-A^*_{FY}X + \nabla^{*t}_X FY + h(X,FY)), \tag{4.18}$$

for any $X, Y \in \Gamma(D_1)$. Taking the tangential components of above equation both sides, we get

$$\nabla_X Y = -fA^*_{FY}X + f\nabla^{*t}_X FY - Bh(X, FY) \tag{4.19}$$

for any $X, Y \in \Gamma(D_1)$. Replacing X by Y and subtracting resulting equation from this equation, we derive

$$\begin{aligned} [X, Y] &= f(A^*_{FX}Y - A^*_{FY}X) + f(\nabla^{*t}_X FY - \nabla^{*t}_Y FX) \\ &\quad -Bh(X, FY) + Bh(Y, FX) \end{aligned} \tag{4.20}$$

thus $[X, Y] \in \Gamma(D_1)$ iff $\nabla^{*t}_X FY - \nabla^{*t}_Y FX \in \Gamma(D_1)$, $Bh(X, FY) = Bh(Y, FX)$, $A^*_{FX}Y = A^*_{FY}X$, this completes the proof. □

From Theorem 4.7 we obtain the following corollary.

Corollary 4.8. Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then D_1 defines a totally geodesic foliation in M iff
(i) $\nabla^{*t}_X FY \in \Gamma(D_1)$,
(ii) $A^*_{FX}Y = 0$,
(iii) $Bh(X, FY) = 0$,
for any $X, Y \in \Gamma(D_1)$.

Theorem 4.9. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then the distribution D_2 is integrable iff*
(i) $\nabla^*_X FY - \nabla^*_Y FX \in \Gamma(F(D_2))$,
(ii) $Bh(X, FY) = Bh(Y, FX)$,
(iii) $h^*(X, FY) = h^*(Y, FX)$,
for any $X, Y \in \Gamma(D_2)$.

Proof. For any $X, Y \in \Gamma(Rad(TM))$ using (3.3), we have $\tilde{\nabla}_X FY = F\tilde{\nabla}_X Y$, applying F to both sides and then using (2.6), (2.13) and (3.1), we get

$$\nabla_X Y + h(X, Y) = F(\nabla^*_X FY + h^*(X, FY) + h(X, FY)), \tag{4.21}$$

for any $X, Y \in \Gamma(D_2)$. Taking the tangential components of above equation both sides, we obtain

$$\nabla_X Y = f\nabla^*_X FY + fh^*(X, FY) + Bh(X, FY) \tag{4.22}$$

for any $X, Y \in \Gamma(D_2)$. Replacing X by Y and subtracting resulting equation from this equation, we get

$$\begin{aligned} [X, Y] &= f(\nabla^{*t}_X FY - \nabla^{*t}_Y FX) + f(h^*(X, FY) - h^*(Y, FX)) \\ &\quad -Bh(X, FY) + Bh(Y, FX) \end{aligned} \tag{4.23}$$

thus $[X, Y] \in \Gamma(D_2)$ iff $\nabla^*_X FY - \nabla^*_Y FX \in \Gamma(F(D_2))$, $Bh(X, FY) = Bh(Y, FX)$, $h^*(X, FY) = h^*(Y, FX)$, which completes the proof. □

From Theorem 4.9 we get the following corollary.

Corollary 4.10. Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then D_2 defines a totally geodesic foliation in M iff
(i) $\nabla^*_X FY \in \Gamma(F(D_2))$,
(ii) $Bh(X, FY) = 0$,
(iii) $h^*(X, FY) = 0$,
for any $X, Y \in \Gamma(D_2)$.

Theorem 4.11. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. Then $F(D_2)$ is integrable iff*
(i) $g(A^*_{\xi}F\xi', FZ) = g(A^*_{\xi'}F\xi, FZ)$,
(ii) $\tilde{g}(h^l(F\xi, \xi'), \xi'') = \tilde{g}(h^l(F\xi', \xi), \xi'')$,
(iii) $\tilde{g}(h^s(F\xi, \xi'), W) = \tilde{g}(h^s(F\xi', \xi), W)$,
(iv) $g(A_N F\xi, F\xi') = g(A_N F\xi', F\xi)$,
for any $Z \in \Gamma(D_0)$, $\xi, \xi', \xi'' \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_2)$.

Proof. Using the definition of GCR-lightlike submanifolds, $F(D_2)$ is integrable iff

$$\tilde{g}([F\xi, F\xi'], Z) = \tilde{g}([F\xi, F\xi'], F\xi'') = \tilde{g}([F\xi, F\xi'], FW) = \tilde{g}([F\xi, F\xi'], N) = 0,$$

for any $Z \in \Gamma(D_0)$, $\xi, \xi', \xi'' \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_2)$. Then, from (2.7), (2.8), (2.9) and (2.14) we obtain

$$\begin{aligned} g([F\xi, F\xi'], Z) &= g(\tilde{\nabla}_{F\xi}F\xi' - \tilde{\nabla}_{F\xi'}F\xi, Z) = g(\tilde{\nabla}_{F\xi}\xi' - \tilde{\nabla}_{F\xi'}\xi, FZ) \\ &= g(A^*_{\xi}F\xi' - A^*_{\xi'}F\xi, FZ), \end{aligned} \tag{4.24}$$

$$\begin{aligned} g([F\xi, F\xi'], F\xi'') &= g(\tilde{\nabla}_{F\xi}F\xi', F\xi'') - g(\tilde{\nabla}_{F\xi'}F\xi, F\xi'') \\ &= g(\tilde{\nabla}_{F\xi}\xi', \xi'') - g(\tilde{\nabla}_{F\xi'}\xi, \xi'') \\ &= \tilde{g}(h^l(F\xi, \xi'), \xi'') - \tilde{g}(h^l(F\xi', \xi), \xi''), \end{aligned} \tag{4.25}$$

$$\begin{aligned} g([F\xi, F\xi'], FW) &= g(\tilde{\nabla}_{F\xi}F\xi', FW) - g(\tilde{\nabla}_{F\xi'}F\xi, FW) \\ &= g(\tilde{\nabla}_{F\xi}\xi', W) - g(\tilde{\nabla}_{F\xi'}\xi, W) \\ &= \tilde{g}(h^s(F\xi, \xi'), W) - \tilde{g}(h^s(F\xi', \xi), W), \end{aligned} \tag{4.26}$$

$$\begin{aligned} \tilde{g}([F\xi, F\xi'], N) &= \tilde{g}(\tilde{\nabla}_{F\xi}F\xi', N) - \tilde{g}(\tilde{\nabla}_{F\xi'}F\xi, N) \\ &= -\tilde{g}(F\xi', \tilde{\nabla}_{F\xi}N) + \tilde{g}(F\xi, \tilde{\nabla}_{F\xi'}N) \\ &= g(A_N F\xi, F\xi') - g(A_N F\xi', F\xi). \end{aligned} \tag{4.27}$$

Thus from (4.24)-(4.27), the result follows. □

From Theorem 4.11 we obtain the following corollary.

Corollary 4.12. Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $(\tilde{M}, \tilde{g}, F)$. The distribution $F(D_2)$ is integrable iff
(i) $\tilde{g}(h^l(F\xi', FZ), \xi) = \tilde{g}(h^l(F\xi, FZ), \xi')$,
(ii) $g(A^*_{\xi''}\xi', F\xi) = g(A^*_{\xi''}\xi, F\xi')$,
(iii) $\tilde{g}(\xi', D^l(F\xi, W)) = \tilde{g}(\xi, D^l(F\xi', W))$,
(iv) $\tilde{g}(h^*(F\xi, F\xi'), N) = \tilde{g}(h^*(F\xi', F\xi), N)$,
for any $Z \in \Gamma(D_0)$, $\xi, \xi', \xi'' \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_2)$.

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